The conservativity problem between fragments of intermediate logics and its application

(中間命題論理の論理断片間の保存性問題とその応用)

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Chapter 1

Introduction

A propositional logic is a rough formalization of deductions in mathematics. In propositional logics, a mathematical proposition is expressed by the language consisting of (propositional) variables and logical symbols, the implication ($\to$), the conjunction ($\wedge$), the disjunction ($\vee$) and the negation ($\neg$). An expression of a mathematical proposition is called a (propositional) formula. Every propositional logic have the following *modus ponens* rule:

if formulas A and $A \to B$ are provable, the formula B is also provable.

For a given propositional logic $\mathbf{L}$, let us treat $\mathbf{L}$ as the set of all provable formulas in $\mathbf{L}$.

The formalization of mathematics we usually use is called the *classical logic* $\mathbf{C}$. It is known that $\mathbf{C}$ has the law of excluded middle, i.e., $A \vee \neg A \in \mathbf{C}$ for every formula A.

The law of excluded middle allows proof of disjunctions $A \vee B$ such that neither A nor B is provable. On the other hand, Brouwer's intuitionism[3, 4] claimed that deductions in mathematics should be constructive and constructive proofs do not allow the law of excluded middle. Heyting[11] formalized Brouwer's intuitionism into a propositional logic, which is called the *intuitionistic logic* $\mathbf{H}$.

There are numerous studies for $\mathbf{C}$ and $\mathbf{H}$ since they have many good properties. In particular, the intuitionistic logic has many constructive properties in contrast to the classical logic. For example, $\mathbf{H}$ has the following *disjunction property*:

if formulas A and B satisfy $A \vee B \in \mathbf{H}$, we have $A \in \mathbf{H}$ or $B \in \mathbf{H}$.

$\mathbf{C}$ does not have the disjunction property ($p \vee \neg p$ is a counter example).

The intuitionistic logic is strictly weaker than the classical logic, i.e., $\mathbf{H} \subsetneq \mathbf{C}$. The propositional logic $\mathbf{L}$ is an *intermediate propositional logic* if $\mathbf{H} \subseteq \mathbf{L} \subseteq \mathbf{C}$. There are infinite intermediate logics. Intermediate logics have been researched until now since studies for intermediate logics satisfying a given property are also studies for the property itself.

In this article, we focus on the separability and the conservativity property. We will survey the definition and a historical background of the separability condition.

Throughout this article, the symbol $\mathcal{S}$ means a set of logical symbols containing $\to$ (i.e., $\to \in \mathcal{S} \subseteq \{\to, \wedge, \vee, \neg\}$). We say a propositional formula is an $\mathcal{S}$-formula if it does not contain logical symbols other than elements of $\mathcal{S}$. We say an $\mathcal{S}$-formula A is $\mathcal{S}$-provable in a given axiomatization $\mathbf{H} + \Gamma$ of an intermediate logic if there exists a deduction (proof figure) of $\mathbf{H} + \Gamma \vdash A$ containing logical symbols only in $\mathcal{S}$.

The notion of separability was introduced by Wajsberg[29]. A given axiomatization $\mathbf{H}+\Gamma$ of an intermediate logic is separable if $\mathbf{H} + \Gamma$ satisfies the following two conditions:

1. (normality) every axiom is $\{\to\}$, $\{\to, \wedge\}$, $\{\to, \vee\}$ or $\{\to, \neg\}$-formula;

2. (completeness) every provable $\mathcal{S}$-formula is $\mathcal{S}$-provable for every $\mathcal{S}$.

Wajsberg claimed that $\mathbf{H}$ is separable in the same paper. However, Church[8] pointed out that the proof in the paper contains a mistake, which is discussed in Kabziński and Porebska[17, 18]. The first correct proof of separability of $\mathbf{H}$ is given by Curry[7]. In [7], Curry gives a method that convert a proof of intuitionistic sequent calculus $\mathbf{LJ}$ to a proof of $\mathbf{H}$. His converting method satisfies that, if a proof of the $\mathbf{LJ}$ satisfies the completeness condition, the proof of $\mathbf{H}$ which is converted from the proof of $\mathbf{LJ}$ also satisfies the completeness condition. Therefore, the cut-elimination theorem (and the subformula property which is a corollary of cut-elimination theorem) shows the completeness condition of $\mathbf{H}$. By this result, we can regard the completeness condition as a weaker form than the subformula property. Therefore, Curry's method can be applied to some intermediate logics with cut-free systems, especially the classical logic $\mathbf{C}$. However, there are few intermediate logics that Curry's method is applicable. By methods different from Curry's one, Hosoi[13, 15] proved that intermediate logics $\mathbf{H} + \neg p \vee \neg\neg p$ and $\mathbf{H} + (p \to q) \vee (q \to p)$ have separable axiomatizations (thus they are separable as logics).

Since Curry and Hosoi's methods are syntactical, it is hard to apply their method to general intermediate logics. On the other hand, Horn[12] proved the separability of $\mathbf{H}$ by a semantical method. Horn constructed $\mathcal{S}$-algebras which characterize the set of $\mathcal{S}$-provable formulas in a given intermediate logic. Notice that the $\mathcal{S}$-algebras are generalizations of the Heyting algebras since an $\mathcal{S}$-algebra means a Heyting algebra if $\mathcal{S} = \{\to, \wedge, \vee, \neg\}$.

Khomich[19, 20, 21, 22, 23, 25] gave many important and general results on separability. He mainly examined the separability of logics each of which is axiomatized by disjunction-free formulas. We introduce two theorems which seem particularly important.

Khomich proved that an intermediate logic $\mathbf{L}$ has a separable axiomatization if $\mathbf{L}$ is tabular (characterized by a finite Kripke frame) and $\mathbf{L}$ has a normal axiomatization. This theorem is proved by McKay[27]. However, Khomich[25] pointed out that the proof in [27] contains a mistake and corrected the proof by the method in [25].

Also, Khomich[22] proved that an intermediate logic $\mathbf{L}$ has a separable axiomatization if $\mathbf{L}$ can be axiomatized only by $\{\to, \wedge\}$-formulas.

Therefore, the remaining problem for the separability is the case Khomich's two theorems above can not apply, i.e., the separability of non-tabular logics each of which needs the disjunction to axiomatize.

The conservativity is more basic concept of the completeness. Let Γ be a set of $\mathcal{S}$-formulas, $\mathbf{H}+\Gamma$ be an axiomatization of an intermediate logic and $\mathcal{S} \subseteq \mathcal{S}'$. We say $\mathbf{H}_{\mathcal{S}'} + \Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}} + \Gamma$ if every $\mathcal{S}'$-provable $\mathcal{S}$-formula is also $\mathcal{S}$-provable. A given intermediate logic $\mathbf{H}+\Gamma$ satisfies the completeness condition if and only if $\mathbf{H}_{\{\to,\wedge,\vee,\neg\}} + \Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}}+\Gamma$ for every $\mathcal{S}$. The conservativity condition suggests that the role of each of logical symbols $\wedge$, $\vee$ and $\neg$ are mutually independent in the axiomatization. For example, in [7], Curry's proved for separability of $\mathbf{H}$ by showing the conservativity for any pair $(\mathcal{S},\mathcal{S}')$ by the subformula property.

Wroński[31] gave a general result for conservativity problem. He proved that $\mathbf{H}_{\mathcal{S}'}+\Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}}+\Gamma$ for every set Γ of $\mathcal{S}$-formulas if and only if $\wedge\in\mathcal{S}$ for every $\mathcal{S}$ and $\mathcal{S}'$ satisfying $\mathcal{S}\subseteq\mathcal{S}'$ and a set Γ of $\mathcal{S}$-formulas. However, when $\wedge\notin\mathcal{S}$, the theorem does not give a method to determine whether a given axiomatization $\mathbf{H}_{\mathcal{S}'}+\Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}}+\Gamma$ for a given Γ.

Khomich and Wroński's theorems above are proved by algebraic methods. In particular, their idea seem to be based on the idea of Jankov's characteristic formula ([16]) [1]. Let $\mathbf{M}$ be a subdirectly irreducible finite $\mathcal{S}$-algebra. Jankov's characteristic formula $X_{\mathbf{M}}$ is constructed from $\mathbf{M}$. Jankov proved the relation between the embeddability of $\mathbf{M}$ and the refutability of $X_{\mathbf{M}}$. Precisely, $\mathbf{M}$ is the "smallest" (see Chapter 4) algebra which does not validate $X_{\mathbf{M}}$. His method is easily extended to subdirectly irreducible $\mathcal{S}$-algebras.

In Chapter 4, we give a general algebraic characterization for conservativity by using Jankov's characteristic formula.

Theorem 1.0.1. *Let $\mathcal{S}\subseteq\mathcal{S}'$ and Γ be a set of $\mathcal{S}$-formulas. The following are equivalent.*

1. $\mathbf{H}_{\mathcal{S}'}+\Gamma$ *is a conservative extension of* $\mathbf{H}_{\mathcal{S}}+\Gamma$*;*
2. *if an $\mathcal{S}$-algebra $\mathbf{M}$ validates every $\gamma\in\Gamma$, $\mathbf{M}$ is $\mathcal{S}$-embeddable in an $\mathcal{S}'$-algebra which validates every $\gamma\in\Gamma$.*

Also in Chapter 4, for the case $\mathcal{S}\cup\{\wedge\}\subseteq\mathcal{S}'$, we give a criteria for the conservativity. For a given $\mathcal{S}$-algebra $\mathbf{M}$, we constructed the smallest $\mathcal{S}\cup\{\wedge\}$-algebra $C(\mathbf{M})$ by using Horn's method ([12]). We proved that the class

$$\{C(\mathbf{M})\mid \mathbf{M}\text{ validates every }\gamma\in\Gamma\}$$

gives a criteria of the conservativity for a given axiomatization $\mathbf{H}+\Gamma$ of an intermediate logic.

Theorem 1.0.2. *$\mathcal{S}\cup\{\wedge\}\subseteq\mathcal{S}'$ and Γ be a set of $\mathcal{S}$-formulas. The following are equivalent.*

1. $\mathbf{H}_{\mathcal{S}'}+\Gamma$ *is a conservative extension of* $\mathbf{H}_{\mathcal{S}}+\Gamma$*;*
2. *every $\mathcal{S}\cup\{\wedge\}$-algebra in $\{C(\mathbf{M})\mid\mathbf{M}\text{ validates every }\gamma\in\Gamma\}$ validates every $\gamma\in\Gamma$.*

In Chapter 5, as an application of our criteria, we give a new separable axiomatization of the Gabbay-de Jongh logics $\mathbf{D}_m$ $(m\geq 2)$.

Theorem 1.0.3. *There is a separable axiomatization of the Gabbay-de Jongh logics $\mathbf{D}_m$ $(m\geq 2)$.*

[1] Jankov's characteristic formula does not appear in the proof of the theorem directly. However, the author suppose that the proof uses the idea of Jankov's characteristic formula (see Chapter 4).

We give the separable axiomatization of the Gabbay-de Jongh logics by Jankov's characteristic formula.

In Chapter 6, we revisit a hypersequent calculus **GLCW** which is defined by Avron[1]. It is known that there are two hypersequent calculi which is equivalent to the intermediate logic $\mathbf{H} + (p \to q) \vee (q \to p)$. They are called **GLCW** and **GLC**. Therefore, $\mathbf{GLCW} \vdash H$ if and only if $\mathbf{GLC} \vdash H$ for every hypersequent H. In [1], he treated $\mathbf{GLCW}_{\mathcal{S}}$ and $\mathbf{GLC}_{\mathcal{S}}$ which are subsystems of **GLCW** and proved that $\mathbf{GLCW}_{\mathcal{S}}$ is strictly weaker than $\mathbf{GLC}_{\mathcal{S}}$ if $\wedge \notin \mathcal{S}$. He wrongly claimed that $\mathbf{GLCW}_{\mathcal{S}}$ admits the cut-elimination theorem if $\wedge \notin \mathcal{S}$. We pointed out an error in his proof for the case $\vee \in \mathcal{S}$. Therefore, we corrected his cut-elimination theorem.

Theorem 1.0.4. $\mathbf{GLCW}_{\mathcal{S}}$ *admits the cut-elimination theorem if and only if* $\mathcal{S} \subseteq \{\to, \neg\}$.

Ciabattoni[5] generalized **GLC** to m-**GLC**. m-**GLC** is a hypersequent calculus equivalent to the intermediate logic which is characterized by the class of Kripke frames whose width is m or less. We generalize **GLCW** to m-**GLC** by using Jankov's characteristic formula. which satisfies the following conditions:

- m-$\mathbf{GLCW}_{\mathcal{S}}$ admits the cut-elimination theorem if and only if $\mathcal{S} \subseteq \{\to, \neg\}$;
- m-$\mathbf{GLCW}_{\mathcal{S}} = m$-$\mathbf{GLC}_{\mathcal{S}}$ if and only if $\wedge \in \mathcal{S}$.

Chapter 2

Preliminaries

In this chapter, we introduce the intermediate logics as sets of formulas satisfying some conditions. Also, we introduce the intuitionistic sequent calculus and the Kripke semantics. The intuitionistic sequent calculus helps to show the Curry's theorem for the conservativity problem in Section 3.1. We show the strong completeness theorem of the intuitionistic for the Kripke semantics. Since all theorems in this chapter are well-known, many textbooks for mathematical logic contain proofs of the theorems. For example, [6] and [8].

2.1 Propositional formulas

Definition 2.1.1 (Propositional formulas)**.** *A propositional formula is defined as follows:*

- *propositional variables* $p_0, p_1, \ldots$ *are formulas;*
- *if* A *and* B *are formulas,* $(A \to B), (A \wedge B), (A \vee B)$ *and* $(\neg A)$ *are also formulas.*

We abbreviate $p_0 \to p_0$ to $\top$ and $\neg\top$ to $\bot$.
Brackets "(" and ")" are usually omitted by the following rules:

- the binding force of logical symbols decreases in strength in the series $\neg, \wedge, \vee, \to$;
- $A \to B \to C$ means the formula $(A \to (B \to C))$.

The *complexity* of a formula A is the number of logical symbols occurring in A. The formulas used in construction of a formula A according to the definition above as well as A itself are called *subformulas* of A. For example, the set of all subformulas of $\neg\neg p \to q$ is $\{\neg\neg p \to q,\ \neg\neg p,\ \neg p,\ p,\ q\}$. A map $\sigma : \mathbf{Var} \longrightarrow \mathbf{Form}$ is a *substitution* if σ satisfies $\sigma(A \odot B) = \sigma(A) \odot \sigma(B)$ ($\odot = \to,\ \wedge,\ \vee$) and $\sigma(\neg A) = \neg\sigma(A)$.

2.2 Intermediate logics

Definition 2.2.1. *An intermediate logic* $\mathbf{L}$ *is a set of formulas satisfying the following conditions:*

1. $\mathbf{L}$ *contains all formulas in the following list of the axioms:*

   List of the axioms

(K) $p \to q \to p$;	(S) $(p \to q \to r) \to (p \to q) \to p \to r$;
$(\wedge 1)$ $p \wedge q \to p$;	$(\wedge 2)$ $p \wedge q \to q$;
$(\wedge 3)$ $(p \to q) \to (p \to r) \to p \to q \wedge r$;	$(\vee 1)$ $p \to p \vee q$;
$(\vee 2)$ $q \to p \vee q$;	$(\vee 3)$ $(p \to r) \to (q \to r) \to p \vee q \to r$;
$(\neg 1)$ $(p \to \neg q) \to q \to \neg p$;	$(\neg 2)$ $\neg p \to p \to q$;

2. $A \in \mathbf{L}$ *and* $A \to B \in \mathbf{L}$ *implies* $B \in \mathbf{L}$ *(we call it modus ponens rule);*

3. $A \in \mathbf{L}$ *implies* $\sigma(A) \in \mathbf{L}$ *for every substitution* σ*;*

4. $\bot \notin \mathbf{L}$.

The smallest intermediate logic $\mathbf{H}$ is called the intuitionistic logic.

For a set Γ of formulas, we write $\mathbf{H} + \Gamma$ for the smallest logic containing $\mathbf{H} \cup \Gamma$. If an intermediate logic $\mathbf{L}$ satisfies $\mathbf{L} = \mathbf{H} + \Gamma$, we say that $\mathbf{H} + \Gamma$ is an *axiomatization* of $\mathbf{L}$. If $A \in \{B \mid B \text{ is an axiom of } \mathbf{H}\} \cup \Gamma$, A is called an axiom of $\mathbf{H} + \Gamma$. If $\Gamma = \{A_1, \ldots, A_m\}$, we may write $\mathbf{H} + A_1 + \cdots + A_m$ instead of $\mathbf{H} + \Gamma$.

Example 2.2.2. *The following axiomatizations are same logics.*

1. $\mathbf{H} + \neg\neg p \to p$*;*

2. $\mathbf{H} + p \vee \neg p$*;*

3. $\mathbf{H} + ((p \to q) \to p) \to p$*.*

The logic which is axiomatized by above three examples is called the classical logic $\mathbf{C}$.

Let $\Gamma \cup \Sigma \cup \{A\}$ be a set of formulas and $\mathbf{H} + \Gamma$ be an axiomatization of an intermediate logic. We define the notion of "$\mathcal{P}$ is a proof $\Sigma \vdash_{\mathbf{H}+\Gamma} A$". The intended to meaning of $\Sigma \vdash_{\mathbf{H}+\Gamma} A$ is that A is provable in $\mathbf{H} + \Gamma$ from the assumption Σ.

Definition 2.2.3 (Proofs)**.** *Let* $\Gamma \cup \Sigma \cup \{A, B\}$ *be a set of formulas and* $\mathbf{H} + \Gamma$ *be an axiomatization of an intermediate logic. Proofs of* $\Sigma \vdash_{\mathbf{H}+\Gamma} A$ *are defined as follows:*

1. *if* B *is one of the axiom and* $\sigma(B) = A$ *for a substitution* σ*,* A *is a proof of* $\Sigma \vdash_{\mathbf{H}+\Gamma} A$*;*

2. *if* $A \in \Sigma$*,* A *is a proof of* $\Sigma \vdash_{\mathbf{H}+\Gamma} A$*;*

3. *if* $\mathcal{P}$ *is a proof of* $\Sigma \vdash_{\mathbf{H}+\Gamma} B$ *and* $\mathcal{Q}$ *is a proof of* $\Sigma \vdash_{\mathbf{H}+\Gamma} A \to B$*,*

$$\dfrac{\begin{matrix} \mathcal{P} & \mathcal{Q} \\ A & A \to B \end{matrix}}{B}$$

is a proof of $\Sigma \vdash_{\mathbf{H}+\Gamma} A$ *(we call it modus ponens rule).*

We say A is provable in $\mathbf{H}+\Gamma$ from the assumption Σ if there exists a proof of $\Sigma \vdash_{\mathbf{H}+\Gamma} A$. We usually write $\mathbf{H}+\Gamma \vdash A$ instead of $\emptyset \vdash_{\mathbf{H}+\Gamma} A$. If there exists a proof of $\emptyset \vdash_{\mathbf{H}+\Gamma} A$, we often say just A is provable in $\mathbf{H}+\Gamma$. It is easy to verify that A is provable in $\mathbf{H}+\Gamma$ if and only if $A \in \mathbf{H}+\Gamma$.

Example 2.2.4. $\mathbf{H} \vdash p \to p$.

Proof. It follows from the following proof.

$$\dfrac{p \to p \to p \qquad \dfrac{p \to (p \to p) \to p \qquad (p \to (p \to p) \to p) \to (p \to p \to p) \to p \to p}{(p \to p \to p) \to p \to p}}{p \to p}$$

□

For a given proof, the *length of the proof* is the number of horizontal lines on it (it is equal to the number of times the modus ponens rule is applied on the proof).

Theorem 2.2.5 (Deduction theorem)**.** *For every an axiomatization* $\mathbf{H}+\Gamma$ *for an intermediate logic,* $\Sigma \vdash_{\mathbf{H}+\Gamma} B$ *if and only if* $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to B$.

Proof. We show only if part. There exists a proof $\mathcal{P}$ of $\Sigma \vdash_{\mathbf{H}+\Gamma} B$ by assumption. We show that there exists a proof of $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to B$ by induction on the length l of $\mathcal{P}$.

If $l = 0$, $\mathcal{P}$ is B itself. We have three cases.

($B = A$) Example 2.2.4 shows $\mathbf{H}+\Gamma \vdash A \to B$. Therefore $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to B$.

(B is an axiom of $\mathbf{H}+\Gamma$)

$$\dfrac{B \qquad B \to A \to B}{A \to B}$$

is a proof of $\mathbf{H}+\Gamma \vdash A \to B$. Therefore we have $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to B$.

(B is not an axiom of $\mathbf{H}+\Gamma$ and $B \neq A$) $\mathcal{P}$ is a proof of $\{B\} \vdash_{\mathbf{H}+\Gamma} B$. Therefore $B \in \Sigma - \{A\}$ since $B \neq A$. Therefore $\mathcal{P}$ is a proof of $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to B$.

We now show the case $l > 0$. $\mathcal{P}$ is the following form.

$$\dfrac{\dfrac{\begin{matrix}\Sigma \\ \vdots\end{matrix}}{C} \qquad \dfrac{\begin{matrix}\Sigma \\ \vdots\end{matrix}}{C \to B}}{B}$$

By induction hypothesis, we have $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to C$ and $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to C \to B$. Therefore, the following proof shows $\Sigma - \{A\} \vdash_{\mathbf{H}+\Gamma} A \to B$.

$$\dfrac{\dfrac{\begin{matrix}\Sigma - \{A\} \\ \vdots\end{matrix}}{A \to C} \qquad \dfrac{\dfrac{\begin{matrix}\Sigma - \{A\} \\ \vdots\end{matrix}}{A \to C \to B} \qquad (A \to C \to B) \to (A \to C) \to A \to B}{(A \to C) \to A \to B}}{A \to B}$$

□

Proposition 2.2.6. *Let $\odot = \wedge, \vee$ and A, B and C be formulas. We have $\mathbf{H} \vdash A \odot B \to B \odot A$ and $\mathbf{H} \vdash (A \odot B) \odot C \to A \odot (B \odot C)$.*

Proposition 2.2.6 above allows the notation $\bigwedge_{i=1}^{m} A_i$ which means $(A_1 \wedge (A_2 \wedge \cdots \wedge (A_{m-1} \wedge A_m)) \cdots)$ while considering the provability of formulas (the case for $\vee$ is treated similarly). Thus we can define $\bigwedge \Gamma = A_1 \wedge \cdots \wedge A_m$ and $\bigvee \Gamma = A_1 \vee \cdots \vee A_m$ for a finite set $\Gamma = \{A_1, \ldots, A_m\}$ of formulas. We define $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

For a given finite set $\Gamma = \{A_1, \ldots, A_m\}$ of formulas and a formula B, we abbreviate $A_1 \to \cdots \to A_m \to B$ to $\Gamma \to B$.

Corollary 2.2.7. *Let $\Sigma \cup \{A\}$ be a finite set of formulas. For every axiomatization $\mathbf{H} + \Gamma$ of an intermediate logic, the following are equivalent:*

1. $\mathbf{H} + \Gamma \vdash \bigwedge \Sigma \to A$;
2. $\mathbf{H} + \Gamma \vdash \Sigma \to A$.

2.3 The intuitionistic sequent calculus LJ

Definition 2.3.1. *Let Γ be a finite multiset of formulas and γ be a formula or the emptyset. A pair which denotes $\Gamma \Rightarrow \gamma$ is called a (intuitionistic) sequent.*

In sequents, we write Γ, A or A, Γ for $\Gamma \cup \{A\}$. For example, $\Gamma, A, B, \Delta \Rightarrow \gamma$ means $\Gamma \cup \{A\} \cup \{B\} \cup \Delta \Rightarrow \gamma$.

Definition 2.3.2. *The intuitionistic (propositional) sequent calculus* **LJ** *is the system defined by the axioms and the inference rules.*

Axioms *$A \Rightarrow A$ for any formula A.*

Inference rules *Let A and B be formulas, γ be a formula or emptyset and Γ and Σ are finite multisets of formulas. The inference rules are divided into two types, structural rules and rules for logical symbols.*

Structural rules

$$\frac{\Gamma \Rightarrow \gamma}{A, \Gamma \Rightarrow \gamma}\ (\mathbf{WeL});$$

$$\frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \gamma}\ (\mathbf{We}\mathrm{R});$$

$$\frac{A, A, \Gamma \Rightarrow \gamma}{A, \Gamma \Rightarrow \gamma}\ (\mathbf{Con});$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma}\ (\mathbf{Cut});$$

Rules for logical symbols

$$\frac{\Gamma \Rightarrow A \qquad B, \Sigma \Rightarrow \gamma}{A \to B, \Gamma, \Sigma \Rightarrow \gamma} \ (\to \mathrm{L});$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \ (\to \mathrm{R});$$

$$\frac{A, \Gamma \Rightarrow \gamma}{A \wedge B, \Gamma \Rightarrow \gamma} \ (\wedge \mathrm{L});$$

$$\frac{B, \Gamma \Rightarrow \gamma}{A \wedge B, \Gamma \Rightarrow \gamma} \ (\wedge \mathrm{L});$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \ (\wedge \mathrm{R});$$

$$\frac{A, \Gamma \Rightarrow \gamma \qquad B, \Gamma \Rightarrow \gamma}{A \vee B, \Gamma \Rightarrow \gamma} \ (\vee \mathrm{L});$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \ (\vee \mathrm{R1});$$

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \ (\vee \mathrm{R2});$$

$$\frac{\Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow} \ (\neg \mathrm{L});$$

$$\frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A} \ (\neg \mathrm{R}).$$

The notation $\vdash_{\mathbf{LJ}}$ *which means provability in* **LJ** *is defined as follows:*

1. $\vdash_{\mathbf{LJ}} S$ *if* S *is an axiom sequent;*

2. $\vdash_{\mathbf{LJ}}$ *is closed under the above inference rules, for example, the rule* $(\vee \mathrm{L})$ *means that* $\vdash_{\mathbf{LJ}} A, \Gamma \Rightarrow \gamma$ *and* $\vdash_{\mathbf{LJ}} B, \Gamma \Rightarrow \gamma$ *implies* $\vdash_{\mathbf{LJ}} A \vee B, \Gamma \Rightarrow \gamma$ *for every formula* A *and* B*, every finite multiset* Γ *of formulas and every formula or emptyset* γ*.*

A proof figure of **LJ** is defined in the same manner as intermediate logics.

Example 2.3.3. $\vdash_{\mathbf{LJ}} \neg\neg\neg p \Rightarrow \neg p$

Proof. It follows from the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p,\ \neg p \Rightarrow}}{p \Rightarrow \neg\neg p}}{\neg\neg\neg p,\ p \Rightarrow}}{\neg\neg\neg p \Rightarrow \neg p}$$

□

Theorem 2.3.4. *Let* Γ *be a multiset of formulas and* γ *be a formula or the emptyset. We have that* $\Gamma \vdash_{\mathbf{H}} T(\gamma)$ *if and only if* $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow \gamma$*, where* $T(\gamma) = \begin{cases} \gamma & (\gamma \neq \emptyset) \\ \bot & (\gamma = \emptyset). \end{cases}$

The notation $\vdash^{cf}_{\mathbf{LJ}} \Gamma \Rightarrow A$ means that there exists a cut-free proof (i.e., a proof such that the cut rule has never been applied in it) of $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow A$.

Theorem 2.3.5 (Cut-elimination theorem)**.** *Let* Γ *be a multiset of formulas and* γ *be a formula or the emptyset. We have that* $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow \gamma$ *if and only if* $\vdash^{cf}_{\mathbf{LJ}} \Gamma \Rightarrow \gamma$.

Corollary 2.3.6 (Subformula property)**.** *Let* Γ *be a multiset of formulas and* γ *be a formula or the emptyset. If* $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow \gamma$*, there exists a proof of* $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow \gamma$ *such that every formula which occurs in the proof is a subformula of a formula which occurs in the sequent* $\Gamma \Rightarrow \gamma$*.*

Proof. Every proof of $\vdash^{cf}_{\mathbf{LJ}} \Gamma \Rightarrow \gamma$ satisfies the condition. □

The following *inversion lemma* for **LJ** is a lemma of the cut-elimination theorem for **LJ**. In Chapter 6, we will show a generalized inversion lemma for a hypersequent calculus to prove the cut-elimination theorem for it.

Lemma 2.3.7 (Inversion lemma)**.** *Let* $\Gamma \cup \{A, B\}$ *be a finite multiset of formulas. We have that* $\vdash^{cf}_{\mathbf{LJ}} \Gamma, A \Rightarrow B$ *if and only if* $\vdash^{cf}_{\mathbf{LJ}} \Gamma \Rightarrow A \to B$*.*

Proof. The only if part is obtained by applying ($\to$ R). We show the converse by induction on the length l of the proof of $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow A \to B$.

If $l = 0$, $\Gamma \Rightarrow A \to B$ is an axiom of **LJ**, i.e., $\Gamma = A \to B$. Therefore this case follows from the fact $\vdash_{\mathbf{LJ}} A, A \to B \Rightarrow B$.

If $l > 0$, $\Gamma \Rightarrow A \to B$ is obtained by applying inference rules. We show the case that $\Gamma \Rightarrow A \to B$ is obtained by applying ($\to$L) finally (the other cases are similarly). Thus we assume $\Gamma = \{C \to D\} \cup \Sigma \cup \Pi$ and the proof is the following form:

$$\dfrac{\begin{array}{c}\mathcal{L}\\ \Sigma \Rightarrow C\end{array} \qquad \begin{array}{c}\mathcal{R}\\ D, \Pi \Rightarrow A \to B\end{array}}{C \to D, \Sigma, \Pi \Rightarrow A \to B}\ (\to\mathrm{L})$$

Therefore we have $\vdash_{\mathbf{LJ}} D, \Pi, A \Rightarrow B$ by induction hypothesis. Consequently, we obtain $C \to D, \Sigma, \Pi, A \Rightarrow B$. □

2.4 Kripke frames

Definition 2.4.1. *A Kripke frame* $(W, \leq)$ *is a partially ordered set, where* $W \neq \emptyset$*.*

For a Kripke frame $(W, \leq)$, a subset $U \subseteq W$ is called hereditary if ($x \in U$ and $x \leq y$) implies $y \in U$. We write $\mathbf{Her}(W)$ for the set of all hereditary subsets of $(W, \leq)$.

Definition 2.4.2. *A Kripke model* $(W, \leq, v)$ *is a pair of a Kripke frame* $(W, \leq)$ *and a map* $v : \mathbf{Var} \longrightarrow \mathbf{Her}(W)$*.*

For a Kripke model $\mathfrak{M} = (W, \leq, v)$, we say that $\mathfrak{M}$ is a model on W.

Definition 2.4.3. *Let* $\mathfrak{M} = (W, \leq, v)$ *be a Kripke model. A binary relation* $\models_{\mathfrak{M}}$ $(\subseteq W \times \mathbf{Form})$ *is defined as follows:*

1. $x \models_{\mathfrak{M}} p$ *if and only if* $x \in v(p)$*;*
2. $x \models_{\mathfrak{M}} A \to B$ *if and only if* $y \models_{\mathfrak{M}} A$ *implies* $y \models_{\mathfrak{M}} B$ *for every* $y \geq x$*;*
3. $x \models_{\mathfrak{M}} A \wedge B$ *if and only if* $x \models_{\mathfrak{M}} A$ *and* $x \models_{\mathfrak{M}} B$*;*
4. $x \models_{\mathfrak{M}} A \vee B$ *if and only if* $x \models_{\mathfrak{M}} A$ *or* $x \models_{\mathfrak{M}} B$*.*
5. $x \models_{\mathfrak{M}} \neg A$ *if and only if* $y \not\models_{\mathfrak{M}} A$ *for every* $y \geq x$*.*

We say $\mathfrak{M} \models A$ if $x \models_{\mathfrak{M}} A$ for every $x \in W$. We say $W \models A$ if $\mathfrak{M} \models A$ for every model $\mathfrak{M}$ on W. If K be a class of frames (models), we say $K \models A$ if $W \models A$ ($\mathfrak{M} \models A$) for every frame (model) $W \in K$ ($\mathfrak{M} \in K$). We say A is valid in W (W validates A) if $W \models A$ and A is refutable in W (W refutes A) if not.

Lemma 2.4.4. *Let* $\mathfrak{M}$ *be a model on* W *and* A *be a formula. If* $x \models_{\mathfrak{M}} A$ *and* $x \leq y$*,* $y \models_{\mathfrak{M}} A$*.*

Proposition 2.4.5. *Let* K *be a class of Kripke frames. The set* $\{A \in \mathbf{Form} \mid K \models A\}$ *is an intermediate logic.*

Corollary 2.4.6 (Soundness theorem)**.** *Let* A *be a formula. We have that* A *is valid in all Kripke frames if* $\mathbf{H} \vdash A$*.*

Proof. Since $\mathbf{H}$ is the smallest intermediate logic and the set $\{A \in \mathbf{Form} \mid K \models A\}$ is an intermediate logic. □

Theorem 2.4.7 (Completeness theorem)**.** *Let* A *be a formula. We have that* $A \in \mathbf{H}$ *if* A *is valid in all Kripke frames.*

To show the complete theorem, we define the *canonical frame* of $\mathbf{H}$.

Definition 2.4.8 (Maximal consistent pair)**.** *Let* U *and* V *are sets of formulas.*

1. *A pair* (U, V) *is consistent if* $\nvdash_{\mathbf{LJ}} \Gamma \Rightarrow \gamma$ *for any finite subset* $\Gamma \subseteq U$ *and any* γ *such that* $\gamma \in V$ *or* $\gamma = \emptyset$*.*

2. *A pair (U, V) is maximal consistent if (U, V) is consistent and $U \cup V = \mathbf{Form}$.*

We say that a pair (U, V) is inconsistent if it is not consistent.

Lemma 2.4.9. *If a pair (U, V) is consistent, there exists a pair (U', V') satisfying $U \subseteq U'$, $V \subseteq V'$ and that (U', V') is maximal consistent.*

Proof. Let $\mathbf{N}$ be a set of all natural numbers and $\mathbf{Form} = \{A_k \mid k \in \mathbf{N}\}$ ($\mathbf{Form}$ is a countable infinity set). We define U_i and V_i ($i \in \mathbf{N}$) as follows:

1. $U_0 = U$ and $V_0 = V$;
2. $(U_{k+1},\ V_{k+1}) = \begin{cases} (U_k \cup \{A_k\},\ V_k) & \text{(if } (U_k \cup \{A_k\},\ V_k) \text{ is consistent)} \\ (U_k,\ V_k \cup \{A_k\}) & \text{(otherwise);} \end{cases}$
3. $(U',\ V') = (\bigcup_{k=1}^{\infty} U_k,\ \bigcup_{k=1}^{\infty} V_k)$.

We first show that $(U_{k+1},\ V_{k+1})$ is consistent for every k. If $(U_{k+1},\ V_{k+1})$ is inconsistent (not consistent), both $(U_k \cup \{A_k\},\ V_k)$ and $(U_k,\ V_k \cup \{A_k\})$ are inconsistent. Thus, there are a finite subset $\Sigma \subseteq U_k \cup \{A_k\}$ and γ such that $\gamma \in V_k$ or $\gamma = \emptyset$ satisfying $\vdash_{\mathbf{LJ}} \Sigma \Rightarrow \gamma$. Similarly, there are a finite subset $\Pi \subseteq U_k$ and δ such that $\delta \in V_k \cup \{A_k\}$ or $\delta = \emptyset$ satisfying $\vdash_{\mathbf{LJ}} \Pi \Rightarrow \delta$. However, we can assume that $C = A_k$ since $C \neq A_k$ and $\vdash_{\mathbf{LJ}} \Pi \Rightarrow C$ shows that $(U_k,\ V_k)$ is inconsistent. Therefore we obtain the following proof.

$$
\dfrac{\begin{array}{c}\mathcal{L} \\ \Pi \Rightarrow A_k\end{array} \qquad \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{R} \\ \Sigma \Rightarrow \gamma\end{array}}{\vdots}\end{array}}{A_k, \Sigma - \{A_k\} \Rightarrow \gamma}\ \text{(some structural rules)}}{\Pi,\ \Sigma - \{A_k\} \Rightarrow \gamma}\ (\mathbf{Cut})
$$

It shows that (U_k, V_k) is inconsistent, contradiction with induction hypothesis.

We next show (U', V') is consistent. If (U', V') is inconsistent, there are a finite subset $\Sigma \subseteq U'$ and γ such that $\gamma \in V_k$ or $\gamma = \emptyset$ satisfying $\vdash_{\mathbf{LJ}} \Sigma \Rightarrow \gamma$. Since Σ is finite, $\vdash_{\mathbf{LJ}} \Sigma \Rightarrow \gamma$ also shows that (U_k, V_k) is inconsistent for some $k \in \mathbf{N}$. It contradicts to the fact we proved above.

(U', V') is maximal since every $A_k \in \mathbf{Form}$ satisfies $A_k \in U_{k+1} \cup V_{k+1} \subseteq U' \cup V'$.

Consequently, we obtained the desired maximal consistent pair (U', V'). □

Definition 2.4.10 (Canonical model)**.** *The canonical model $(\mathbf{Can_H}, \subseteq, \models_{\mathfrak{M}})$ is the Kripke frame defined as follows:*

1. $\mathbf{Can_H} = \{U \subseteq \mathbf{Form} \mid (U, \mathbf{Form} - U)$ is maximal consistent$\}$*;*
2. *$U \models_{\mathfrak{M}} p$ if and only if $p \in U$.*

Lemma 2.4.11 (Truth lemma)**.** *Let A be a formula. We have that $U \models_{\mathfrak{M}} A$ if and only if $A \in U$.*

Proof. Induction on the complexity c of A. The case $c = 0$ is guaranteed by the definition of the canonical model.

In the case $c \geq 0$, we give a proof for the case that $A = B \to C$ (the other cases are similarly or much easier).

We show the contraposition of the only if part. Suppose that $B \to C \notin U$. We first verify that $(U \cup \{B\}, \{C\})$ is consistent. If not, we have a finite subset $\Sigma \subseteq U$ satisfying $\vdash_{\mathbf{LJ}} \Sigma, B \Rightarrow C$. Therefore we obtain $\vdash_{\mathbf{LJ}} \Sigma \Rightarrow B \to C$. It implies that $(U, \{A\})$ is inconsistent. Therefore, $(U, \mathbf{Form} - U)$ is also inconsistent, which contradicts the definition of the canonical model. Consequently, $(U \cup \{B\}, \{C\})$ is consistent. Then, by Lemma 2.4.8, there exists a maximal consistent pair (U', V') satisfying both $U \cup \{B\} \subseteq U'$ and $C \in V'$. It implies that $U' \in \mathbf{Can_H}$. Moreover, we have $U' \models_{\mathfrak{M}} B$ and $U' \not\models_{\mathfrak{M}} C$ by induction hypothesis. Therefore we obtain $U \not\models_{\mathfrak{M}} A$.

We show the contraposition of the if part. Suppose that $U \not\models_{\mathfrak{M}} B \to C$. Then we have $V \supseteq U$ satisfying $V \models_{\mathfrak{M}} B$ and $V \not\models_{\mathfrak{M}} C$. By induction hypothesis, we have $B \in V$ and $C \notin V$. Therefore, we obtain $B \to C \notin V$ since $B \to C \in V$ implies V is inconsistent in spite of $V \in \mathbf{Can_H}$. Consequently, we obtain $B \to C \notin U$. $\square$

We now show the completeness theorem.

Suppose $A \notin \mathbf{H}$. Then we have $\not\vdash_{\mathbf{LJ}} A$ which implies that $(\emptyset,\ \{A\})$ is consistent. Therefore we have $U \in \mathbf{Can_H}$ satisfying $A \notin U$ by Lemma 2.4.8. By Lemma 2.4.11, we have $U \not\models_{\mathfrak{M}} A$. Consequently, A is not valid in $\mathbf{Can_H}$. Therefore we proved that $\mathbf{Can_H}$ is the desired Kripke frame. $\square$

We note that the soundness theorem and completeness theorem for $\mathbf{H}$ can be enhanced with the following results.

Corollary 2.4.12 (Strong completeness theorem)**.** *The following are equivalent even if Γ is an infinite set of formulas:*

1. $\Gamma \vdash_{\mathbf{H}} A$*;*
2. *for every Kripke model $\mathfrak{M}$ and $x \in \mathfrak{M}$, the following holds: $x \models_{\mathfrak{M}} A$ if $x \models_{\mathfrak{M}} \gamma$ for every $\gamma \in \Gamma$.*

Definition 2.4.13. *Let $\mathbf{L}$ be an intermediate logic and K be a class of Kripke frames. We say $\mathbf{L}$ is characterized by K if the following holds: $A \in \mathbf{L}$ if and only if $K \models A$.*

The soundness theorem and completeness theorem for $\mathbf{H}$ shows that $\mathbf{H}$ is characterized by the class of all Kripke frames. Similarly, some intermediate logics are characterized by Kripke frames. The table below introduces some famous intermediate logics each of which is characterized by a class of Kripke frames.

The notation $|W|$ means the number of elements of W. The width of W means the maximum number of elements of mutually incomparable subset of W. The depth of W means the maximum number of elements of totally ordered subset of W. For example, $\mathbf{H} + (p \to q) \vee (q \to p)$ is characterized by all totally ordered (linear) Kripke frames.

logic	Kripke frames
$\mathbf{H}$	all frames
$\mathbf{H} + \neg\neg p \to p$	$\|W\| = 1$
$\mathbf{H} + (p \to q) \vee (q \to p)$	totally ordered frames
$\mathbf{H} + \bigvee_{i=0}^{n}(\bigwedge_{j \neq i} p_j \to p_i)$	width $\leq n$
$\mathbf{H} + p_n \vee (p_n \to (p_{n-1} \vee (p_{n-1} \to (\cdots \to (p_2 \vee (p_2 \to (p_1 \vee \neg p_1) \cdots)$	depth $\leq n$

Chapter 3

Propositional logics in restricted languages

In this chapter, we consider $\mathcal{S}$-logics which is obtained from intermediate logics by restricting logical symbols to $\mathcal{S}$ satisfying $\to \in \mathcal{S} \subseteq \{\to,\ \wedge,\ \vee,\ \neg\}$ and $\mathcal{S}$-algebras which characterize $\mathcal{S}$-logics. To show some important theorems for $\mathcal{S}$-algebras, we introduce some definition and fundamental results for universal algebras. Also, we introduce Khomich's theorem for $\mathcal{S}$-algebras. Khomich's theorem explains generating set in finite $\mathcal{S}$-algebras which is based on Stone's representation.

3.1 A conservativity result for the intuitionistic logic

Throughout the article, $\mathcal{S}$ ($\mathcal{S}'$ or these with subscriptions) means a set of propositional logical symbols containing $\to$, i.e., $\to \in \mathcal{S} \subseteq \{\to,\ \wedge,\ \vee,\ \neg\}$. A formula A is an $\mathcal{S}$-formula if A contains only elements in $\mathcal{S}$ among the propositional logical symbols. For example, $\neg\neg p \to q \vee r$ is a $\{\to,\ \vee,\ \neg\}$-formula. Thus, every formula is an $\{\to,\ \wedge,\ \vee,\ \neg\}$-formula. We write $\mathbf{Form}_{\mathcal{S}}$ for the set of all $\mathcal{S}$-formulas.

We first introduce a theorem for the intuitionistic logic $\mathbf{H}$.

Lemma 3.1.1 (Curry[7])**.** *Let $\Gamma \cup \{A\}$ be a set of $\mathcal{S}$-formulas satisfying $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow A$ and that $\mathcal{P}$ be a proof figure of $\vdash_{\mathbf{LJ}} \Gamma \Rightarrow A$ such that every formula in $\mathcal{P}$ is an $\mathcal{S}$-formula. Then there exists a proof figure $\mathcal{Q}$ of $\Gamma \vdash_{\mathbf{H}} A$ such that every formula in $\mathcal{Q}$ is an $\mathcal{S}$-formula.*

Proof. Induction on the length of $\mathcal{P}$. □

Theorem 3.1.2 (Curry[7])**.** *Let A be an $\mathcal{S}$-formula $\mathbf{H} \vdash A$. Then there exists a proof figure $\mathcal{P}$ of $\mathbf{H} \vdash A$ such that every formula in $\mathcal{P}$ is an $\mathcal{S}$-formula.*

Proof. We have a proof $\mathcal{Q}$ of $\mathbf{H} \vdash A$ by the assumption. Then we obtain a proof $\mathcal{Q}'$ of $\vdash_{\mathbf{LJ}} \emptyset \Rightarrow A$ such that every formula in $\mathcal{Q}'$ is $\mathcal{S}$-formula by the subformula property. Therefore, by Lemma 3.1.1, we obtain desired proof $\mathcal{P}$ from $\mathcal{Q}'$. □

In Chapter 4, we will consider the analogue of Curry's theorem for intermediate logics.

3.2 $\mathcal{S}$-logics

A substitution σ is an $\mathcal{S}$-substitution if $\sigma(p)$ is an $\mathcal{S}$-formula for every propositional variable p. The definition (intermediate) $\mathcal{S}$-logics is a generalization of the definition of intermediate logics.

Definition 3.2.1. *An $\mathcal{S}$-logic $\mathbf{L}$ is a set of $\mathcal{S}$-formulas satisfying the following conditions:*

1. *every $\mathcal{S}$-formula in the following axiom list is an element of $\mathbf{L}$;*

   Axioms

$(K)\ p \to q \to p;$	$(S)\ (p \to q \to r) \to (p \to q) \to (p \to r);$
$(\wedge 1)\ p \wedge q \to p;$	$(\wedge 2)\ p \wedge q \to q;$
$(\wedge 3)\ (p \to q) \to (p \to r) \to (p \to q \wedge r);$	$(\vee 1)\ p \to p \vee q;$
$(\vee 2)\ q \to p \vee q;$	$(\vee 3)\ (p \to r) \to (q \to r) \to (p \vee q \to r);$
$(\neg 1)\ (p \to \neg q) \to (q \to \neg p);$	$(\neg 2)\ \neg p \to p \to q;$

2. *$A \in \mathbf{L}$ and $A \to B \in \mathbf{L}$ implies $B \in \mathbf{L}$ (we call it modus ponens rule);*

3. *$A \in \mathbf{L}$ implies $\sigma(A) \in \mathbf{L}$ for every $\mathcal{S}$-substitution σ;*

4. *$\bot \notin \mathbf{L}$.*

We write $\mathbf{H}_{\mathcal{S}}$ for the smallest $\mathcal{S}$-logic.

If an $\mathcal{S}$-logic $\mathbf{L}$ is the smallest logic containing $\mathbf{H}_{\mathcal{S}} \cup \Gamma$ for a set Γ of $\mathcal{S}$-formulas, we say that $\mathbf{H}_{\mathcal{S}} + \Gamma$ is an axiomatization of $\mathbf{L}$.

If Γ is a set of formulas, we define $\Gamma_{\mathcal{S}} = \{A \in \Gamma \mid A \text{ is an } \mathcal{S}\text{-formula}\}$.

Proposition 3.2.2. *Let $\mathcal{S} \subseteq \mathcal{S}'$ and $\mathbf{L}$ be an $\mathcal{S}'$-logic. Then $\mathbf{L}_{\mathcal{S}}$ is an $\mathcal{S}$-logic.*

If $\mathbf{L}$ is an $\mathcal{S}'$-logic and $\mathcal{S} \subseteq \mathcal{S}'$, we call an $\mathcal{S}$-logic $\mathbf{L}_{\mathcal{S}}$ the $\mathcal{S}$-*fragment* of $\mathbf{L}$.

By definitions above, $\mathbf{H}_{\mathcal{S}}$ can be defined by the following two ways:

1. the smallest $\mathcal{S}$-logic;

2. the $\mathcal{S}$-fragment of $\mathbf{H}$.

However, Theorem 3.1.2 shows that the two definitions are equivalent. Thus, $\mathbf{H}_{\mathcal{S}}$ is well-defined in both senses.

3.3 Universal algebras

To show theorems for $\mathcal{S}$-algebras which characterize $\mathcal{S}$-logics in Section 3.4, we summarize some basic concepts and results in the theory of universal algebras.

Definition 3.3.1. *An algebra* $\mathbf{M}$ *is a pair* $(M, \langle f_1, \ldots, f_k \rangle)$ *defined as follows:*

1. M *is a set;*

2. *for each* f_i, $\alpha(f_i)$ *is a non-negative integer and* f_i *is an* $\alpha(f_i)$*-ary function, i.e., each* f_i *is a function such that* $f_i : M^{\alpha(f_i)} \longrightarrow M$.

We define a type for a row of non-negative integers. For example, an algebra $(M, \langle f_1, \ldots, f_k \rangle)$ has the type $\langle \alpha(f_1), \ldots, \alpha(f_k) \rangle$. Thus,

Definition 3.3.2. *Let* $\mathbf{M} = (M, \langle f_1, \ldots, f_k \rangle)$ *and* $\mathbf{N} = (N, \langle g_1, \ldots, g_l \rangle)$ *are algebras. We say* $\mathbf{M}$ *and* $\mathbf{N}$ *are same type if* $k = l$ *and* $\langle \alpha(f_1), \ldots, \alpha(f_k) \rangle = \langle \alpha(g_1), \ldots, \alpha(g_l) \rangle$.

We note that each 0-ary function is a constant. If $\mathbf{M} = (M, \langle f_1, \ldots, f_k \rangle)$, we may write $x \in \mathbf{M}$ instead of $x \in M$.

Definition 3.3.3. *For a given type* $\langle n_1, \ldots, n_m \rangle$*, we define a term of a type* $\langle n_1, \ldots, n_m \rangle$ *as follows:*

1. *every* a_i $(i \in I)$ *is a term (*I *is an infinite set of symbols);*

2. *there exists a symbol* O_i *for each* $i = 1, \ldots, m$*;*

3. *if* $x_1, \ldots, x_{n_i}$ *are terms, a string of symbols* $O_i(x_1, \ldots, x_{n_i})$ *is a term for every* $i = 1, \ldots, m$.

Let $\mathbf{Term}_{\langle n_1, \ldots, n_m \rangle}$ be the set of all terms of a type $\langle n_1, \ldots, n_m \rangle$.

Definition 3.3.4 (Term algebra). *Let* $\langle n_1, \ldots, n_k \rangle$ *be a type. The term algebra*

$$\mathbf{TermAlg}_{\langle n_1, \ldots, n_k \rangle} = (\mathbf{Term}_{\langle n_1, \ldots, n_k \rangle}, \langle T_1, \ldots, T_k \rangle)$$

is defined as follows:

1. $\langle \alpha(T_1), \ldots, \alpha(T_k) \rangle = \langle n_1, \ldots, n_k \rangle$*;*

2. *For every* T_i $(i = 1, \ldots k)$, $T_i(x_1, \ldots, x_{\alpha(T_i)}) = O_i(x_1, \ldots, x_{n_i})$.

Definition 3.3.5. *Let* $\mathbf{M} = (M, \langle f_1, \ldots, f_k \rangle)$ *and* $\mathbf{N} = (N, \langle g_1, \ldots, g_k \rangle)$ *be algebras of same type. A map* $h : \mathbf{M} \longrightarrow \mathbf{N}$ *is a homomorphism if* $h(f(x_1, \ldots, x_{\alpha(f_i)}) = g_i(h(x_1), \ldots, f(x_{\alpha(g_i)}))$ *for every* $i = 1, \ldots, k$.

We say a homomorphism h is an *embedding* if it is injective. We say a homomorphism h is an *isomorphism* if it is bijective.

A equation of a type $\langle n_1, \ldots, n_m \rangle$ is a string $s = t$. where $s, t \in \mathbf{Term}_{\langle n_1, \ldots, n_m \rangle}$.

Definition 3.3.6. *Let* $\mathbf{M} = (M, \langle f_1, \ldots, f_k \rangle)$ *be an algebra and* $s, t \in \mathbf{Term}_{\langle \alpha(f_1), \ldots, \alpha(f_k) \rangle}$. *We say that* $\mathbf{M}$ *satisfies the equation* $s = t$ *if* $h(s) = h(t)$ *holds on* $\mathbf{M}$ *for every homomorphism* $h : \mathbf{TermAlg}_{\langle \alpha(f_1), \ldots, \alpha(f_k) \rangle} \longrightarrow \mathbf{M}$.

Definition 3.3.7. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be algebras of same type.* $\mathbf{N}$ *is a subalgebra of* $\mathbf{M}$ *if there exists an embedding* $h : \mathbf{N} \longrightarrow \mathbf{M}$.

Lemma 3.3.8. *Let* $\mathbf{M} = (M, \langle f_1, \ldots, f_k \rangle)$ *and* $\mathbf{N} = (N, \langle g_1, \ldots, g_k \rangle)$ *be algebras of same type. If a map* $h : \mathbf{M} \longrightarrow \mathbf{N}$ *is a homomorphism,* $h(M)$ *is closed under functions* $g_1, \ldots, g_k$ *and* $(h(M), \langle g_1, \ldots, g_k \rangle)$ *is a subalgebra of* $\mathbf{N}$.

Theorem 3.3.9. *Let* $\mathbf{M} = (M,\ \langle f_1, \ldots, f_k \rangle)$ *be an algebra and* $s, t \in \mathbf{Term}_{\langle \alpha(f_1), \ldots, \alpha(f_k) \rangle}$ *such that* $\mathbf{M}$ *satisfies the equation* $s = t$.

1. $h(\mathbf{M})$ *satisfies the equation* $s = t$ *for every homomorphism* h *from* $\mathbf{M}$;

2. $\mathbf{N}$ *satisfies the equation* $s = t$ *for every subalgebra* $\mathbf{N}$ *of* $\mathbf{M}$.

Definition 3.3.10. *Let* $\mathbf{M}_j = (M_j,\ \langle f_1, \ldots, f_k \rangle)$ $(j \in J)$ *be algebras of same type. A direct product* $\prod_{j \in J} \mathbf{M}_j = (\prod_{j \in J} M_j,\ \langle f_1, \ldots, f_k \rangle)$ *is defined as follows:*

1. $\prod_{j \in J} M_j$ *is a direct product (as a set) of* $(M_j)_{j \in J}$;

2. *every function* $f \in \{f_1, \ldots, f_k\}$ *is defined by*

$$f((x_j)^1_{j \in J}, \ldots, (x_j)^{\alpha(f)}_{j \in J}) = (f(x^1_j, \ldots, x^{\alpha(f)}_j))_{j \in J},$$

   where $(x^1_j)_{j \in J}, \ldots, (x^{\alpha(f)}_j)_{j \in J} \in \prod_{j \in J} M_j$.

Theorem 3.3.11. *Let* $\mathbf{M}_i = (M_i,\ \langle f_1, \ldots, f_k \rangle)$ *be algebras of same type and* $s, t \in \mathbf{Term}_{\langle \alpha(f_1), \ldots, \alpha(f_k) \rangle}$ *such that every* $\mathbf{M}_i$ *satisfies the equation* $s = t$. *Then, the direct product* $\prod_{i \in I} \mathbf{M}_i$ *also satisfies* $s = t$.

Definition 3.3.12. *A class of algebras* V *of same type is called a variety if* V *satisfies the following:*

1. V *is closed under homomorphisms, i.e., if* $\mathbf{M} \in V$ *and* h *is a homomorphism from* $\mathbf{M}$, $h(\mathbf{M}) \in V$;

2. V *is closed under subalgebras;*

3. V *is closed under direct products.*

Theorem 3.3.13 (Birkhoff)**.** *Let* $\mathbf{M}_i = (M_i,\ \langle f_1, \ldots, f_k \rangle)$ $(i \in I)$ *be algebras of same type and a class of algebra* $\mathbf{F} = \{\mathbf{N}$: *an algebra* $\mid \mathbf{M}_i$ *and* $\mathbf{N}$ *are of same type* $\}$. *The following are equivalent:*

1. $\{\mathbf{M}_i \mid i \in I\}$ *is a variety;*

2. *there are a set of equations* E *of* $\mathbf{Term}_{\langle \alpha(f_1), \ldots, \alpha(f_k) \rangle}$ *such that* $\{\mathbf{M}_i \mid i \in I\} = \{\mathbf{M} \in \mathbf{F} \mid \mathbf{M}$ *satisfies all equations of* $E\}$.

Definition 3.3.14. *Let* $\mathbf{M}$ *and* $\mathbf{M}_j$ $(j \in J)$ *be algebras of same type.* $\mathbf{M}$ *is a subdirect product of* $\mathbf{M}_j$ $(j \in J)$ *if the following hold:*

1. *there exists a embedding* $h : \mathbf{M} \longrightarrow \prod_{j\in J} \mathbf{M}_j$*; (thus,* $\mathbf{M}$ *is a subalgebra of* $\prod_{j\in J} \mathbf{M}_j$*);*
2. *every projection* $\pi_k \circ h : \mathbf{M} \longrightarrow \mathbf{M}_k$ $(k \in J)$ *defined by* $\pi_k((x_k)_{j\in J}) = x_k$ *is surjective.*

Moreover, we say $\mathbf{M}$ *is a proper subdirect product if* $\mathbf{M}$ *is not isomorphic to* $\mathbf{M}_j$ *for any* $j \in J$*. If* $\mathbf{M}$ *is not a proper subdirect product of any set of algebras, we say* $\mathbf{M}$ *is subdirectly irreducible .*

Theorem 3.3.15 (Birkhoff's factorization theorem)**.** *Every algebra is a subdirect product of a set of subdirectly irreducible algebras.*

To show Birkhoff's factorization theorem, we prepare some definition and lemmas.

Definition 3.3.16. *An equivalence relation* θ *is a congruence on an algebra* $\mathbf{M}$ *if, for every function* f *on* $\mathbf{M}$*,* $x_1\theta y_1, \dots, x_{\alpha(f)}\theta y_{\alpha(f)}$ *implies* $f(x_1, \dots, x_{\alpha(f)})\theta f(y_1, \dots, y_{\alpha(f)})$*.*

For every algebra $\mathbf{M}$, we write $\mathbf{E}$ for the smallest congruence defined by $x\mathbf{E}y$ if and only if $x = y$.

Definition 3.3.17. *Let* $\mathbf{M} = (M, \langle f_1, \dots, f_k\rangle)$ *be an algebra and* θ *be a congruence on* $\mathbf{M}$*. We define the quotient algebra* $\mathbf{M}/\theta = (M/\theta, \{f_i \mid i \in I\})$ *as follows:*

1. $M/\theta = \{[x]_\theta \mid x \in M\}$*;*
2. $f_i([x_1]_\theta, \dots, [x_{\alpha(f_i)}]_\theta) = [f_i(x_1, \dots, x_{\alpha(f_i)})]_\theta$*,*

where $[x]_\theta$ is the equivalence class of x with respect to θ. The definition of $\mathbf{M}/\theta$ is well-defined.

Proposition 3.3.18. *Let* $\mathbf{M}$ *be an algebra. A map* $h : \mathbf{M} \longrightarrow \mathbf{M}/\theta$ *is a homomorphic surjection.*

Proof. For each $x \in \mathbf{M}/\theta$, there exists $y \in \mathbf{M}$ such that $[y]_\theta = x$. Hence h is surjective since $h(y) = x$. Let f be a function of $\mathbf{M}$. Then

$$\begin{aligned} & h(f(x_1, \dots, x_{\alpha(f)})) \\ =\ & [f(x_1, \dots, x_{\alpha(f)})]_\theta \\ =\ & f([x_1]_\theta, \dots, [x_{\alpha(f)}]_\theta) \\ =\ & f(h(x_1), \dots, h(x_\theta)). \end{aligned}$$

It proved that h is a homomorphism. □

Proposition 3.3.19. *Let* $\mathbf{M}$ *be an algebra and* h *be a homomorphism from* $\mathbf{M}$*. Then there exists a congruence* θ *on* $\mathbf{M}$ *such that* $h(\mathbf{M})$ *and* $\mathbf{M}/\theta$ *are isomorphic.*

Proof. The equivalence relation θ defined by $x\theta y$ if and only if $h(x) = h(y)$ is the congruence we desired. $\square$

Corollary 3.3.20. *If an algebra $\mathbf{M}$ is a subdirect product of $\mathbf{M}_j$ $(j \in J)$, there are congruences θ_j $(j \in J)$ on $\mathbf{M}$ such that $\mathbf{M}_j$ is isomorphic to $\mathbf{M}/\theta_j$ for every $j \in J$.*

Proof. By definition of subdirect products, there exists a homomorphic surjection $h_j : \mathbf{M} \longrightarrow \mathbf{M}_j$ for each j. Thus, we obtain the corollary by Proposition 3.3.19. $\square$

Lemma 3.3.21. *Let $\mathbf{M}$ be an algebra and $a,\ b \in \mathbf{M}$ such that $a \neq b$. There is a maximal congruence θ' such that $a \not\theta' b$.*

Proof. Let $A = \{\theta :$ congruence on $\mathbf{M} \mid a \not\theta b\}$ (notice that $A \neq \emptyset$ since $\mathbf{E} \in A$) and $C \subseteq A$ be a chain (totally ordered subset). We show $\theta = \bigcup C$ is an upper bound of C in A.

We show θ is transitive (the other cases are similarly). Suppose $x\theta y$ and $y\theta z$. Then there are $\theta_1,\ \theta_2 \in C$ such that $x\theta_1 y$ and $y\theta_2 z$. Hence we have $x(\theta_1 \cup \theta_2)y$ and $y(\theta_1 \cup \theta_2)z$ which implies $x(\theta_1 \cup \theta_2)z$. Since C is a chain, $\theta_1 \cup \theta_2 = \theta_1$ or θ_2. Therefore $\theta_1 \cup \theta_2 \in C$, i.e., $\theta_1 \cup \theta_2 \subseteq \theta$. Consequently, we obtain $x\theta z$.

Consequently, by Zorn's lemma, there exists a maximal congruence θ' such that $a \not\theta' b$. $\square$

Lemma 3.3.22. *Let $\mathbf{M}$ be an algebra and $\{\theta_i \mid i \in I\}$ be a set of all congruences on $\mathbf{M}$ except $\mathbf{E}$. If $\bigcap\{\theta_i \mid i \in I\} = \mathbf{E}$, $\mathbf{M}$ is a subdirect product of $(\mathbf{M}/\theta_j)_{j\in J}$.*

Proof. Let a map $h : \mathbf{M} \longrightarrow \prod_{i\in I} \mathbf{M}/\theta_i$ be defined by $h(x) = ([x]_{\theta_i})_{i\in I}$. We show h is injective (Proposition 3.3.18 shows that h is homomorphic). Let $x \neq y$. Then we have some θ_k $(k \in I)$ satisfying $x \not\theta_j y$. Therefore $h(x) \neq h(y)$ since $[x]_{\theta_k} \neq [y]_{\theta_k}$.

We show $\pi_j \circ h : \mathbf{M} \longrightarrow \mathbf{M}/\theta_j$ $(j \in I)$ is surjective. Let $x \in \mathbf{M}/\theta_j$. Then $x = [y]_{\theta_j}$ for some $y \in \mathbf{M}$. Hence $\pi_j \circ h(y) = \pi_j(([y]_{\theta_i})_{i\in I}) = [y]_{\theta_j}$. Therefore, $\pi_j \circ h$ is surjective.

Consequently, $\mathbf{M}$ is a subdirect product of $(\mathbf{M}/\theta_i)_{i\in I}$. $\square$

Lemma 3.3.23. *Let $\mathbf{M}$ be an algebra and $\{\theta_i \mid i \in I\}$ be the set of all congruences on $\mathbf{M}$ except $\mathbf{E}$. If $\bigcap\{\theta_i \mid i \in I\} \neq \mathbf{E}$, $\mathbf{M}$ is subdirectly irreducible .*

Proof. We have $a,\ b \in \mathbf{M}$ such that $a\theta_i b$ for every $i \in I$ but $a \neq b$ by the assumption. Suppose that $\mathbf{M}$ is a proper subdirect product of $\mathbf{M}_j$ $(j \in J)$. Then, by Corollary 3.3.20, there are θ_j such that $\mathbf{M}_j = \mathbf{M}/\theta_j$ for each $j \in J$. Thus there exists an embedding $h : \mathbf{M} \longrightarrow \prod_{j\in J} \mathbf{M}/\theta_j$. However, we have $h(a) = h(b)$ since $[a]_\theta = [b]_\theta$ for every congruence θ on $\mathbf{M}$. Therefore, h cannot be injective, contradiction. $\square$

Lemma 3.3.24. *Let $\mathbf{M}$ be an algebra, θ be a congruence on $\mathbf{M}$ and π be a congruence on $\mathbf{M}/\pi$. Then a binary relation π^θ on $\mathbf{M}$ defined by $a\pi^\theta b$ if and only if $[a]_\theta \pi [b]_\theta$ is a congruence on $\mathbf{M}$.*

Lemma 3.3.25. *Let $\mathbf{M}$ be an algebra and $\theta_{a,b}$ be a maximal congruence on $\mathbf{M}$ such that $a \not\theta_{a,b} b$ for $a,\ b \in \mathbf{M}$. Then $\mathbf{M}/\theta_{a,b}$ is subdirectly irreducible .*

Proof. In the proof, we write $[x]$ instead of $[x]_{\theta_{a,b}}$. Let $\{\pi_i \mid i \in I\}$ be the set of all congruences on $\mathbf{M}/\theta_{a,b}$ except $\mathbf{E}$. We show that $[a]\pi_i[b]$ for every $i \in I$. Suppose not, we have $j \in I$ satisfying $[a] \not\pi_j [b]$. We define a binary relation π_j^θ on $\mathbf{M}/\theta_{a,b}$ by $x\pi_j^\theta y$ if and only if $[x]\pi_j[y]$. Then π_j^θ is a congruence on $\mathbf{M}/\theta_{a,b}$ by Lemma 3.3.24. Since $\pi_j \neq \mathbf{E}$, we have $x, y \in \mathbf{M}$ such that $[x] \neq [y]$ and $[x]\pi_j[y]$. Therefore we have $x \not\theta_{a,b} y$ and $a\pi_j^\theta y$ which implies $\theta_{a,b} \subsetneq \pi_j^\theta$. Therefore we have $a\pi_j^\theta b$ by maximality of $\theta_{a,b}$. Consequently, we obtain $[a]\pi_j[b]$. The lemma is proved. $\square$

Now we show Birkhoff's factorization theorem.

Let $\mathbf{M}$ be an algebra such that $\mathbf{M}$ is not subdirectly irreducible. Then, for every $a,\ b \in \mathbf{M}$ such that $a \neq b$, there are maximal congruence $\theta_{a,b}$ such that $a \not\theta_{a,b} b$.

We verify that $\theta_{a,b} \neq \mathbf{E}$ for any $a, b \in \mathbf{M}$. If $\theta_{a,b} = \mathbf{E}$, $\mathbf{E}$ is a maximal congruence satisfying $a \not\mathbf{E} b$. On the other hand, $\mathbf{E}$ is the smallest congruence on $\mathbf{M}$. Therefore, $\mathbf{E}$ is the unique congruence satisfying $a \not\mathbf{E} b$. Consequently, by Lemma 3.3.23, $\mathbf{M}$ is subdirectly irreducible, which contradicts with the assumption.

By the fact $\bigcap\{\theta_{a,b} \mid a,\ b \in \mathbf{M},\ a \neq b\} = \mathbf{E}$ and Lemma 3.3.22, $\mathbf{M}$ is a subdirect product of $\mathbf{M}/\theta_{a,b}$. By Lemma 3.3.25, $\mathbf{M}/\theta_{a,b}$ is subdirectly irreducible for every $a, b \in \mathbf{M}$. The theorem is proved. $\square$

Definition 3.3.26. *Let* $\mathbf{M} = (M,\ \langle f_1, \ldots, f_k\rangle)$ *be an algebra and* $\langle g_1, \ldots, g_l\rangle$ *be a substring of* $\langle f_1, \ldots, f_k\rangle$ *(i.e.,* $l \leq k$*). Then, an algebra* $\mathbf{M}' = (M,\ \langle g_1, \ldots, g_l\rangle)$ *is called the (*$\langle g_1, \ldots, g_l\rangle$*-)reduct of* $\mathbf{M}$.

3.4 $\mathcal{S}$-algebras

We now define $\mathcal{S}$-algebras for $\to \in \mathcal{S} \subseteq \{\to, \wedge, \vee, \neg\}$. $\mathcal{S}$-algebras are generalizations of *Heyting algebras.* If $\mathcal{S} = \{\to,\ \wedge,\ \vee,\ \neg\}$, $\mathcal{S}$-algebras are Heyting algebras. Many concepts of Heyting algebras can be defined for $\mathcal{S}$-algebras in similar ways. However, we give detailed definition and proofs for $\mathcal{S}$-algebras in this section since there are few textbooks which focus on $\mathcal{S}$-algebras.

Definition 3.4.1 ($\mathcal{S}$-algebra)**.** *An algebra* $\mathbf{M} = (M, \{f_\odot \mid \odot \in \mathcal{S}\} \cup \{1\})$ *is an* $\mathcal{S}$*-algebra if* $\mathbf{M}$ *satisfies the following conditions:*

1. *for each* $\odot \in \mathcal{S}$, $\mathbf{M}$ *has the function* $f_\odot$ *and the arity of* $f_\odot$ *is the same as the one of* $\odot$ *(in these algebras, we write simply* $\odot$ *as* $f_\odot$*);*
2. $\mathbf{M}$ *has a constant element* 1*;*
3. $\mathbf{M}$ *satisfies the equations* $1 \to x = x$, $x \to 1 = 1$ *and* $(x \to y) \to (y \to x) \to y = (x \to y) \to (y \to x) \to x$*;*
4. *if an* $\mathcal{S}$*-formula* A *is one of the axioms of* $\mathbf{H}$, $\mathbf{M}$ *satisfies the equation* $\delta(A) = 1$, *where* δ *is conversion of a formula to a term* [1]*.*

[1]For example, $\delta(p \to q \to p) = x \to y \to x$, where $x \to y \to x$ is a term which is an abbreviation of $x\ f_\to\ (y\ f_\to\ x)$.

Definition 3.4.2. *Let* $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra. A map* $v : \mathbf{Form}_{\mathcal{S}} \longrightarrow \mathbf{M}$ *is a valuation if* v *satisfies the following:*

1. $v(x \to y) = v(x) \to v(y)$;
2. *if* $\wedge \in \mathcal{S}$, $v(x \wedge y) = v(x) \wedge v(y)$;
3. *if* $\vee \in \mathcal{S}$, $v(x \vee y) = v(x) \vee v(y)$;
4. *if* $\neg \in \mathcal{S}$, $v(\neg x) = \neg v(x)$.

If $\neg \in \mathcal{S}$, we abbreviate $\neg 1$ to 0.

A is valid in $\mathbf{M}$ ($\mathbf{M}$ validates A) if $v(A) = 1$ for every valuation v on $\mathbf{M}$. A is refutable in $\mathbf{M}$ ($\mathbf{M}$ refutes A) if A is not valid in $\mathbf{M}$. If a valuation w on $\mathbf{M}$ satisfies $w(A) \neq 1$, we say w is a refutation of A on $\mathbf{M}$. Let $\mathbf{L}$ be an $\mathcal{S}$-logic. If an $\mathcal{S}$-algebra $\mathbf{M}$ validates all $\mathcal{S}$-formulas $A \in \mathbf{L}$, we say $\mathbf{M}$ be an $\mathbf{L}$-algebra. Thus, the soundness theorem between $\mathcal{S}$-logics and $\mathcal{S}$-algebras is obvious.

Theorem 3.4.3. *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-logic and* A *be an* $\mathcal{S}$*-formula. Then,* $\mathbf{L} \vdash A$ *implies that every* $\mathbf{L}$*-algebra validates* A.

Proof. It follows from the definition of $\mathcal{S}$-algebras. □

Moreover, by Theorem 3.3.13, we obtain the following proposition.

Proposition 3.4.4. *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-logic. Then the class of all* $\mathbf{L}$*-algebras is a variety.*

Proof. Since $\mathbf{L}$-algebras are defined by equations (see Definition 3.4.1). □

Corollary 3.4.5. *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-logic and* $\mathbf{M}$ *be an* $\mathbf{L}$*-algebra. Then the following hold:*

1. *every* $\mathcal{S}$*-subalgebra of* $\mathbf{M}$ *is an* $\mathbf{L}$*-algebra;*
2. *for every* $\mathcal{S}$*-homomorphism* $h : \mathbf{M} \longrightarrow h(\mathbf{M})$, $h(\mathbf{M})$ *is an* $\mathbf{L}$*-algebra;*
3. *if* $\mathbf{M}_i$ *is an* $\mathbf{L}$*-algebra for each* $i \in I$, $\prod_{i \in I} \mathbf{M}_i$ *is an* $\mathbf{L}$*-algebra.*

Let Γ be a set of $\mathcal{S}$-formulas. If an $\mathcal{S}$-algebra $\mathbf{M}$ validates all $\gamma \in \Gamma$, we say Γ is an $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra.

Theorem 3.4.6 (Soundness theorem)**.** *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-logic and* Γ *be a set of* $\mathcal{S}$*-formulas such that* $\mathbf{H}_{\mathcal{S}} + \Gamma$ *is an axiomatization of* $\mathbf{L}$*. Then, If an* $\mathcal{S}$*-algebra* $\mathbf{M}$ *validates any* $\gamma \in \Gamma$, $\mathbf{M}$ *is an* $\mathbf{L}$*-algebra.*

Proof. Let v be a valuation on an $\mathcal{S}$-algebra $\mathbf{M}$ and $A \in \mathbf{L}$. We show $v(A) = 1$ by induction on the length l of the proof of $\mathbf{H}_{\mathcal{S}} + \Gamma \vdash A$. If $l = 0$, $A = \sigma(B(p_1, \ldots, p_m))$, where B is an axiom of $\mathbf{H}_{\mathcal{S}}$ or an element of Γ, σ is an $\mathcal{S}$-substitution and $p_1, \ldots, p_m$ are all propositional variables occurring in B. We define a valuation w by $w(p_i) = v(\sigma(p_i))$. Thus, we have $v(A) = v(\sigma(B(p_1, \ldots, p_m)) = v(B(\sigma(p_1), \ldots, \sigma(p_m)) = B(v(\sigma(p_1), \ldots, v(\sigma(p_m)))) =$

$B(w(p_1), \ldots, w(p_m))) = w(B) = 1$ since B itself is valid in every $\mathcal{S}$-algebra. If $l > 0$, A is obtained by the modus ponens rule from C and $C \to A$ for some $\mathcal{S}$-formula C. Then, by induction hypothesis, $v(C) = v(C \to A) = 1$. Thus, we have $v(A) = 1 \to v(A) = v(C) \to v(A) = v(C \to A) = 1$. □

To show the completeness theorem, we define the *Lindenbaum algebra* for a given $\mathcal{S}$-logic.

Lemma 3.4.7. *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-logic and* $\equiv_{\mathbf{L}}$ *be a binary relation on* $\mathbf{Form}_{\mathcal{S}}$ *defined by* $A \equiv_{\mathbf{L}} B$ *if and only if* $A \to B \in \mathbf{L}$ *and* $B \to A \in \mathbf{L}$*. Then* $\equiv_{\mathbf{L}}$ *is a congruence on* $\mathbf{Form}_{\mathcal{S}}$*.*

Proof. It is clear that $\equiv_{\mathbf{L}}$ is an equivalence relation. If $A \equiv_{\mathbf{L}} B$ and $C \equiv_{\mathbf{L}} D$, Thus we have $(A \to C) \to (B \to D) \in \mathbf{L}$ by the following proof and the deduction theorem:

$$\dfrac{\dfrac{\dfrac{B \qquad \begin{array}{c}\vdots\\ B \to A\end{array}}{A} \qquad A \to C}{C} \qquad \begin{array}{c}\vdots\\ C \to D\end{array}}{D}$$

Similarly, we also have $(B \to D) \to (A \to C) \in \mathbf{L}$. Therefore, $A \to C \equiv_{\mathbf{L}} B \to D$. The other cases ($\wedge$, $\vee$, $\neg$) are similarly. □

Definition 3.4.8. *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-logic. We define an* $\mathcal{S}$*-algebra* $\mathbf{Lin}_{\mathbf{L}}$*, the Lindenbaum algebra of* $\mathbf{L}$*, as follows:*

1. $\mathbf{Lin}_{\mathbf{L}} = \mathbf{Form}_{\mathcal{S}} / \equiv_{\mathbf{L}}$;
2. $1 = [\top]$;
3. $[A] \to [B] = [A \to B]$;
4. *if* $\wedge \in \mathcal{S}$, $[A] \wedge [B] = [A \wedge B]$;
5. *if* $\vee \in \mathcal{S}$, $[A] \vee [B] = [A \vee B]$;
6. *if* $\neg \in \mathcal{S}$, $\neg[A] = [\neg A]$,

where $[A]$ *is the equivalence class of* A*.*

Proposition 3.4.9. $\mathbf{Lin}_{\mathbf{L}}$ *is an* $\mathbf{L}$*-algebra.*

Proof. Let $A(p_1, \ldots, p_m) \in \mathbf{L}$, v be a valuation on $\mathbf{Lin}_{\mathbf{L}}$ and $v(p_i) = [B_i]$. We define a $\mathcal{S}$-substitution σ by $\sigma(p_i) = B_i$. Thus

$$\begin{aligned}
& v(A(p_1, \ldots, p_m)) \\
&= A(v(p_1), \ldots, v(p_m)) \\
&= A([B_1], \ldots, [B_m]) \\
&= [A(B_1, \ldots, B_m)] \\
&= [A(\sigma(p_1), \ldots, \sigma(p_m)] \\
&= [\sigma(A)] \equiv_{\mathbf{L}} [\top] \\
&= 1,
\end{aligned}$$

since $A \in \mathbf{L}$ implies $\sigma(A) \in \mathbf{L}$. $\square$

Theorem 3.4.10 (Completeness theorem)**.** *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-algebra. If an* $\mathcal{S}$*-formula* A *satisfies* $A \notin \mathbf{L}$*, there exists an* $\mathcal{S}$*-logic which refutes* A*.*

Proof. $\mathbf{Lin_L}$ is an $\mathbf{L}$-algebra refutes A by the valuation v defined by $v(p) = [p]$ for all propositional variable p. $\square$

Proposition 3.4.11. *Let* $\mathbf{L}$ *and* $\mathbf{L}'$ *be* $\mathcal{S}$*-logics. Then* $\mathbf{L} \subseteq \mathbf{L}'$ *if and only if* $V(\mathbf{L}') \subseteq V(\mathbf{L})$*, where* $V(\mathbf{L})$ *and* $V(\mathbf{L}')$ *are the set of all* $\mathbf{L}$*-algebras and the set of all* $\mathbf{L}'$*-algebras respectively.*

Proof. Assume $\mathbf{L} \subseteq \mathbf{L}'$ and let $\mathbf{M}$ be an $\mathbf{L}'$-algebra. Then $\mathbf{M}$ validates all theorems of $\mathbf{L}'$. Thus $\mathbf{M}$ also validates all theorems of $\mathbf{L}$ by assumption. Conversely, suppose that $V(\mathbf{L}') \subseteq V(\mathbf{L})$ and an $\mathcal{S}$-formula A satisfies $\mathbf{L} \vdash A$. Then every $\mathbf{L}'$-algebra $\mathbf{M}$ validates A since $\mathbf{M} \in V(\mathbf{L}') \subseteq V(\mathbf{L})$. Therefore $\mathbf{L}' \vdash A$. We proved $\mathbf{L} \subseteq \mathbf{L}'$. $\square$

Let us introduce the definition of *lattices* to explain properties of $\mathcal{S}$-algebras.

Definition 3.4.12. *An algebra* $\mathbf{M}$ *is a semilattice if* $\mathbf{M}$ *has a binary function* $\circ$ *and satisfies the following equations:*

1. $(x \circ y) \circ z = x \circ (y \circ z)$*;*
2. $x \circ y = y \circ x$*;*
3. $x \circ x = x$*.*

Precisely, "$\mathbf{M}$ has a binary function $\circ$" means "there exists a binary function $\circ$ and an algebra $(M,\ \langle\circ\rangle)$ is a restriction of $\mathbf{M}$." We use this phrase throughout the article.

Proposition 3.4.13. *Let* $\mathbf{M} = (M,\ \{\circ\})$ *be a semilattice. Binary relations* $\leq_1$ *and* $\leq_2$ *on* M *defined as follows are partial order:*

- $x \leq_1 y$ *if and only if* $x \circ y = x$*;*
- $x \leq_2 y$ *if and only if* $x \circ y = y$*.*

$\leq_1$ and $\leq_2$ are called *semilattice order.*

Definition 3.4.14. *An algebra* $\mathbf{M}$ *is a lattice if* $\mathbf{M}$ *has binary functions* $\wedge$ *and* $\vee$ *satisfy the following:*

1. $(M,\ \wedge)$ *and* $(M,\ \vee)$ *are semilattices;*
2. $\mathbf{M}$ *satisfies the equations* $x \wedge (x \vee y) = x$ *and* $x \vee (x \wedge y) = x$*.*

Moreover, a lattice $\mathbf{M}$ *is called a distributive lattice if it has the axioms* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*.*

Proposition 3.4.15. *Let* $\mathbf{M} = (M,\ \wedge,\ \vee)$ *be a lattice. Partial orders* $\leq_1$ *and* $\leq_2$ *on* M *defined as follows are equivalent:*

1. $x \leq_1 y$ *if and only if* $x \wedge y = x$;

2. $x \leq_2 y$ *if and only if* $x \vee y = y$.

Proof. ($1 \Longrightarrow 2$) is shown by $x \vee y = (x \wedge y) \vee y = y$. The converse is similarly. □

We now introduce some famous properties of $\mathcal{S}$-algebras.

Proposition 3.4.16. *Let* $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra. The following hold:*

1. $(M, \leq_{\rightarrow})$ *is a partial order set with the biggest element* 1*, where* $\leq_{\rightarrow}$ *is defined by* $x \leq y$ *if and only if* $x \rightarrow y = 1$ *(we call* $\leq_{\rightarrow}$ *the implicational order);*

2. *if* $\wedge$ $(\vee) \in \mathcal{S}$, $(M,\ \wedge\ (\vee))$ *is a semilattice;*

3. *if* $\wedge$ $(\vee) \in \mathcal{S}$, *the semilattice order on* $\mathbf{M}$ *is equivalent to the implicational order on* $\mathbf{M}$*;*

Proof.
1. We have $x \rightarrow x = 1$ by the fact $\mathbf{H} \vdash x \rightarrow x$ and the condition 4 of Definition 3.4.1. We have that $x \rightarrow y = y \rightarrow x = 1$ implies $x = y$ by the condition 3 of Definition 3.4.1. We show the transitivity. We have $(x \rightarrow y) \rightarrow (y \rightarrow z) \rightarrow x \rightarrow z = 1$ by the fact $\mathbf{H} \vdash (x \rightarrow y) \rightarrow (y \rightarrow z) \rightarrow x \rightarrow z$ and the condition 4 of Definition 3.4.1. Thus, if $x \rightarrow y = y \rightarrow z = 1$, we have $x \rightarrow z = (x \rightarrow y) \rightarrow (y \rightarrow z) \rightarrow x \rightarrow z = 1$.

2. It follows from the condition 4 of Definition 3.4.1.

3. We show the case $\odot = \vee$ (the other case ($\odot = \wedge$) is similarly). Assume $x \leq_{\vee} y$, where $\leq_{\vee}$ is the semilattice order on $\mathbf{M}$ defined by $\vee$. Thus we have $x \vee y = y$. Then we have $x \rightarrow y = x \rightarrow x \vee y = 1$ by the condition 4 of Definition 3.4.1. Conversely, suppose $x \leq_{\rightarrow} y$, i.e., $x \rightarrow y = 1$. Then, we have $1 = (x \rightarrow y) \rightarrow (y \rightarrow y) \rightarrow x \vee y \rightarrow y = 1 \rightarrow 1 \rightarrow x \vee y \rightarrow y = x \vee y \rightarrow y$ by the condition 3 and 4 of Definition 3.4.1. Also we have $y \rightarrow x \vee y = 1$ by the condition 4 of Definition 3.4.1. Consequently, we obtain $x \vee y = y$.

□

By the theorem above, the semilattice order and the implicational order are equivalent. Thus, we will write just $\leq$ for these partial orders on $\mathcal{S}$-algebras.

Proposition 3.4.17. *Let* $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra. The following hold:*

1. *if* $\wedge(\vee) \in \mathcal{S}$, $x \wedge y$ *is the largest lower bound (the least upper bound) of* $\{x, y\}$ *on* $\mathbf{M}$*;*

2. *if* $\wedge,\ \vee \in \mathcal{S}$, $(M, \wedge, \vee)$ *is a distributive lattice;*

3. *if* $\wedge \in \mathcal{S}$, $x \wedge y \leq z$ *if and only if* $y \leq x \rightarrow z$ *holds for every* $x,\ y,\ z \in M$

Proof. 1. We show the case $\vee$ (the case $\wedge$ is similarly). By axioms of $\mathbf{H}_{\mathcal{S}}$, we have $x \to x \vee y = 1$ and $y \to x \vee y = 1$ which implies $x, y \leq x \vee y$. Moreover, if $x, y \leq z$, then we have $x \to z = y \to z = 1$. Therefore, by axioms of $\mathbf{H}_{\mathcal{S}}$, we have $x \vee y \to z = (x \to z) \to (y \to z) \to (x \vee y \to z) = 1$. Consequently, we proved that $x \vee y$ is the least upper bound of $\{x, y\}$ on $\mathbf{M}$.

2. It follows from the fact $\mathbf{H} \vdash x \wedge (y \vee z) \to (x \wedge y) \vee (x \wedge z)$, $\mathbf{H} \vdash (x \wedge y) \vee (x \wedge z) \to x \wedge (y \vee z)$, $\mathbf{H} \vdash x \vee (y \wedge z) \to (x \vee y) \wedge (x \vee z)$ and $\mathbf{H} \vdash (x \vee y) \wedge (x \vee z) \to x \vee (y \wedge z)$.

3. It follows from the fact that $\mathbf{H}_{\mathcal{S}} \vdash (x \wedge y \to z) \to (y \to x \to z)$, $\mathbf{H}_{\mathcal{S}} \vdash (y \to x \to z) \to (x \wedge y \to z)$ and the soundness theorem.

□

Proposition 3.4.18. *Let $\mathbf{M}$ be an $\mathcal{S}$-algebra and $x,\ x',\ y,\ y' \in \mathbf{M}$ satisfy $x \leq x'$ and $y \leq y'$. Then $x \to y \leq x \to y'$ and $x' \to y \leq x \to y$ holds.*

Proof. They follow from $(x \to y) \to x \to y' = 1 \to (x \to y) \to x \to y' = (x \to y \to y') \to (x \to y) \to x \to y = 1$ and $(x' \to y) \to x \to y = 1 \to (x' \to y) \to x \to y = (x' \to x) \to (x' \to y) \to x \to y = 1$ respectively. □

Proposition 3.4.19. *Let $\wedge \in \mathcal{S}$. Every $\mathcal{S}$-algebra satisfy the following equations:*

1. $x \wedge y \to z = x \to y \to z$;

2. $x \to y \wedge z = (x \to y) \wedge (x \to z)$;

3. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ *(if $\vee \in \mathcal{S}$)*;

4. $\neg(x \wedge y) = (x \to y \to z) \wedge (x \to y \to \neg z)$ *(if $\neg \in \mathcal{S}$)*.

Proof. It follows from the condition 4 of Definition 3.4.1. □

We recall the basic definition of homomorphisms between $\mathcal{S}$-algebras. Let $\mathbf{M}$ be an $\mathcal{S}_{\mathbf{M}}$-algebra and $\mathbf{N}$ be an $\mathcal{S}_{\mathbf{N}}$-algebra, where $\mathcal{S} \subseteq \mathcal{S}_{\mathbf{M}},\ \mathcal{S}_{\mathbf{N}}$. A map $h : \mathbf{M} \longrightarrow \mathbf{N}$ is called an $\mathcal{S}$-homomorphism if, for every $x,\ y \in \mathbf{M}$, h satisfies the following:

1. $h(x \to_{\mathbf{M}} y) = h(x) \to_{\mathbf{N}} h(y)$;

2. if $\wedge \in \mathcal{S}$, $h(x \wedge_{\mathbf{M}} y) = h(x) \wedge_{\mathbf{N}} h(y)$;

3. if $\vee \in \mathcal{S}$, $h(x \vee_{\mathbf{M}} y) = h(x) \vee_{\mathbf{N}} h(y)$;

4. if $\neg \in \mathcal{S}$, $h(\neg_{\mathbf{M}} x) = \neg_{\mathbf{N}} h(x)$,

We note that, if $\neg \notin \mathcal{S}$ and each of $\mathbf{M}$ and $\mathbf{N}$ has the minimum element (we put $m_{\mathbf{M}}$ and $m_{\mathbf{N}}$ are the minimum elements of $\mathbf{M}$ and $\mathbf{N}$ respectively), an $\mathcal{S}$-homomorphism $h : \mathbf{M} \longrightarrow \mathbf{N}$ does not always satisfy $h(m_{\mathbf{M}}) = m_{\mathbf{N}}$. An $\mathcal{S}$-homomorphism h is called an $\mathcal{S}$-embedding if h is an injection.

Proposition 3.4.20. *Let $\neg \in \mathcal{S}$ and $\mathbf{M}$ be an $\mathcal{S}$-algebra. Then $\neg x = x \to 0$ holds.*

Proof. We have $\neg x \leq x \to 0$ since $\neg x \to x \to 0 = 1$. Conversely, we have $(x \to 0) \leq \neg x$ since $1 = (x \to \neg 1) \to 1 \to \neg x = (x \to 0) \to \neg x$. □

Proposition 3.4.21. *Let $\neg \in \mathcal{S}$, $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{S}$-algebras and $h : \mathbf{M} \longrightarrow \mathbf{N}$ be a map. h is a $\{\to,\ \neg\}$-homomorphism if and only if h is a $\{\to\}$-homomorphism and h satisfies $h(0_{\mathbf{M}}) = 0_{\mathbf{N}}$.*

Proof. Let $h : \mathbf{M} \longrightarrow \mathbf{N}$ be a $\{\to,\ \neg\}$-homomorphism. Then we obtain $h(0_{\mathbf{M}}) = h(\neg(x \to x)) = \neg(h(x) \to h(x)) = \neg 1_{\mathbf{N}} = 0_{\mathbf{N}}$.

Conversely, let $h : \mathbf{M} \longrightarrow \mathbf{N}$ be a $\{\to\}$-homomorphism and h satisfies $h(0_{\mathbf{M}}) = 0_{\mathbf{N}}$. We obtain $h(\neg x) = h(x \to 0_{\mathbf{M}}) = h(x) \to h(0_{\mathbf{M}}) = h(x) \to 0_{\mathbf{N}} = \neg h(x)$ by Proposition 3.4.20. □

Theorem 3.4.10 showed the (not strong) completeness theorem between $\mathcal{S}$-logics and $\mathcal{S}$-algebras by virtue of the Lindenbaum algebra. Moreover, we will show the strong completeness between between $\mathcal{S}$-logics and $\mathcal{S}$-algebras.

Definition 3.4.22 (Boolean algebra)**.** $\mathbf{M}$ *is a Boolean algebra if* $\mathbf{M}$ *is a* $\{\to,\ \wedge,\ \vee,\ \neg\}$*-algebra satisfying an equation* $\neg\neg x = x$.

For a given set Φ, $\mathcal{P}(\Phi)$ means the power set of Φ.

Proposition 3.4.23. *Let Φ be an arbitrary set. Then, $(\mathcal{P}(\Phi), \to, \wedge, \vee, \neg)$ is a Boolean algebra by defining each function as follows:*

1. $\alpha \to \beta = \alpha^c \cup \beta$;
2. $\alpha \wedge \beta = \alpha \cap \beta$;
3. $\alpha \vee \beta = \alpha \cup \beta$;
4. $\neg \alpha = \alpha^c$,

where $\alpha,\ \beta \in \mathcal{P}(\Phi)$, $\cup$ means the union, $\cap$ means the intersection and α^c means the complement of α.

Proof. We can easily verify that $\mathcal{P}(\Phi)$ is a $\{\to,\ \wedge,\ \vee,\ \neg\}$-algebra, notice that the maximum element 1 of $\mathcal{P}(\Phi)$ is Φ itself. The equation $\neg\neg\alpha = \alpha$ immediately follows from the fact $(\alpha^c)^c = \alpha$. □

In the rest of the section, we write $\mathcal{P}(\Phi)$ for a Boolean algebra $(\mathcal{P}(\Phi), \to, \wedge, \vee, \neg)$ defined above.

By the proposition above, we can define filter on $\mathcal{P}(\Phi)$. Recall that $F \subseteq \mathcal{P}(\Phi)$ is a filter if F satisfies following:

1. $\Phi \in F$;

2. $\alpha \in F$ and $\alpha \subseteq \beta$ implies $\beta \in F$;

3. $\alpha,\ \beta \in F$ implies $\alpha \cap \beta \in F$.

Definition 3.4.24. *Let Φ be an arbitrary set and $\mathcal{P}(\Phi)$ be a Boolean algebra. A filter $F \in \mathcal{P}(\Phi)$ is an ultra filter if $\alpha \in F$ or $\alpha^c \in F$ for every $\alpha \in \mathcal{P}(\Phi)$ and $\emptyset \notin F$.*

Notice that the above definition is equivalent to "exactly one of α and α^c is an element of F for every $\alpha \in \mathcal{P}(\Phi)$" since, if both of $\alpha \in F$ and $\alpha^c \in F$ holds, we have $\emptyset = \alpha \cap \alpha^c \in F$.

Definition 3.4.25 (Ultra product)**.** *Let $\mathbf{M}_i$ $(i \in I)$ be $\mathcal{S}$-algebras. We define the ultraproduct of $\mathbf{M}_i$ $(i \in I)$ as follows:*

1. *define a equivalence relation $\equiv_F$ the direct product $\prod_{i\in I} \mathbf{M}_i$ by $(x_i) \equiv_F (y_i)$ if and only if $\{j \in I \mid x_j = y_j\} \in F$;*

2. *The ultra product of $\mathbf{M}_i$ $(i \in I)$ is $\prod_{i\in I} \mathbf{M}_i / \equiv_F$.*

Theorem 3.4.26 (Łós's theorem)**.** *Let $\mathbf{M}_i$ $(i \in I)$ be $\mathcal{S}$-algebras, $F \subseteq \mathcal{P}(I)$ be an ultra filter, $\prod_{i\in I} \mathbf{M}_i / \equiv_F$ be the ultra product, v be a valuation on $\prod_{i\in I} \mathbf{M}_i / \equiv_F$ and, for each $i \in I$, v_i is a valuation defined by $v(p) = [(v_i(p))]_{\equiv_F}$ for every propositional variable p, where $[\cdots]_{\equiv_F}$ means an equivalent class with respect to $\equiv_F$. Then, for a given formula A, $v(A) = 1$ if and only if $\{i \in I \mid v_i(A) = 1\} \in F$.*

Proof.

$$\begin{aligned} & v(A) = 1 \\ \Longleftrightarrow\ & (v_i(A)) \equiv_F 1 \\ \Longleftrightarrow\ & \{i \in I \mid v_i(A) = 1\} \in F. \end{aligned}$$

□

Theorem 3.4.27 (Strong completeness theorem)**.** *Let $\mathbf{L}$ be an $\mathcal{S}$-logic and $\Gamma \cup \{A\}$ be a set of $\mathcal{S}$-formulas. The following are equivalent:*

1. *$\mathbf{L} \not\vdash \Sigma \to A$ holds for any finite subset $\Sigma \subseteq \Gamma$;*

2. *there exist an $\mathbf{L}$-algebra $\mathbf{M}$ and a valuation v on $\mathbf{M}$ satisfying $v(\gamma) = 1$ for any $\gamma \in \Gamma$ and $v(A) \neq 1$.*

Proof. $(1 \Longrightarrow 2)$ Let $I = \{\Sigma \subseteq \Gamma \mid \Sigma \text{ is finite}\}$ and assume $\mathbf{L} \not\vdash \Sigma \to A$ for any $\Sigma \in I$. Thus, for any $\Sigma \in I$, we have an $\mathbf{L}$-algebra $\mathbf{M}_\Sigma$ and a valuation v_Σ on $\mathbf{M}_\Sigma$ satisfying $v_\Sigma(\Sigma \to A) \neq 1$ by Theorem 3.4.34. We can assume $v_\Sigma(\sigma) = 1$ for any $\sigma \in \Sigma$ by Proposition 3.4.34. Let $V_\gamma = \{\Sigma \in I \mid v_\Sigma(\gamma) = 1\} (\supseteq \{\Sigma \in I \mid \gamma \in \Sigma\})$. Then $F = \{V_\gamma \mid \gamma \in \Gamma\}$ has the finite intersection property since $V_{\gamma_1} \cap \cdots \cap V_{\gamma_m} = \{\Sigma \in I \mid v_\Sigma(\gamma_j) = 1\ (j = 1, \ldots, m)\} \ni \{\gamma_1, \ldots, \gamma_m\}$. Therefore, there exists an ultra filter $G \supseteq F$ on $(\mathcal{P}(I), \subseteq)$. Let $\mathbf{M} = (\prod_{\Sigma\in I} \mathbf{M}_\Sigma)/G$ be the ultra product and define a valuation v on $\mathbf{M}$ by $v(A) = (\prod_{\Sigma\in I} v_\Sigma(A))/G$. Thus $v(A) =$

$1 \Longleftrightarrow \{\Sigma \in I \mid v_\Sigma(A) = 1\} \in G$ holds by Łós's theorem. Since $\mathbf{M}_\Sigma$ is an $\mathbf{L}$-algebra for any $\Sigma \in I$, $\mathbf{M}$ is $\mathbf{L}$-algebra. If $\gamma \in \Gamma$, $v(\gamma) = 1$ since $\{\Sigma \in I \mid v_\Sigma(\gamma) = 1\} = V_\gamma \in G$. On the other hand, $\{\Sigma \in I \mid v_\Sigma(A) = 1\} = \emptyset$ by the assumption. Hence $\{\Sigma \in I \mid v_\Sigma(A) = 1\} \notin G$ since G is a proper filter. Therefore $v(A) \neq 1$. Consequently, $\mathbf{M}$ and v are the $\mathbf{L}$-algebra and valuation we wanted.

($2 \Longrightarrow 1$) By the assumption, v satisfies that $v(\Sigma \to A) \neq 1$ which implies $\mathbf{L} \not\vdash \Sigma \to A$. □

We will consider the question: what kind of $\mathcal{S}$-algebras are subdirectly irreducible ?

In $\mathcal{S}$-algebras, congruences and *implicational filters* defined below are equivalent.

Definition 3.4.28. *Let $\mathbf{M}$ be an $\mathcal{S}$-algebra. A subset $F \subseteq \mathbf{M}$ is a filter if ($x,\ x \to y \in F$ implies $y \in F$ and $1 \in F$).*

Proposition 3.4.29. *Let $\wedge \in \mathcal{S}$ and $\mathbf{M}$ be an $\mathcal{S}$-algebra. Then, $F \subseteq \mathbf{M}$ is a filter if and only if F satisfies the following:*

1. *$1 \in F$;*
2. *$x \in F$ and $x \leq y$ imply $y \in F$;*
3. *$x,\ y \in F$ implies $x \wedge y \in F$.*

Proof. ($\Longrightarrow$) follows from the facts that ($x \leq y$ implies $x \to y = 1 \in F$) and $x \to y \to x \wedge y = 1 \in F$

($\Longleftarrow$) Let $x, x \to y \in F$. Then we have $x \wedge x \to y \in F$ by the assumption. Therefore we obtain $y \in F$ since $x \wedge (x \to y) \to y = 1 \in F$.

□

Proposition 3.4.30. *Let $\mathbf{M}$ be an $\mathcal{S}$-algebra and $F \subseteq \mathbf{M}$ be a filter. Then, the binary relation $\sim_F$ on $\mathbf{M}$ by $x \sim_F y$ if and only if $x \to y,\ y \to x \in F$ is a congruence.*

Proof. Similar to the proof of Lemma 3.4.7. □

Therefore, by Definition 3.3.17, we have the quotient algebra $\mathbf{M}/\sim_F$ for each $\mathcal{S}$-algebra $\mathbf{M}$ and filter F on $\mathbf{M}$. We abbreviate $\mathbf{M}/\sim_F$ to $\mathbf{M}/F$.

Proposition 3.4.31. *Let $\mathbf{M}$ be an $\mathcal{S}$-algebra and θ be a congruence on $\mathbf{M}$. Then there exists a filter F of $\mathbf{M}$ such that $\sim_F = \theta$.*

Proof. The filter $F = \{x \in \mathbf{M} \mid [x]_\theta = [1]_\theta\}$ satisfies $\sim_F = \theta$ since

$$\begin{aligned} & x \sim_F y \\ \Longleftrightarrow\ & x \to y,\ y \to x \in F \\ \Longleftrightarrow\ & [x \to y]_\theta = [y \to x]_\theta = [1]_\theta \\ \Longleftrightarrow\ & [x]_\theta \to [y]_\theta = [y]_\theta \to [x]_\theta = [1]_\theta \\ \Longleftrightarrow\ & [x]_\theta = [y]_\theta \\ \Longleftrightarrow\ & x\theta y. \end{aligned}$$

□

Theorem 3.4.32. *Let* $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra.* $\mathbf{M}$ *is subdirectly irreducible if and only if* $\mathbf{M}$ *has the unique second greatest element* ω*. Precisely,* ω *satisfies* $\omega \leq 1$ *and* $x \leq \omega$ *for all* $x \in \mathbf{M} - \{1\}$*.*

Proof. By Lemma 3.3.22 and 3.3.23, we will show that $\bigcap_i F_i \neq \{1\}$ if and only if $\mathbf{M}$ has the unique second greatest element ω, where F_i $(i \in I)$ is all filters on $\mathbf{M}$ except $\{1\}$, since the minimum congruence $\mathbf{E}$ on $\mathbf{M}$ is $\sim_{\{1\}}$.

Assume that $\mathbf{M}$ does not have the unique second greatest element ω. For every $x \in \mathbf{M} - \{1\}$, there exists $y \in \mathbf{M} - \{1\}$ such that $y \not\leq x$. Thus, a filter $y\uparrow = \{z \in \mathbf{M} \mid y \leq z\}$ on $\mathbf{M}$ satisfies $x \notin y\uparrow$ and $y\uparrow \neq \{1\}$. Therefore, $\bigcap_{i \in I} F_i \subseteq \bigcap\{z\uparrow \mid z \in \mathbf{M} - \{1\}\} = \{1\}$. Consequently, $\bigcap_{i \in I} F_i = \{1\}$. we proved the contraposition.

Conversely, suppose that $\mathbf{M}$ has the unique second greatest element ω. Then, if $F \neq \{1\}$ is a filter on $\mathbf{M}$, there exists $x \in F - \{1\}$. Thus, $\omega \in F$ since $x \leq \omega$. Therefore, $\bigcap_{i \in I} F_i \supseteq \{1,\ \omega\}$ (in fact, we have $\bigcap_{i \in I} F_i = \{1,\ \omega\}$ since $\{1,\ \omega\}$ is a filter on $\mathbf{M}$). □

Let A be an $\mathcal{S}$-formula, $\mathbf{M}$ be an $\mathcal{S}$-algebra and w be a valuation of $\mathbf{M}$. If $\mathbf{M}$ is subdirectly irreducible and $w(A) = \omega$, we say w is an ω-refutation of A on $\mathbf{M}$.

Proposition 3.4.33 (c.f., Jankov[16] pp.28, Lemma)**.** *On a subdirectly irreducible* $\mathcal{S}$*-algebra,* $x \to y = \omega$ *if and only if* $x = 1$ *and* $y = \omega$*.*

Proof. Assume $x \to y = \omega$. Since $x \to x \to y = x \to y$ holds in every $\mathcal{S}$-algebra (since it is provable in $\mathbf{H}$), we have $x \to \omega = \omega$. Therefore we have $x = 1$. Then we also have $y = \omega$ immediately. The converse is obvious. □

Proposition 3.4.34 (c.f., Wroński[30], Lemma 1)**.** *Let* $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra and* $a \in \mathbf{M} - \{1\}$*. There is an* $\mathcal{S}$*-homomorphism* h *from* $\mathbf{M}$ *such that* $h(a)$ *is the unique second greatest element of* $h(\mathbf{M})$ *(thus,* $h(\mathbf{M})$ *is a subdirectly irreducible* $\mathcal{S}$*-algebra).*

Proof. Let $F \subseteq \mathbf{M}$ be a maximal filter satisfying $a \notin F$ (such F is guaranteed to exist by Zorn's Lemma). We show that h defined by $h(x) = x/F$ is the $\mathcal{S}$-homomorphism we wanted. We can easily verify $h(a) \neq 1$ since $a \notin F$. Thus, it is sufficient to prove that $b \leq h(a)$ for every $b \in h(\mathbf{M}) - \{1\}$. Let $b \in h(\mathbf{M}) - \{1\}$. Then $b\uparrow = \{y \in h(\mathbf{M}) \mid b \leq y\}$ is a filter of $h(\mathbf{M})$. Thus, $h^{-1}(b\uparrow)$ is a filter of $\mathbf{M}$ such that $F \subsetneq h^{-1}(b\uparrow)$ since $h(F) = \{1\} \subseteq b\uparrow$ and $b \in b\uparrow - \{1\}$. Therefore, $a \in h^{-1}(b\uparrow)$ since F is maximal. Thus we obtain $h(a) \in b\uparrow$, i.e., $b \leq h(a)$. □

By, Proposition 3.4.34, we can refine the strong completeness theorem.

Theorem 3.4.35 (Strong completeness theorem for subdirectly irreducible algebras)**.** *Let* $\mathbf{L}$ *be an* $\mathcal{S}$*-logic and* $\Gamma \cup \{A\}$ *be a set of* $\mathcal{S}$*-formulas. The following are equivalent:*

1. $\mathbf{L} \not\vdash \Sigma \to A$ *holds for any finite subset* $\Sigma \subseteq \Gamma$*;*

2. *there exist an subdirectly irreducible* $\mathbf{L}$*-algebra* $\mathbf{M}$ *and a valuation* v *on* $\mathbf{M}$ *satisfying* $v(\gamma) = 1$ *for any* $\gamma \in \Gamma$ *and* $v(A) \neq 1$*.*

Proof. We show ($1 \Rightarrow 2$). The converse can be proved in the same way as Theorem 3.4.27. By the proof of Theorem 3.4.27, we obtain an $\mathbf{L}$-algebra $\mathbf{M}$ and a valuation v on $\mathbf{M}$ satisfying $v(\gamma) = 1$ for any $\gamma \in \Gamma$ and $v(A) \neq 1$. By Proposition 3.4.34, we obtain a quotient algebra $\mathbf{N}$ of $\mathbf{M}$ satisfying the following:

1. $\mathbf{N}$ is subdirectly irreducible ;

2. there exists an ω-refutation of A on $\mathbf{N}$.

Therefore, $\mathbf{N}$ is the subdirectly irreducible $\mathbf{L}$-algebra we wanted. □

We note that the theorem above implies that we just need to consider subdirectly irreducible $\mathcal{S}$-algebras, i.e., we can assume that each $\mathcal{S}$-algebra has the unique second greatest element while considering relation between $\mathcal{S}$-logics and $\mathcal{S}$-algebras.

3.5 Khomich's results

Khomich studied $\mathcal{S}$-algebras by using indecomposable elements of $\mathcal{S}$-algebras. In this section, we summarize Khomich's result ([24]) for indecomposable elements.

Definition 3.5.1. *Let $\mathbf{M}$ be an $\mathcal{S}$-algebra. If $y \in \mathbf{M}$ satisfies $x \to y = y$ or $x \to y = 1$ for every $x \in \mathbf{M}$, we say that y is indecomposable.*

Let $\mathbf{M}$ be an $\mathcal{S}$-algebra, $x \in \mathbf{M}$ and $\phi \subseteq \mathbf{M}$ (notice that ϕ does not have to be a subalgebra of $\mathbf{M}$). We define $I(x) = \{y \in \mathbf{M} \mid y \geq x \text{ and } y \text{ is indecomposable}\}$, $m(\phi) = \{y \in \phi \mid \forall z \in I(x),\ z \leq y \text{ implies } z = y\}$ (i.e., $m(\phi)$ is the set of all minimal elements of ϕ) and $r(x) = m(I(x))$.

Lemma 3.5.2 (Khomich[24])**.** *Let $\mathbf{M}$ be a finite $\mathcal{S}$-algebra and $x,\ y \in \mathbf{M}$. Then $I(x \to y) = \{z \in \mathbf{M} \mid z \text{ is indecomposable and there exists } w \in I(y) - I(x) \text{ such that } w \leq z\} \cup \{1\}$.*

Proof. We show $\supseteq$. Let $z \in \mathbf{M}$ be indecomposable and $w \in I(y) - I(x)$ satisfies $w \leq z$. Then we have $z \geq w \geq y$ and $w \not\geq x$. Thus, since w is indecomposable, we have $x \to w = w$. Therefore, we obtain $z \geq w = x \to w \geq x \to y$ which implies $z \in I(x \to y)$.

Conversely, we have $(I(y) - I(x)) \to x \to y = I(y) \to x \to y = 1$ since x is smaller than each element of $I(x)$. Thus, for every $z \in I(x \to y)$, we have $(I(x) - I(y)) \to z = 1$ since $x \to y \leq z$. Since z is indecomposable, there exists $w \in I(y) - I(x)$ satisfying $w \leq z$. We obtained the desired element w. □

Theorem 3.5.3 (Khomich[24])**.** *Let $\mathbf{M}$ be a finite $\mathcal{S}$-algebra and $x,\ y \in \mathbf{M}$. Then*

$$r(x \to y) = \begin{cases} r(y) - I(x) & (x \to y \neq 1) \\ \{1\} & (x \to y = 1). \end{cases}$$

Proof. We have $r(x \to y) = m(I(x \to y)) = m(\{z \in \mathbf{M} \mid z$ is indecomposable and there exists $w \in I(y) - I(x)$ such that $w \leq z\} \cup \{1\}) = m(I(y) - I(x))$. Thus, it is sufficient to show that $m(I(y) - I(x)) = r(y) - I(x)$.

We show $\supseteq$. Let $z \in r(y) - I(x)$. Then z is a minimal element in $I(y)$. Therefore, z is also a minimal element in $I(y) - I(z)$. We obtained that $z \in m(I(y) - I(x))$.

Conversely, suppose that $z \in m(I(y) - I(x))$ and $z \notin r(y) - I(z)$. Then we have $z \notin r(y)$ since $z \notin I(x)$. Moreover, there exists $w \in r(y)$ satisfying $w < z$ since $z \in m(I(y) - I(x)) \subseteq I(y)$. The element w satisfies $w \notin I(y) - I(x)$ since $w < z \in m(I(y) - I(x))$. Therefore, we have $w \in I(x)$, i.e., $x \leq w$. Hence we have $x \leq z$, which contradicts to assumption that $z \notin I(x)$. □

Proposition 3.5.4. *Let* $\mathbf{M}$ *be a finite* $\mathcal{S}$*-algebra and* $x \in \mathbf{M}$*. Then* $r(x) \to x = 1$.

Proof. Induction on $|r(x)|$, the number of elements in $r(x)$. If $|r(x)| = 1$, x itself is indecomposable, i.e., $r(x) = \{x\}$. Then $r(x) \to x = x \to x = 1$.

If $|r(x)| > 1$, let $r(x) = \{y_1, \ldots, y_m\}$. Since $y_1, \ldots, y_m$ are mutually incomparable, we have that $y_2, \ldots, y_m \notin I(y_1)$. Therefore, we have $r(y_1 \to x) = r(x) - I(y_1) = \{y_2, \ldots, y_m\}$. Thus, by induction hypothesis, we obtain $r(x) \to x = r(y_1 \to x) \to y_1 \to x = 1$. □

Proposition 3.5.5. *Let* $\mathbf{M}$ *be a finite* $\mathcal{S}$*-algebra and* $x,\ y \in \mathbf{M}$*. Then* $r(x) = r(y)$ *implies* $x = y$.

Proof. $x \to y = x \to r(x) \to y = x \to r(y) \to y = 1$. We proved $x \leq y$. The converse $y \leq x$ can be shown in the same way. □

If $\wedge \in \mathcal{S}$, the definition of indecomposable elements becomes easy to understand,

Proposition 3.5.6. *Let* $\wedge \in \mathcal{S}$, $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra and* $x \in \mathbf{M}$*. Then the following are equivalent:*

1. *x is indecomposable;*
2. *if $x = y \wedge z$, $x = y$ or $x = z$ hold.*

Chapter 4

Algebraic characterization for the conservativity

This chapter contains two results for conservativity problem. The former result gives a general algebraic condition which is equivalent to conservativity condition between $\mathbf{H}_{\mathcal{S}'} + \Gamma$ and $\mathbf{H}_{\mathcal{S}} + \Gamma$ for $\mathcal{S} \subseteq \mathcal{S}'$ and a set Γ of $\mathcal{S}$-formulas. We used Jankov's characteristic formula ([16]) for this characterization. The first result can be strengthen for the case $\wedge \in \mathcal{S}'$. Our second result is a criteria for conservativity problem by using a concrete class of $\mathcal{S} \cup \{\wedge\}$-algebras which are constructed by Horn[12]. As an application of our results, we give algebraic proofs for some Khomich's results ([19, 21, 22]). We also give a detailed proof for Wroński's theorem ([31]) for conservativity problem. This chapter is based on [33].

4.1 Jankov's characteristic formula

Jankov[16] defined the *characteristic formulas* for subdirectly $\mathcal{S}$-algebras. Jankov's characteristic formula $X_{\mathbf{M}}$ shows the relation between the embeddability of $\mathbf{M}$ and the refutability of $X_{\mathbf{M}}$ for a given $\mathcal{S}$-algebra $\mathbf{M}$. In [16], Jankov needed to define only for finite subdirectly $\mathcal{S}$-algebras for the case $\wedge \in \mathcal{S}$. However, it is clear that his definition and theorems can be applied for any $\mathcal{S}$. Moreover, we can obtain similar results for infinite $\mathcal{S}$-algebras.

Definition 4.1.1 (Jankov[16] Section 3)**.** *Let* $\mathbf{M}$ *be a subdirectly irreducible* $\mathcal{S}_{\mathbf{M}}$*-algebra and* $\mathcal{S} \subseteq \mathcal{S}_{\mathbf{M}}$*. We construct the diagram of* $\mathbf{M}$ *and the* $\mathcal{S}$*-characteristic formula* $X^{\mathcal{S}}_{\mathbf{M}}$ *by the following processes:*

1. *for every* $x \in \mathbf{M}$*, pick a propositional variable* p_x *which is distinct each other;*

2. *the set $Y^{\mathcal{S}}_{\mathbf{M}}$ of $\mathcal{S}$-formulas is defined as follows:*

$$\begin{aligned} Y^{\mathcal{S}}_{\mathbf{M}} = \{ & p_a \odot p_b \to p_{a\odot b} \\ & |\forall a, b \in \mathbf{M}, \odot \in \mathcal{S} \cap \{\to, \wedge, \vee\}\} \\ \cup \{ & p_{a\odot b} \to p_a \odot p_b \\ & |\forall a, b \in \mathbf{M}, \odot \in \mathcal{S} \cap \{\to, \wedge, \vee\}\} \\ \cup \{ & p_1\} \\ \cup \{ & \neg p_a \to p_{\neg a} | \forall a \in \mathbf{M}\} \text{ (if } \neg \in \mathcal{S}) \\ \cup \{ & p_{\neg a} \to \neg p_a | \forall a \in \mathbf{M}\} \text{ (if } \neg \in \mathcal{S}) \end{aligned}$$

3. *if $\mathbf{M}$ is finite, we define $X^{\mathcal{S}}_{\mathbf{M}} = Y^{\mathcal{S}}_{\mathbf{M}} \to p_\omega$.*

We can not define $X^{\mathcal{S}}_{\mathbf{M}}$ if $\mathbf{M}$ is infinite. Thus we define an (ω-)refutation for a pair of a set of $\mathcal{S}$-formulas and an $\mathcal{S}$-formula as follows.

Definition 4.1.2. *Let Γ be a set of $\mathcal{S}$-formulas, A be an $\mathcal{S}$-formula and $\mathbf{M}$ be an $\mathcal{S}$-algebra.*

1. *A valuation v on $\mathbf{M}$ is a refutation of $(\Gamma,\ A)$ if $v(\Delta \to A) \neq 1$ for any finite subset $\Delta \subseteq \Gamma$.*

2. *A valuation v on $\mathbf{M}$ is an ω-refutation of $(\Gamma,\ A)$ if $v(\gamma) = 1$ for every $\gamma \in \Gamma$, $\mathbf{M}$ is subdirectly irreducible and $v(A) = \omega$.*

Lemma 4.1.3 (Wroński[30] Lemma 7)**.** *Let $\mathbf{M}$ and $\mathbf{N}$ be subdirectly irreducible $\mathcal{S}$-algebras. If there is an ω-refutation v of $(Y^{\mathcal{S}}_{\mathbf{M}},\ p_\omega)$ on $\mathbf{N}$, there is $\mathcal{S}$-embedding $h : \mathbf{M} \longrightarrow \mathbf{N}$.*

Proof. Notice that we have following equations for every $a,\ b \in \mathbf{M}$ by Proposition 3.4.33:

$$\begin{aligned} v(p_a) \odot v(p_b) &= v(p_a \odot p_b) = v(p_{a\odot b}) \ (\odot \in \mathcal{S} \cap \{\to, \wedge, \vee\}); \\ \neg v(p_a) &= v(\neg p_a) = v(p_{\neg a}) \text{ (if } \neg \in \mathcal{S}); \\ v(p_1) &= 1; \\ v(p_\omega) &= \omega. \end{aligned}$$

We show that $h : \mathbf{M} \longrightarrow \mathbf{N}$ defined by $h(x) = v(p_x)$ is the $\mathcal{S}$-embedding we wanted.

Let $x,\ y \in \mathbf{M}$ satisfying $x \neq y$. Thus, we can assume $x \not\leq y$, especially, $x \to y \leq \omega$. Then we have $v(p_{x\to y}) \to v(p_\omega) = v(p_1) = 1$. Hence we obtain $h(x) \to h(y) = v(p_x) \to v(p_y) = v(p_{x\to y}) \leq v(p_\omega) = \omega \neq 1$. We proved that h is injective.

Since v is a valuation, we can easily show that h preserves every function of $\mathcal{S}$: $h(x) \odot h(y) = v(p_x) \odot v(p_y) = v(p_x \odot p_y) = h(p_x \odot p_y)$ (the case $\neg$ is similarly). $\square$

Lemma 4.1.4 (Wroński[30] Lemma 1)**.** *Let $\mathbf{M}$ be a subdirectly irreducible $\mathcal{S}$-algebra and $\mathbf{N}$ be an $\mathcal{S}$-algebra. If there exists a refutation v of $(Y^{\mathcal{S}}_{\mathbf{M}},\ p_\omega)$ on $\mathbf{N}$, there is an $\mathcal{S}$-homomorphism h satisfying the following:*

1. *$h(\mathbf{N})$ is subdirectly irreducible ;*

2. $h \circ v$ *is an* ω*-refutation of* $(Y_{\mathbf{M}}^{\mathcal{S}},\ p_\omega)$ *on* $h(\mathbf{N})$*.*

Proof. It can be shown similarly to Proposition 3.4.34. Let $F \subseteq \mathbf{N}$ be a maximal filter such that $v(p_\omega) \notin F$ and $h : \mathbf{N} \longrightarrow \mathbf{N}/F$ be a canonical homomorphism, i.e., h is defined by $h(x) = [x]$, where $[x]$ means the equivalent class of x with respect to $\equiv_F$. Then, $h \circ v$ is the desired valuation on $h(\mathbf{N})$ $(= \mathbf{N}/F)$ satisfying $(h \circ v)(p_\omega) = \omega$ and therefore $(h \circ v)(\alpha) = 1$ for every $\alpha \in Y_{\mathbf{M}}^{\mathcal{S}}$. □

Theorem 4.1.5 (Jankov[16] Section 3)**.** *Let* $\mathbf{M}$ *be a subdirectly irreducible* $\mathcal{S}$*-algebra and* $\mathbf{N}$ *be an* $\mathcal{S}$*-algebra. the following are equivalent:*

1. $(Y_{\mathbf{M}}^{\mathcal{S}},\ p_\omega)$ *is refutable in* $\mathbf{N}$*;*

2. $\mathbf{M}$ *is* $\mathcal{S}$*-embeddable in an* $\mathcal{S}$*-homomorphic image of* $\mathbf{N}$*.*

Proof. $(1 \Longrightarrow 2)$ Let v be a refutation of $(Y_{\mathbf{M}}^{\mathcal{S}},\ p_\omega)$ on $\mathbf{N}$. Then, by Lemma 4.1.4, there is an $\mathcal{S}$-homomorphism h satisfying the following:

(1) $h(\mathbf{N})$ is subdirectly irreducible ;

(2) $h \circ v$ is an ω-refutation of $(Y_{\mathbf{M}}^{\mathcal{S}},\ p_\omega)$ on $h(\mathbf{N})$.

Therefore, by Lemma 4.1.3, $\mathbf{M}$ is $\mathcal{S}$-embeddable in $\mathbf{N}$.

$(2 \Longrightarrow 1)$ Let $h : \mathbf{M} \longrightarrow \mathbf{N}$ be an $\mathcal{S}$-embedding. Then the valuation v on $\mathbf{N}$ defined by $v(p_x) = h(x)$ is a refutation on $\mathbf{N}$ since we can verify $v(\alpha) = 1$ for every $\alpha \in Y_{\mathbf{M}}^{\mathcal{S}}$ and $v(p_\omega) \neq 1$. Precisely, they follows from $v(p_x \odot p_y) = v(p_x) \odot v(p_y) = h(x) \odot h(y) = h(x \odot y) = v(p_{x \odot y})$ (the case $\neg$ is similarly) and $v(p_\omega) = h(\omega) \neq 1$. □

4.2 Horn's construction

For a given $\mathcal{S}$-algebra $\mathbf{M}$, Horn constructed an $\mathcal{S} \cup \{\wedge\}$-algebra containing $\mathbf{M}$.

Definition 4.2.1 (Horn[12] pp.395-397)**.** *Let* $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra. The* $\mathcal{S} \cup \{\wedge\}$*-algebra* $C(\mathbf{M})$ *is constructed from* $\mathbf{M}$ *as follows:*

1. $\mathbf{M}_\wedge = \{\Gamma \subseteq \mathbf{M} \mid \Gamma \text{ is finite and not empty}\}$*; with the functions defined as follows:*

$$\begin{aligned}
\Gamma \to \Delta &= \{\Gamma \to_{\mathbf{M}} \delta \mid \delta \in \Delta\};\\
\Gamma \wedge \Delta &= \Gamma \cup \Delta;\\
\Gamma \vee \Delta &= \{\gamma \vee_{\mathbf{M}} \delta \mid \gamma \in \Gamma,\ \delta \in \Delta\} \text{ (if } \vee \in \mathcal{S});\\
\neg\Gamma &= \Gamma \to \{p, \neg_{\mathbf{M}} p\} \text{ (if } \neg \in \mathcal{S}).
\end{aligned}$$

2. define a binary relation $\approx$ *by* $\Gamma \approx \Delta \Longleftrightarrow \forall\gamma \in \Gamma,\ \Delta \to \gamma = 1$ *and* $\forall\delta \in \Delta,\ \Gamma \to \delta = 1$*;*

3. $C(\mathbf{M}) = \mathbf{M}_\wedge/{\approx}$ *with the functions defined as follows:*

$$[\Gamma] \odot [\Delta] = [\Gamma \odot \Delta] \ (\odot \in \{\to,\ \wedge\})$$
$$[\Gamma] \vee [\Delta] = [\Gamma \vee \Delta] \ (\text{if } \vee \in \mathcal{S});$$
$$\neg[\Gamma] = [\neg\Gamma] \ (\text{if } \neg \in \mathcal{S}),$$

where $[\Gamma]$ *is the equivalence class of* Γ *with respect to* $\approx$*.*

We verify that well-definedness of $\approx$.

Lemma 4.2.2 (c.f., Horn[12])**.** $\approx$ *is a congruence with respect to* $\to,\ \wedge,\ \vee$ *and* $\neg$*.*

Proof. Let finite subsets $\Gamma,\ \Delta,\ \Sigma,\ \Pi \subseteq \mathbf{M}$ satisfy $[\Gamma] \approx [\Delta]$ and $[\Sigma] \approx [\Pi]$.

We show $[\Gamma \to \Sigma] \to [\Delta \to \Pi] = 1$. We have

$$\begin{aligned} &[\Gamma \to \Sigma] \to [\Delta \to \Pi] \\ &= [\{\Gamma \to \sigma \mid \sigma \in \Sigma\}] \to [\{\Delta \to \pi \mid \pi \in \Pi\}] \\ &= [\{\{\Gamma \to \sigma \mid \sigma \in \Sigma\} \to \Delta \to \pi \mid \pi \in \Pi\}]. \end{aligned}$$

Therefore, it is enough to show $\{\Gamma \to \sigma \mid \sigma \in \Sigma\} \to \Delta \to \pi = 1$ for all $\pi \in \Pi$. It follows from the assumption $\Delta \to \gamma = 1$ for every $\gamma \in \Gamma$ and $\Sigma \to \pi = 1$ for every $\pi \in \Pi$. The case $\wedge$ can be verified by the following:

$$\begin{aligned} &[\Gamma \wedge \Sigma] \to [\Delta \wedge \Pi] \\ &= [\{\Gamma \cup \Sigma \to x \mid x \in \Delta \cup \Pi\}] \\ &= [\{1\}]. \end{aligned}$$

The case $\vee$ is proved in the same way. We have

$$\begin{aligned} &[\Gamma \vee \Sigma] \to [\Delta \vee \Pi] \\ &= [\{\gamma \vee \sigma \mid \gamma \in \Gamma,\ \sigma \in \Sigma\}] \to [\{\delta \vee \pi \mid \delta \in \Delta,\ \pi \in \Pi\}] \\ &= [\{\{\gamma \vee \sigma \mid \gamma \in \Gamma,\ \sigma \in \Sigma\} \to \delta \vee \pi \mid \delta \in \Delta,\ \pi \in \Pi\}] \end{aligned}$$

Therefore, it is enough to show $\{\gamma \vee \sigma \mid \gamma \in \Gamma,\ \sigma \in \Sigma\} \to \delta \vee \pi = 1$ for all $\delta \in \Delta$ and $\pi \in \Pi$. We can verify it by the following facts:

1. $\{\gamma \vee \sigma \mid \gamma \in \Gamma,\ \sigma \in \Sigma\} \to \Gamma \vee \Sigma = 1$ holds by distributivity;
2. $(\Gamma \to \delta \vee \pi) \to (\Sigma \to \delta \vee \pi) \to (\Gamma \vee \Sigma \to \delta \vee \pi) = 1$ holds by the axiom of $\mathbf{H}_{\{\to\}}$;
3. $\Gamma \to \delta \vee \pi = \Sigma \to \delta \vee \pi = 1$ holds by assumption.

The case $\neg$ is similar to the case $\to$. □

Lemma 4.2.3 (Horn[12] Theorem 8.4)**.** *If* $\mathbf{M}$ *is an* $\mathcal{S}$*-algebra,* $C(\mathbf{M})$ *is an* $\mathcal{S} \cup \{\wedge\}$*-algebra, i.e., it satisfies all axioms of* $\mathbf{H}_{\mathcal{S}\cup\{\wedge\}}$*.*

Lemma 4.2.4 (Horn[12] Theorem 8.5)**.** *For every $\mathcal{S}$-algebra $\mathbf{M}$, the map $f : \mathbf{M} \longrightarrow C(\mathbf{M})$ defined by $f(x) = [\{x\}]$ is an $\mathcal{S}$-embedding.*

Theorem 4.2.5 (c.f., Horn[12])**.** *Let $\mathbf{M}$ be an $\mathcal{S}$-algebra and $\mathbf{N}$ be an $\mathcal{S} \cup \{\wedge\}$-algebra. If there exists $\mathcal{S}$-embedding $f : \mathbf{M} \longrightarrow \mathbf{N}$, we have an $\mathcal{S} \cup \{\wedge\}$-embedding $g : C(\mathbf{M}) \longrightarrow \mathbf{N}$ defined by $g([\Gamma]) = \bigwedge_{\gamma \in \Gamma} f(\gamma)$.*

Proof. We can verify that g is well-defined and injection since $\approx$ is a congruence. Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ and $\Delta = \{\delta_1, \ldots, \delta_n\}$ be finite subsets of $\mathbf{M}$. We verify that g is an $\mathcal{S} \cup \{\wedge\}$-homomorphism by the following:

$$
\begin{aligned}
& g([\Gamma] \to [\Delta]) \\
&= g([\{\Gamma \to \delta_j \mid j = 1, \ldots, n\}]) \\
&= \bigwedge_{j=1,\ldots,n} f(\Gamma \to \delta_j) \\
&= \bigwedge_{j=1,\ldots,n} (f(\gamma_1) \to \cdots f(\gamma_m) \to f(\delta_j)) \\
&= \bigwedge_{i=1,\ldots,m} f(\gamma_i) \to \bigwedge_{j=1,\ldots,n} f(\delta_j) \\
&= g([\Gamma]) \to g([\Delta]);
\end{aligned}
$$

$$
\begin{aligned}
& g([\Gamma] \wedge [\Delta]) \\
&= g([\Gamma \cup \Delta]) \\
&= \bigwedge_{x \in \Gamma \cup \Delta} f(x) \\
&= \bigwedge_{i=1,\ldots,m} f(\gamma_i) \wedge \bigwedge_{j=1,\ldots,n} f(\delta_j) \\
&= g([\Gamma]) \wedge g([\Delta]);
\end{aligned}
$$

$$
\begin{aligned}
& g([\Gamma] \vee [\Delta]) \\
&= g([\{\gamma_i \vee \delta_j \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}]) \\
&= \bigwedge_{i=1,\ldots,m,\ j=1,\ldots,n} f(\gamma_i \vee \delta_j) \\
&= \bigwedge_{i=1,\ldots,m,\ j=1,\ldots,n} (f(\gamma_i) \vee f(\delta_j)) \\
&= \bigwedge_{i=1,\ldots,m} f(\gamma_i) \vee \bigwedge_{j=1,\ldots,n} f(\delta_j) \\
&= g([\Gamma]) \vee g([\Delta]).
\end{aligned}
$$

The case $\neg$ can be shown by the similar way to the case $\to$. Therefore, g is an $\mathcal{S} \cup \{\wedge\}$-homomorphism. Consequently, g is an $\mathcal{S} \cup \{\wedge\}$-embedding. $\square$

4.3 Separability and conservativity

We define the separability and conservativity for intermediate logics.

Let $\mathcal{S} \subseteq \mathcal{S}'$. If an $\mathcal{S}$-formula A satisfies both of $\mathbf{H}_{\mathcal{S}} + \Gamma \not\vdash A$ and $\mathbf{H}_{\mathcal{S}'} + \Gamma \vdash A$, the proof of $\mathbf{H}_{\mathcal{S}'} + \Gamma \vdash A$ must contain a logical symbol $\odot \in \mathcal{S}' - \mathcal{S}$ nevertheless $\odot$ does not occur in A. When the above situation does not occur, in other words, every $\mathcal{S}$-formula A which is provable in $\mathbf{H}_{\mathcal{S}'} + \Gamma$ is $\mathcal{S}$-provable, we say that $\mathbf{H}_{\mathcal{S}'} + \Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}} + \Gamma$.

More formally stated,

Definition 4.3.1 (Conservativity)**.** *Let $\mathcal{S} \subseteq \mathcal{S}'$ and Γ be a set of $\mathcal{S}$-formulas. $\mathbf{H}_{\mathcal{S}'} + \Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}} + \Gamma$ if $(\mathbf{H}_{\mathcal{S}'} + \Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma$.*

Notice that conservativity condition is a property for axiomatization of logics.

Definition 4.3.2 (Separability)**.** *An axiomatization $\mathbf{H} + \Gamma$ of a logic $\mathbf{L}$ is a separable axiomatization if it satisfies the following conditions:*

1. *$\mathbf{H}+\Gamma$ is normal, i.e., any formula in Γ is a $\{\to\}$, $\{\to, \wedge\}$, $\{\to, \vee\}$ or $\{\to, \neg\}$-formula;*
2. *$\mathbf{H} + \Gamma$ is $\mathcal{S}$-complete [1] for every $\mathcal{S}$, i.e., $(\mathbf{H} + \Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma_{\mathcal{S}}$ holds for every $\mathcal{S}$.*

The conservativity is a part of the conditions of the separability. If Γ is a set of $\mathcal{S}$-formulas, $\mathcal{S}$-completeness is equivalent to the conservativity between $\mathbf{H} + \Gamma$ and $\mathbf{H}_{\mathcal{S}} + \Gamma$.

Definition 4.3.3. *An intermediate logic $\mathbf{L}$ is separable if there is a finite set Γ of formulas such that $\mathbf{H} + \Gamma$ is a separable axiomatization of $\mathbf{L}$.*

The definition of separability is introduced by M. Wajsberg[29]. The separability of some famous intermediate logics has been proved until in the mid of 1960's.

Theorem 4.3.4. *The following logics are separable:*

1. *the intuitionistic propositional logic $\mathbf{H}$ (Curry[7])*
2. *the classical propositional logic $\mathbf{C}$ [2];*
3. *$\mathbf{H} + \neg p \vee \neg\neg p$; (Hosoi[15]);*
4. *$\mathbf{H} + (p \to q) \vee (q \to p)$ (Hosoi[13], pp.537, Corollary).*

We note that the previous results proved the separability of logics. For example, Hosoi did not proved the separability of the axiomatization $\mathbf{H} + (p \to q) \vee (q \to p)$ but the logic represented by $\mathbf{H} + (p \to q) \vee (q \to p)$.

However, there are few general results for the separability and conservativity. We introduce two general results for the separability which seem particularly important.

[1] The name of this condition is due to Khomich ([19]). Thus, it does not mean the completeness theorem between logics and Kripke frames.

[2] The author have not found the first paper which shows this fact. However, this fact can be easily obtain by results in the present-day. For example, McKay's result([27]) shows the separability since the classical propositional logic is a tabular logic which can be normally axiomatized,

Theorem 4.3.5 (McKay [27], Theorem 3 and Khomich [25], Theorem 1)**.** *Any tabular (characterized by a finite Kripke frame) logic which has normal axiomatization is separable.*

The theorem above is proved by McKay[27]. However, Khomich[25] pointed out that the proof in [27] contains a mistake and corrected the proof by the method in [25].

V. I. Khomich[19, 20, 21, 22, 23, 25] examined the separability of logics each of which is axiomatized by disjunction-free formulas. In particular, he proved the following theorem.

Theorem 4.3.6 (Khomich[22], Theorem 17)**.** *If an intermediate propositional logic* $\mathbf{L}$ *can be axiomatized by formulas without neither* $\vee$ *nor* $\neg$*,* $\mathbf{L}$ *is separable.*

Therefore, the remaining problem for the separability is the case which McKay and Khomich's two theorems above can not apply, i.e., the separability of non-tabular logics each of which needs the disjunction to axiomatize.

4.4 Wroński's theorem

Wroński[31] gave the following theorem by algebraic methods. The following theorem gives the answer for the question that what $\mathcal{S}$ satisfies the following condition:

$$(\mathbf{H}+\Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma \text{ for every set } \Gamma \text{ of } \mathcal{S}\text{-formulas.}$$

However, if $\wedge \notin \mathcal{S}$, the following theorem does not give any criteria for $(\mathbf{H}+\Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma$ or not.

Theorem 4.4.1 (Wroński[31])**.** *The following hold.*

1. *If* $\wedge \in \mathcal{S}$*,* $(\mathbf{H}+\Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma$ *for every set* Γ *of* $\mathcal{S}$*-formulas.*
2. *If* $\wedge \notin \mathcal{S}$*, there exists a set* Γ *of* $\mathcal{S}$*-formulas such that* $(\mathbf{H}+\Gamma)_{\mathcal{S}} \neq \mathbf{H}_{\mathcal{S}} + \Gamma$*.*

In this section, we give a proof of Theorem 4.4.1 in detail since in the original paper [31], there is only an outline of the proof for the case $\wedge \in \mathcal{S}$.

4.4.1 The case $\wedge \in \mathcal{S}$

We show 1 of Theorem (4.4.1). In the original paper ([31]), all of the case of $\wedge \in \mathcal{S}$ follow from the McKay's theorem 4.4.3. However, we write a detailed proof since The McKay's theorem seems not to be applied in the case $\mathcal{S} = \{\to, \wedge, \vee\}$.

Theorem 4.4.2 (McKay[26] Section 2)**.** *Every finitely generated* $\mathcal{S}$*-algebra is finite if* $\mathcal{S} = \{\to,\ \wedge\}$, $\{\to,\ \neg\}$ *or* $\{\to,\ \wedge,\ \neg\}$ *(in other words,* $\vee \notin \mathcal{S}$*).*

Theorem 4.4.3 (McKay[26] Section 2)**.** *Every finite* $\{\to, \wedge\}$*-algebra* $\mathbf{M}$ *can be expanded to a Heyting algebra uniquely.*

Lemma 4.4.4 (Wroński[31]). *For every set Γ of $\mathcal{S}$-formulas, $\mathbf{H}_{\mathcal{S}} + \Gamma$ is the $\mathcal{S}$-fragment of $\mathbf{H} + \Gamma$ if $\mathcal{S} = \{\to,\ \wedge\}$ or $\{\to,\ \wedge,\ \neg\}$.*

Proof. Let A be an $\mathcal{S}$-formula satisfying $\mathbf{H}_{\mathcal{S}} + \Gamma \nvdash A$. Then there exists the subalgebra $\mathbf{M}$ of the Lindenbaum algebra of $\mathbf{H}_{\mathcal{S}} + \Gamma$ generated by the propositional variables occurring in A. $\mathbf{M}$ is finitely generated $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra which refutes A. Hence $\mathbf{M}$ is finite by Theorem 4.4.2. Therefore, $\mathbf{M}$ can be expanded to a Heyting algebra $\mathbf{M}'$ by Theorem 4.4.3. Consequently, $\mathbf{M}'$ is a $\mathbf{H} + \Gamma$-algebra which refutes A. It implies $\mathbf{H} + \Gamma \nvdash A$. The lemma is proved. $\square$

Lemma 4.4.5 (Wroński[31]). *For every set Γ of $\mathcal{S}$-formulas, $\mathbf{H}_{\mathcal{S}} + \Gamma$ is the $\mathcal{S}$-fragment of $\mathbf{H} + \Gamma$ if $\mathcal{S} = \{\to,\ \wedge,\ \vee\}$.*

Proof. Let $A = A(p_1, \ldots, p_n)$ be an $\mathcal{S}$-formula satisfying $\mathbf{H}_{\mathcal{S}} + \Gamma \nvdash A$. There is an $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra $\mathbf{M}$ which refutes A by a refutation v. Let $v(p_i) = a_i$ ($\forall i \in \{1, \ldots, n\}$) and $\mathbf{N} = \{x \in \mathbf{M} \mid x \geq a_1 \wedge \cdots \wedge a_n\}$. Then we can verify that $\mathbf{N}$ is closed under the functions of $\mathbf{M}$. Therefore, $\mathbf{N}$ is an $\mathcal{S}$-subalgebra of $\mathbf{M}$ which implies that $\mathbf{N}$ is an $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra. We can expand $\mathbf{N}$ to a Heyting algebra $\mathbf{N}'$ by defining $\neg$ as follows: $\neg x = x \to (a_1 \wedge \cdots \wedge a_n)$, since $a_1 \wedge \cdots \wedge a_n$ is the smallest element of $\mathbf{N}$. Hence $\mathbf{N}'$ is a $\mathbf{H} + \Gamma$ algebra which refutes A. Consequently, $\mathbf{H} + \Gamma \nvdash A$. The lemma is proved. $\square$

Theorem 4.4.1 for the case $\wedge \in \mathcal{S}$ is now proved by Lemma 4.4.4 and 4.4.5.

Corollary 4.4.6. *Let Γ be a set of $\mathcal{S}$-formulas. The following are equivalent:*

1. $(\mathbf{H} + \Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma$;

2. $(\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma$.

Proof. ($1 \Longrightarrow 2$) $(\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma)_{\mathcal{S}} \subseteq (\mathbf{H} + \Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \Gamma \subseteq (\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma)_{\mathcal{S}}$. The first inclusion follows from $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} \subseteq \mathbf{H}$.

($2 \Longrightarrow 1$) We obtain $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma = (\mathbf{H} + \Gamma)_{\mathcal{S} \cup \{\wedge\}}$ by Theorem 4.4.1. Thus, we have $\mathbf{H}_{\mathcal{S}} + \Gamma = (\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma)_{\mathcal{S}} = ((\mathbf{H} + \Gamma)_{\mathcal{S} \cup \{\wedge\}})_{\mathcal{S}} = (\mathbf{H} + \Gamma)_{\mathcal{S}}$. $\square$

4.4.2 The case $\wedge \notin \mathcal{S}$

We give a detailed proof 2 of Theorem 4.4.1 by using Jankov's characteristic formulas. We suppose that the idea in original paper ([31]) is based on Jankov's characteristic formulas too.

Let $\mathbf{N}_1 = \{1,\ \omega,\ a,\ b,\ 0\}$ is the Heyting algebra defined by the diagram of the Figure 1 and $\mathbf{M}_1 = \{1,\ \omega,\ a,\ b\}$ is the $\{\to,\ \vee,\ \neg\}$-subalgebra of $\mathbf{N}_1$ displayed in Figure 2. The calculation table of $\mathbf{N}_1$ is on Table 1. We note that $a \to 0 = b \to 0 = 0$ are defined in $\mathbf{M}_1$. Thus, the $\{\to,\ \vee,\ \neg\}$-algebra defined by the diagram on Figure 3 is not $\mathbf{M}_1$.

Table 4.1: Calculation table of $\mathbf{N}_1$

$\to$	1	ω	a	b	c	0
1	1	ω	a	b	0	0
ω	1	1	a	b	0	0
a	1	1	1	b	b	0
b	1	1	a	1	a	0
c	1	1	1	1	1	0
0	1	1	1	1	1	1

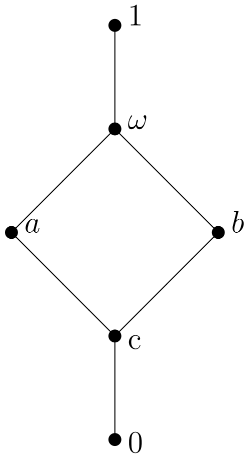


Figure 1 $\mathbf{N}_1$

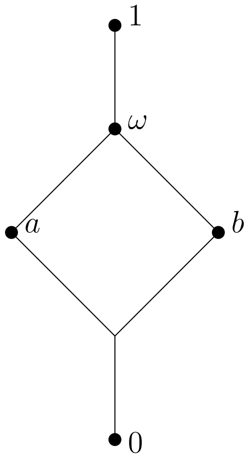


Figure 2 $\mathbf{M}_1$

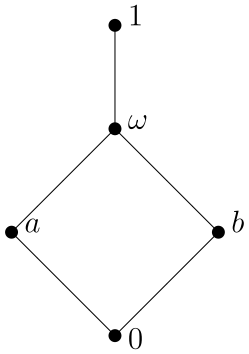


Figure 3
This is *not* $\mathbf{M}_1$

Lemma 4.4.7. $\mathbf{M}_1$ *validates* $X^{\{\to\}}_{\mathbf{N}_1}$ *and refutes* $X^{\{\to\}}_{\mathbf{M}_1}$.

Proof. The Heyting algebra $\mathbf{N}_1$ is not $\{\to\}$-embeddable in any $\{\to,\ \vee,\ \neg\}$-homomorphic image of $\mathbf{M}_1$ since any embedding is injective and $|h(\mathbf{M}_1)| \leq |\mathbf{M}_1| < |\mathbf{N}_1|$ holds for every $\{\to,\ \vee,\ \neg\}$-homomorphism h. Therefore, by Theorem 4.1.5, $\mathbf{M}_1$ validates $X^{\{\to\}}_{\mathbf{N}_1}$.

On the other hand, since $\mathbf{M}_1$ is $\{\to\}$-embeddable in $\mathbf{M}_1$ itself, $\mathbf{M}_1$ refutes $X^{\{\to\}}_{\mathbf{M}_1}$ by Theorem 4.1.5. □

Corollary 4.4.8. $\mathbf{H}_{\{\to,\ \vee,\ \neg\}} + X^{\{\to\}}_{\mathbf{N}_1} \nvdash X^{\{\to\}}_{\mathbf{M}_1}$.

Proof. It follows from Lemma 4.4.7 and Theorem 3.4.3 and 3.4.6. □

Lemma 4.4.9. $\mathbf{H}_{\{\to,\ \wedge\}} + X^{\{\to\}}_{\mathbf{N}_1} \vdash X^{\{\to\}}_{\mathbf{M}_1}$.

Proof. We show the (contraposition of) following claim which shows the lemma by (the contraposition of) Theorem 3.4.10:

- if an $\{\to,\ \wedge\}$-algebra $\mathbf{M}$ is an $\mathbf{H}_{\{\to,\ \wedge\}} + X^{\{\to\}}_{\mathbf{N}_1}$-algebra (i.e., $\mathbf{M}$ validates $X^{\{\to\}}_{\mathbf{N}_1}$), $\mathbf{M}$ also validates $X^{\{\to\}}_{\mathbf{M}_1}$.

Let $\mathbf{M}$ be a $\{\to,\ \wedge\}$-algebra refutes $X^{\{\to\}}_{\mathbf{M}_1}$. Then, $\mathbf{M}_1$ is $\{\to\}$-embeddable in $f(\mathbf{M})$ for some $\{\to,\ \wedge\}$-homomorphism f, i.e., there is an $\{\to\}$-embedding $h : \mathbf{M}_1 \longrightarrow f(\mathbf{M})$. We define a map $g : \mathbf{N}_1 \longrightarrow f(\mathbf{M})$ by

$$\begin{cases} g(0) = h(a) \wedge h(b) \\ g(x) = h(x) \text{ (if } x \neq 0). \end{cases}$$

Then, it is easy to verify that g is a $\{\to\}$-embedding. Therefore, $\mathbf{M}$ also refutes $X^{\{\to\}}_{\mathbf{N}_1}$. The lemma is proved. □

Theorem 4.4.10. *Let $\wedge \notin \mathcal{S}$. Then, there exists a set Γ of $\mathcal{S}$-formulas such that $(\mathbf{H}_{\mathcal{S}\cup\{\wedge\}} + \Gamma)_{\mathcal{S}} \neq \mathbf{H}_{\mathcal{S}\cup\wedge} + \Gamma$.*

Proof. Since $\wedge \notin \mathcal{S}$ implies $(\mathbf{H}_{\{\to,\ \wedge\}} + X^{\{\to\}}_{\mathbf{N}_1})_{\mathcal{S}} \subseteq (\mathbf{H}_{\mathcal{S}\cup\{\wedge\}} + X^{\{\to\}}_{\mathbf{N}_1})_{\mathcal{S}}$ and $\mathbf{H}_{\mathcal{S}} + X^{\{\to\}}_{\mathbf{N}_1} \subseteq \mathbf{H}_{\{\to,\ \vee,\ \neg\}} + X^{\{\to\}}_{\mathbf{N}_1}$, the following calculation shows the theorem:

$$\begin{aligned} & X^{\{\to\}}_{\mathbf{M}_1} \\ & \in (\mathbf{H}_{\{\to,\ \wedge\}} + X^{\{\to\}}_{\mathbf{N}_1})_{\mathcal{S}} - \mathbf{H}_{\{\to,\ \vee,\ \neg\}} + X^{\{\to\}}_{\mathbf{N}_1} \\ & \subseteq (\mathbf{H}_{\mathcal{S}\cup\{\wedge\}} + X^{\{\to\}}_{\mathbf{N}_1})_{\mathcal{S}} - \mathbf{H}_{\mathcal{S}} + X^{\{\to\}}_{\mathbf{N}_1}. \end{aligned}$$

□

Let $\mathbf{N}_2 = \{1,\ \omega,\ a,\ b,\ c\ d.\ e,\ 0\}$ is the Heyting algebra defined by the diagram of the Figure 3. and $\mathbf{M}_2 = \{1,\ \omega,\ a,\ b,\ d,\ e,\ 0\}$ is the $\{\to,\ \neg\}$-subalgebra of $\mathbf{N}_2$ displayed in Figure 4. The calculation table of $\mathbf{N}_2$ is in Table 2.

Table 4.2: Calculation table of $\mathbf{N}_2$

$\to$	1	ω	a	b	c	d	e	0
1	1	ω	a	b	c	d	e	0
ω	1	1	a	b	c	d	e	0
a	1	1	1	b	b	d	e	0
b	1	1	a	1	a	d	e	0
c	1	1	1	1	1	d	e	0
d	1	1	1	1	1	1	e	e
e	1	1	1	1	1	d	1	d
0	1	1	1	1	1	1	1	1

We will define $Z^{\{\to\}}_{\mathbf{N}_2}$ which is similar to Jankov's characteristic formula by

$$Z^{\{\to\}}_{\mathbf{N}_2} = Y^{\{\to\}}_{\mathbf{M}_2} \cup \{p_c \to p_a,\ p_c \to p_b,\ p_d \to p_c,\ p_e \to p_c\} \to p_\omega.$$

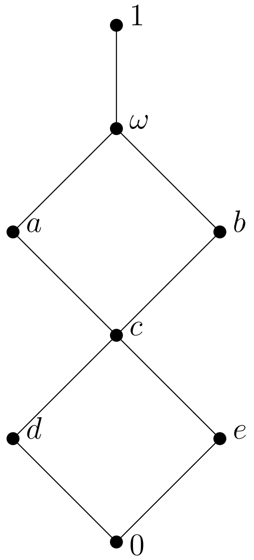


Figure 4 $\mathbf{N}_2$

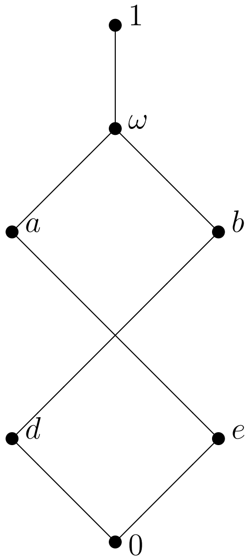


Figure 5 $\mathbf{M}_2$

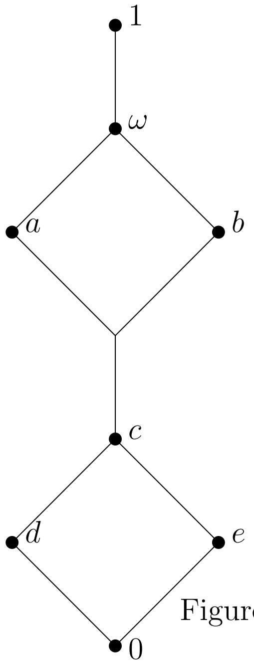


Figure 6 $\mathbf{J}$

The formula $Z^{\{\to\}}_{\mathbf{N}_2}$ determines the $\{\to\}$-reduct of the $\{\to,\ \neg\}$-algebra $\mathbf{M}_2$ and there is an element c such that $d,\ e < c < b,\ a$. We have the following lemma similar to Theorem 4.1.5.

Lemma 4.4.11. *Let $\vee \in \mathcal{S}$. An $\mathcal{S}$-algebra $\mathbf{M}$ refutes $Z^{\{\to\}}_{\mathbf{N}_2}$ if and only if the following hold:*

1. *there are an $\mathcal{S}$-homomorphism g and an $\{\to\}$-embeddable $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$;*
2. *there is $c' \in g(\mathbf{M})$ such that $h(d),\ h(e) < c' < h(a),\ h(b)$.*

Proof. We show the if part first. Let v be a valuation on $\mathbf{M}$ by

$$\begin{cases} v(p_c) = c' \\ v(p_x) = g(x) \text{ (if } x \neq c). \end{cases}$$

Then, similar to the proof of Theorem 4.1.5, we obtain $v(Z^{\{\to\}}_{\mathbf{N}_2}) = \omega$.

We show the only if part. By Theorem 3.4.34, there is a $\mathcal{S}$-homomorphism g such that there is ω-refutation v of $Z^{\{\to\}}_{\mathbf{N}_2}$ on $g(\mathbf{M})$. Therefore, by Theorem 4.1.5, there is $\{\to\}$-embeddable $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$. Moreover, since v is an ω-refutation, v satisfies $v(p_c \to p_a) = v(p_c \to p_b) = v(p_d \to p_c) = v(p_e \to p_c) = 1$, which implies $v(d),\ v(e) \leq v(c) \leq v(a),\ v(b)$. Therefore, we have $v(d),\ v(e) < v(c) < v(a),\ v(b)$ since $v(d)$ and $v(e)$, and $v(a)$ and $v(b)$ are incomparable. Consequently, $v(c)$ is the element c' we wanted. $\square$

We note that $Z^{\{\to\}}_{\mathbf{N}_2}$ does not determine calculation table with respect to c. $Z^{\{\to\}}_{\mathbf{N}_2}$ determines only $d,\ e < c < b,\ a$. For example, both of $\mathbf{N}_2$ and a $\{\to,\ \neg\}$-algebra $\mathbf{J}$ displayed in Figure 6 refute $Z^{\{\to\}}_{\mathbf{N}_2}$ ($\mathbf{J}$ defines $a \to c = b \to c = c$ unlike in the $\{\to,\ \neg\}$-reduct of $\mathbf{N}_2$).

Lemma 4.4.12. $\mathbf{H}_{\{\to,\ \wedge,\ \neg\}} + Z^{\{\to\}}_{\mathbf{N}_2} \nvdash X^{\{\to\}}_{\mathbf{M}_2}$.

Proof. Since the $\{\to,\ \wedge,\ \neg\}$-algebra $\mathbf{M}_2$ validates $Z^{\{\to\}}_{\mathbf{N}_2}$, and refutes $X^{\{\to\}}_{\mathbf{M}_2}$. See the proof of Lemma 4.4.7. □

Lemma 4.4.13. $\mathbf{H}_{\{\to,\ \vee\}} + Z^{\{\to\}}_{\mathbf{N}_2} \vdash X^{\{\to\}}_{\mathbf{M}_2}$.

Proof. Similar to Lemma 4.4.9, it is sufficient to show the following claim:

- if an $\{\to,\ \vee\}$-algebra $\mathbf{M}$ validates $Z^{\{\to\}}_{\mathbf{N}_2}$, $\mathbf{M}$ also validates $X^{\{\to\}}_{\mathbf{M}_2}$.

We show the contraposition. Thus, we assume that there are a $\{\to,\ \vee\}$-algebra $\mathbf{M}$ and $\{\to,\ \vee\}$-homomorphism g such that there is a $\{\to\}$-embedding $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$.

By Lemma 4.4.11, the following two claims show that $\mathbf{M}$ refutes $Z^{\{\to\}}_{\mathbf{N}_2}$:

1. there is an $\{\to\}$-embedding $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$;
2. there is $c' \in g(\mathbf{M})$ such that $h(d),\ h(e) < c' < h(a),\ h(b)$.

We have already obtained an $\{\to\}$-embedding $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$. Thus we will show the second claim above. Put $c' = h(d) \vee h(e)$. Then we have $h(d),\ h(e) \leq h(d) \vee h(e) \leq h(a),\ h(b)$ since $h(d) \vee h(e)$ is the least upper bound of $\{h(d),\ h(e)\}$. Moreover, we have $h(d),\ h(e) < h(d) \vee h(e) < h(a),\ h(b)$ since $h(d)$ and $h(e)$, and $h(a)$ and $h(b)$ are incomparable. The lemma is proved. □

Consequently, we obtain the following theorem in the same way as Theorem 4.4.10.

Theorem 4.4.14. *Let* $\wedge,\ \vee \notin \mathcal{S}$. *Then, there exists a set* Γ *of* $\mathcal{S}$*-formulas such that* $(\mathbf{H}_{\mathcal{S}\cup\{\vee\}} + \Gamma)_{\mathcal{S}} \neq \mathbf{H}_{\mathcal{S}\cup\{\vee\}} + \Gamma$.

Let $\mathbf{N}_3 = \{1,\ \omega,\ a,\ b,\ 0\}$ be the Heyting algebra defined by the diagram of the Figure 7 and $\mathbf{M}_3 = \{1,\ \omega,\ a,\ b\}$ is the $\{\to,\ \vee\}$-subalgebra of $\mathbf{N}_3$ displayed in Figure 8. The calculation table of $\mathbf{N}_3$ is in Table 3.

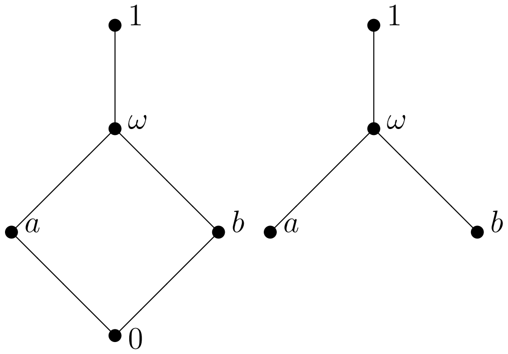


figure-7 $\mathbf{N}_3$ figure-8 $\mathbf{M}_3$

Table 4.3: Calculation table of $\mathbf{N}_3$

$\to$	1	ω	a	b	0
1	1	ω	a	b	0
ω	1	1	a	b	0
a	1	1	1	b	b
b	1	1	a	1	a
c	1	1	1	1	1

Similar to $Z^{\{\to\}}_{\mathbf{N}_2}$, we define that

$$Z^{\{\to\}}_{\mathbf{N}_3} = Y^{\{\to\}}_{\mathbf{M}_3} \cup \{p_0 \to p_x \mid x = a,\ b\} \to p_\omega.$$

Lemma 4.4.15. *Let $\neg \in \mathcal{S}$. An $\mathcal{S}$-algebra $\mathbf{M}$ refutes $Z^{\{\to\}}_{\mathbf{N}_3}$ if and only if the following hold:*

1. *there are an $\mathcal{S}$-homomorphism g and an $\{\to,\ \neg\}$-embeddable $h : \mathbf{M}_3 \longrightarrow g(\mathbf{M})$;*
2. *there is $0' \in g(\mathbf{M})$ such that $0' < h(a),\ h(b)$.*

Proof. We show the if part first. Let v be a valuation on $\mathbf{M}$ by

$$\begin{cases} v(p_0) = 0' \\ v(p_x) = g(x) \text{ (if } x \neq 0). \end{cases}$$

Then, similar to the proof of Theorem 4.1.5, we obtain $v(Z^{\mathcal{S}}_{\mathbf{N}_2}) = \omega$.

We show the only if part. By Theorem 3.4.34, there is a $\mathcal{S}$-homomorphism g such that there is ω-refutation v of $Z^{\mathcal{S}}_{\mathbf{N}_3}$ on $g(\mathbf{M})$. Therefore, by Theorem 4.1.5, there is $\{\to\}$-embeddable $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$. Moreover, since v is an ω-refutation, v satisfies $v(p_0 \to p_a) = v(p_0 \to p_b) = 1$, which implies $v(p_0) \leq v(p_a),\ v(p_b)$. Therefore, we have $v(p_0) < v(p_a),\ v(p_b)$ since $v(p_a)$ and $v(p_b)$ are incomparable. Consequently, $v(p_0)$ is the element $0'$ we wanted. □

Lemma 4.4.16. $\mathbf{H}_{\{\to,\ \vee\}} + Z^{\{\to\}}_{\mathbf{N}_3} \nvdash X^{\{\to\}}_{\mathbf{M}_3}$.

Proof. Since the $\{\to,\ \vee\}$-algebra $\mathbf{M}_3$ validates $Z^{\{\to\}}_{\mathbf{N}_3}$, and refutes $X^{\{\to\}}_{\mathbf{M}_3}$. See the proof of Lemma 4.4.7. □

Lemma 4.4.17. $\mathbf{H}_{\{\to,\ \neg\}} + Z^{\{\to\}}_{\mathbf{N}_3} \vdash X^{\{\to\}}_{\mathbf{M}_3}$.

Proof. Similar to Lemma 4.4.9, it is sufficient to show the following claim:

- if an $\{\to,\ \neg\}$-algebra $\mathbf{M}$ validates $Z^{\{\to\}}_{\mathbf{N}_3}$, $\mathbf{M}$ also validates $X^{\{\to\}}_{\mathbf{M}_3}$.

We show the contraposition. Thus, we assume that there is a $\{\to,\ \neg\}$-algebra $\mathbf{M}$ and $\{\to,\ \neg\}$-homomorphism g such that there is a $\{\to\}$-embedding $h : \mathbf{M}_3 \longrightarrow g(\mathbf{M})$. By Lemma 4.4.15, the following two claims show that $\mathbf{M}$ refutes $Z^{\{\to\}}_{\mathbf{N}_3}$:

1. there is a $\{\to\}$-embedding $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$;
2. there is $0' \in g(\mathbf{M})$ such that $0' < h(a),\ h(b)$.

We already obtained a $\{\to\}$-embedding $h : \mathbf{M}_2 \longrightarrow g(\mathbf{M})$. Thus we will show the second claim above. Put $0' = \neg h(1)$. Then we have $\neg h(1) \leq h(a),\ h(b)$. since $\neg h(1) = \neg 1$ is the smallest element of $g(\mathbf{M})$. Moreover, we have $\neg h(1) < h(a),\ h(b)$ since $h(a)$ and $h(b)$ are incomparable. The lemma is proved. □

Consequently, we obtain the following theorem in the same way as Theorem 4.4.10.

Theorem 4.4.18. *Let $\wedge,\ \neg \notin \mathcal{S}$. Then, there exists a set Γ of $\mathcal{S}$-formulas such that $(\mathbf{H}_{\mathcal{S}\cup\{\neg\}} + \Gamma)_{\mathcal{S}} \neq \mathbf{H}_{\mathcal{S}\cup\{\neg\}} + \Gamma$.*

We obtain 2 (the case $\wedge \notin \mathcal{S}$) of Theorem 4.4.1 by Theorem 4.4.10, 4.4.14 and 4.4.18.

4.5 Algebraic characterization of the conservativity

When $\mathcal{S} \subsetneq \mathcal{S}'$ and $\wedge \notin \mathcal{S}$, Theorem 4.4.1 states that there is a set Γ of $\mathcal{S}$-formulas such that $\mathbf{H}_{\mathcal{S}'} + \Gamma$ is not a conservative extension of $\mathbf{H}_{\mathcal{S}} + \Gamma$. However, for a given Γ, Theorem 4.4.1 does not gives an answer to the question:

Is $\mathbf{H}_{\mathcal{S}'} + \Gamma$ a conservative extension of $\mathbf{H}_{\mathcal{S}} + \Gamma$?

As our first result for conservativity problem, we give a characterization which answers the above question.

Theorem 4.5.1. *Let $\mathcal{S} \subseteq \mathcal{S}'$ and Γ be a set of $\mathcal{S}$-formulas. The following are equivalent.*

1. $\mathbf{H}_{\mathcal{S}} + \Gamma = (\mathbf{H}_{\mathcal{S}'} + \Gamma)_{\mathcal{S}}$.
2. *Every $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra is $\mathcal{S}$-embeddable in an $\mathbf{H}_{\mathcal{S}'} + \Gamma$-algebra.*
3. *Every finitely generated $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra is $\mathcal{S}$-embeddable in an $\mathbf{H}_{\mathcal{S}'} + \Gamma$-algebra.*

Proof. ($1 \Longrightarrow 2$) We can assume that $\mathbf{M}$ is subdirectly irreducible (i.e., $\mathbf{M}$ is Gödelian). Let $\mathbf{M}$ be a subdirectly irreducible $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra. We have an ω-refutation v of $(Y^{\mathcal{S}}_{\mathbf{M}}, p_\omega)$ on $\mathbf{M}$ defined by $v(p_x) = x$. Hence we have $\mathbf{H}_{\mathcal{S}} + \Gamma \nvdash \Delta \to p_\omega$ for every finite subset $\Delta \subseteq Y^{\mathcal{S}}_{\mathbf{M}}$. We have $\mathbf{H}_{\mathcal{S}'} + \Gamma \nvdash \Delta \to p_\omega$ for every finite subset $\Delta \subseteq Y^{\mathcal{S}}_{\mathbf{M}}$ by assumption. Therefore we obtain an $\mathcal{S}'$-algebra $\mathbf{N}$ and a valuation w on $\mathbf{N}$ satisfying $w(A) = 1$ for any $A \in Y^{\mathcal{S}}_{\mathbf{M}}$ and $w(p_\omega) \neq 1$ by Theorem 3.4.27. There exists an $\mathcal{S}'$-homomorphism h satisfying

$h(\mathbf{N})$ is subdirectly irreducible and $h \circ w$ is an ω-refutation of $(Y^{\mathcal{S}}_{\mathbf{M}},\ p_\omega)$ on $h(\mathbf{N})$ by Lemma 4.1.4. Hence $\mathbf{M}$ is $\mathcal{S}$-embeddable in $h(\mathbf{N})$ by Lemma 4.1.3. We proved that every subdirectly irreducible $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra is $\mathcal{S}$-embeddable in an $\mathbf{H}_{\mathcal{S}'} + \Gamma$-algebra.

(2 $\Longrightarrow$ 3) is obvious.

(3 $\Longrightarrow$ 1) Assume $\mathbf{H}_{\mathcal{S}} + \Gamma \not\vdash A$. Let $\mathbf{M}$ be the subalgebra of the Lindenbaum algebra of $\mathbf{H}_{\mathcal{S}} + \Gamma$ generated by the propositional variables occurring in A. $\mathbf{M}$ is a finitely generated $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra which refutes A. Therefore, $\mathbf{M}$ is $\mathcal{S}$-embeddable in an $\mathbf{H}_{\mathcal{S}'} + \Gamma$-algebra $\mathbf{N}$ by assumption. Hence $\mathbf{N}$ refutes A. Therefore, $\mathbf{H}_{\mathcal{S}'} + \Gamma \not\vdash A$. Consequently, $\mathbf{H}_{\mathcal{S}} + \Gamma \supseteq (\mathbf{H}_{\mathcal{S}'} + \Gamma)_{\mathcal{S}}$ ($\subseteq$ is obvious). $\Box$

Corollary 4.5.2. *Let $\mathcal{S} \subseteq \mathcal{S}'$ and Γ be a set of $\mathcal{S}$-formulas. If $\vee \notin \mathcal{S}$, the following are equivalent.*

1. $\mathbf{H}_{\mathcal{S}} + \Gamma = (\mathbf{H}_{\mathcal{S}'} + \Gamma)_{\mathcal{S}}$*;*
2. *Every finite $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra is $\mathcal{S}$-embeddable an $\mathbf{H}_{\mathcal{S}'} + \Gamma$-algebra.*

Proof. (1 $\Longrightarrow$ 2) follows from Theorem 4.5.1.

(2 $\Longrightarrow$ 1) follows from the proof of (3 $\Longrightarrow$ 1) of Theorem 4.5.1 and Theorem 4.4.2 since $\vee \notin \mathcal{S}$. $\Box$

Our second result is that, for the case $\mathcal{S} \cup \{\wedge\} \subseteq \mathcal{S}'$, we can strengthen the first result by using concrete $\mathcal{S} \cup \{\wedge\}$-algebras which are constructed by Horn[12].

Theorem 4.5.3. *For every $\mathcal{S}$ and set Γ of $\mathcal{S}$-formulas, the following are equivalent.*

1. $\mathbf{H}_{\mathcal{S}} + \Gamma = (\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma)_{\mathcal{S}}$*;*
2. *for every $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra $\mathbf{M}$, $C(\mathbf{M})$ is an $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma$-algebra.*

Proof. (1 $\Longrightarrow$ 2) Let $\mathbf{M}$ be an $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra. Then there exists an $\mathcal{S} \cup \{\wedge\}$-algebra $\mathbf{N}$ such that $\mathbf{M}$ is $\mathcal{S}$-embeddable in $\mathbf{N}$ since the assumption and Theorem 4.5.1. By Theorem 4.2.5, $C(\mathbf{M})$ is $\mathcal{S} \cup \{\wedge\}$-embeddable in $\mathbf{N}$. Therefore, $C(\mathbf{M})$ is an $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma$-algebra.

(2 $\Longrightarrow$ 1) Let $\mathbf{M}$ be an $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra. By Lemma 4.2.4, $\mathbf{M}$ is $\mathcal{S}$-embeddable in $C(\mathbf{M})$. Furthermore, $C(\mathbf{M})$ is an $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma$-algebra by assumption. Therefore, $\mathbf{H}_{\mathcal{S}} + \Gamma = (\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma)_{\mathcal{S}}$ by Theorem 4.5.1. $\Box$

In conclusion, we obtain the following characterization.

Theorem 4.5.4. *Let Γ is a set of $\mathcal{S}$-formulas and $\mathcal{S} \cup \{\wedge\} \subseteq \mathcal{S}'$. All the following seven statements are equivalent:*

1. $\mathbf{H}_{\mathcal{S}} + \Gamma = (\mathbf{H}_{\mathcal{S}'} + \Gamma)_{\mathcal{S}}$*;*
2. $\mathbf{H}_{\mathcal{S}} + \Gamma = (\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \Gamma)_{\mathcal{S}}$*;*
3. *For every $\mathbf{H}_{\mathcal{S}} + \Gamma$-algebra $\mathbf{M}$, $\mathbf{M}$ is $\mathcal{S}$-embeddable in an $\mathbf{H}_{\mathcal{S}'} + \Gamma$-algebra;*

4. *For every* $\mathbf{H}_{\mathcal{S}} + \Gamma$*-algebra* $\mathbf{M}$*,* $\mathbf{M}$ *is* $\mathcal{S}$*-embeddable in an* $\mathbf{H}_{\mathcal{S}\cup\{\wedge\}} + \Gamma$*-algebra;*

5. *For every finitely generated* $\mathbf{H}_{\mathcal{S}} + \Gamma$*-algebra* $\mathbf{M}$*,* $\mathbf{M}$ *is* $\mathcal{S}$*-embeddable in an* $\mathbf{H}_{\mathcal{S}'} + \Gamma$*-algebra;*

6. *For every finitely generated* $\mathbf{H}_{\mathcal{S}} + \Gamma$*-algebra* $\mathbf{M}$*,* $\mathbf{M}$ *is* $\mathcal{S}$*-embeddable in an* $\mathbf{H}_{\mathcal{S}\cup\{\wedge\}} + \Gamma$*-algebra;*

7. *For every* $\mathbf{H}_{\mathcal{S}} + \Gamma$*-algebra* $\mathbf{M}$*,* $C(\mathbf{M})$ *is an* $\mathbf{H}_{\mathcal{S}\cup\{\wedge\}} + \Gamma$*-algebra.*

Furthermore, if $\vee \notin \mathcal{S}$*, all the above seven statements and the following two are equivalent:*

8. *For every finite* $\mathbf{H}_{\mathcal{S}} + \Gamma$*-algebra* $\mathbf{M}$*,* $\mathbf{M}$ *is* $\mathcal{S}$*-embeddable in an* $\mathbf{H}_{\mathcal{S}'} + \Gamma$*-algebra;*

9. *For every finite* $\mathbf{H}_{\mathcal{S}} + \Gamma$*-algebra* $\mathbf{M}$*,* $\mathbf{M}$ *is* $\mathcal{S}$*-embeddable in an* $\mathbf{H}_{\mathcal{S}\cup\{\wedge\}} + \Gamma$*-algebra.*

Proof. We obtain ($1 \iff 3 \iff 5$) and ($2 \iff 4 \iff 6$) by Theorem 4.5.1, ($2 \iff 7$) by Theorem 4.5.3, ($1 \iff 2$) by Corollary 4.4.6. If $\vee \notin \mathcal{S}$, we obtain ($1 \iff 8$) and ($2 \iff 9$) by Corollary 4.5.2. □

4.6 Applications of the characterization for the conservativity

As applications of our theorems, we obtain algebraic proofs of some Khomich's theorems ([19, 21, 22]) which are proved by syntactical methods in original papers.

Proposition 4.6.1. *Let* $\mathcal{S} \subseteq \mathcal{S}'$*,* $\mathbf{M}'$ *be an* $\mathcal{S}'$*-algebra and* $\mathbf{M}$ *be the* $\mathcal{S}$*-reduct of* $\mathbf{M}'$*. Then* $C(\mathbf{M})$ *is the* $\mathcal{S} \cup \{\wedge\}$*-reduct of* $C(\mathbf{M}')$*.*

Proof. Let $x = \{x_1, \dots, x_m\},\ y = \{y_1, \dots, y_n\} \in \mathbf{M}_\wedge$ (see Definition 4.2.1). Then $x \approx_{C(\mathbf{M})} y$ if and only if $x \approx_{C(\mathbf{M}')} y$ since the $\{\to\}$-reduct of $C(\mathbf{M})$ and the $\{\to\}$-reduct of $C(\mathbf{M}')$ are equivalent. Therefore

$$\begin{aligned}
& x \wedge_{C(\mathbf{M})} y \\
&= [\{x_1, \dots, x_m,\ y_1, \dots, y_n\}]_{C(\mathbf{M})} \\
&= [\{x_1, \dots, x_m,\ y_1, \dots, y_n\}]_{C(\mathbf{M}')} \\
&= x \wedge_{C(\mathbf{M}')} y,
\end{aligned}$$

where $[\cdots]_{C(\mathbf{M})}$ and $[\cdots]_{C(\mathbf{M}')}$ mean equivalence classes of $C(\mathbf{M})$ and $C(\mathbf{M}')$ respectively.

Also, we show the case $\vee$. The case $\to$ and $\neg$ are similarly.

$$\begin{aligned}
& x \vee_{C(\mathbf{M})} y \\
&= [\{x_1, \dots, x_m\}]_{C(\mathbf{M})} \vee_{C(\mathbf{M})} [\{y_1, \dots, y_n\}]_{C(\mathbf{M})} \\
&= [\{x_i \vee_{\mathbf{M}} y_j \mid i = 1, \dots, m,\ j = 1, \dots, n\}]_{C(\mathbf{M})} \\
&= [\{x_i \vee_{\mathbf{M}'} y_j \mid i = 1, \dots, m,\ j = 1, \dots, n\}]_{C(\mathbf{M}')} \\
&= [\{x_1, \dots, x_m\}]_{C(\mathbf{M}')} \vee_{C(\mathbf{M}')} [\{y_1, \dots, y_n\}]_{C(\mathbf{M}')} \\
&= x \vee_{C(\mathbf{M}')} y.
\end{aligned}$$

□

Theorem 4.6.2 (c.f., Khomich[22] Theorem 1)**.** *For $i = 1, \dots, m$, assume that*

1. *Γ_i is a set of $\mathcal{S}_i$-formulas;*
2. *$(\mathbf{H} + \Gamma_i)_{\mathcal{S}_i} = \mathbf{H}_{\mathcal{S}_i} + \Gamma_i$.*

Then $(\mathbf{H} + \bigcup_{i=1}^m \Gamma_i)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \bigcup_{i=1}^m \Gamma_i$ holds for any $\mathcal{S} \supseteq \bigcup_{i=1}^m \mathcal{S}_i$.

Proof. Let $\mathbf{M}$ be an $\mathbf{H}_{\mathcal{S}} + \bigcup_{i=1}^m \Gamma_i$-algebra and $\mathbf{M}_i$ is the $\mathcal{S}_i$-reduct of $\mathbf{M}$. Then $C(\mathbf{M}_i)$ validates Γ_i since $\mathbf{M}_i$ validates any $\gamma \in \Gamma_i$ and ($1 \iff 7$) of Theorem 4.5.4. Hence $C(\mathbf{M})$ validates any $\gamma \in \Gamma_i$ since $C(\mathbf{M}_i)$ is the $\mathcal{S}_i \cup \{\wedge\}$-reduct of $C(\mathbf{M})$. Consequently, $C(\mathbf{M})$ is an $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + \bigcup_{i=1}^m \Gamma_i$-algebra. Therefore, we have $(\mathbf{H} + \bigcup_{i=1}^m \Gamma_i)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}} + \bigcup_{i=1}^m \Gamma_i$ by applying ($1 \iff 7$) of Theorem 4.5.4 again. □

Theorem 4.6.2 contains the following result proved by Khomich[22].

Corollary 4.6.3 (Khomich[19] Corollary 1)**.** *Let Γ_1 and Γ_2 be sets of $\mathcal{S}$-formulas. If $\mathbf{H}_{\mathcal{S}} + \Gamma_i = (\mathbf{H} + \Gamma_i)_{\mathcal{S}}$ holds for $i = 1,\ 2$, $\mathbf{H}_{\mathcal{S}} + \Gamma_1 \cup \Gamma_2 = (\mathbf{H} + \Gamma_1 \cup \Gamma_2)_{\mathcal{S}}$ also holds.*

Also, we obtain the following corollary.

Corollary 4.6.4. *Let Γ be $\mathcal{S}$-formulas and $\mathcal{S} \subseteq \mathcal{S}'$. If $(\mathbf{H}+\Gamma)_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}}+\Gamma$ holds, $(\mathbf{H}+\Gamma)_{\mathcal{S}'} = \mathbf{H}_{\mathcal{S}'} + \Gamma$ also holds.*

Khomich[21] gave a syntactical property equivalent to the properties of Theorem 4.5.4. We show the equivalence between his property and the property 7 of Theorem 4.5.4 by an algebraic method.

Definition 4.6.5 (Khomich[21])**.** *Let an $\mathcal{S}$-formula $Q = Q(q_1, \dots, q_n)$ and $T_1, \dots, T_n$ are sets of propositional variables. We fix a method of constructing from Q a formula $\widetilde{Q}$ satisfying the following conditions :*

1. *$\mathbf{H} \vdash Q(\bigwedge T_1, \dots, \bigwedge T_m) \to \widetilde{Q}$;*
2. *$\mathbf{H} \vdash \widetilde{Q} \to Q(\bigwedge T_1, \dots, \bigwedge T_m)$;*
3. *$\widetilde{Q}$ is a conjunction of formulas constructed from variables in the list $T_1 \cup \cdots \cup T_n$ with the help of $\to$ and the logical symbols occurring in Q and different from $\wedge$* [3]*.*

We say that $\mathbf{H} + Q_1 + \cdots + Q_n$ possesses the property $\mathfrak{C}_{\mathcal{S}}$ if, for any i such that Q_i is an $\mathcal{S}$-formula, every conjunct A of $\widetilde{Q_i}$ satisfies $\mathbf{H}_{\mathcal{S}} + \{Q_1, \dots, Q_n\} \vdash A$.

Lemma 4.6.6. *Let $\mathbf{M}$ be an $\mathcal{S}$-algebra and $Q = Q(q_1, \dots, q_m)$ be an $\mathcal{S}$-formula. The following are equivalent:*

[3] $\widetilde{Q}$ can be constructed by Proposition 3.4.19.

1. Q is valid on $C(\mathbf{M})$;

2. if $T_1, \ldots, T_m$ are finite sets of propositional variables, $v(Q(\bigwedge T_1, \ldots, \bigwedge T_m)) = 1$ for every valuation v on $C(\mathbf{M})$ satisfying $v(p) \in \{[\{x\}] \mid x \in \mathbf{M}\}$ for every propositional variable p occurring in $T_1, \ldots, T_m$.

Proof. ($1 \Longrightarrow 2$) The formula $Q(\bigwedge T_1, \ldots, \bigwedge T_m)$ is a substitution instance of Q. Hence $Q(\bigwedge T_1, \ldots, \bigwedge T_m)$ is valid on $C(\mathbf{M})$ by assumption. Therefore we obtain the condition 2 of the lemma.

($2 \Longrightarrow 1$) Let v be an arbitrary valuation on $C(\mathbf{M})$. By definition of $C(\mathbf{M})$, we can assume $v(q_i) = [S_i]$, where $S_i = \{s^i_1, \ldots, s^i_{k_i}\} \subseteq \mathbf{M}$ is a finite subset for every $i = 1, \ldots, m$. Therefore we have

$$\begin{aligned} & v(Q(q_1, \ldots, q_m)) \\ = & Q(v(q_1), \ldots, v(q_m)) \\ = & Q([S_1], \ldots, [S_m]) \\ = & Q(\bigwedge_{s \in S_1} [\{s\}], \ldots, \bigwedge_{s \in S_m} [\{s\}]). \end{aligned}$$

On the other hand, for each i, pick the set $T_i = \{t^i_1 \ldots, t^i_{k_i}\}$ of distinct propositional variables and define the valuation w on $C(\mathbf{M})$ by $w(t^i_j) = [\{s^i_j\}]$. Thus we have $w(Q(\bigwedge T_1, \ldots, \bigwedge T_m)) = Q(\bigwedge_{s \in S_1} [\{s\}], \ldots, \bigwedge_{s \in S_m} [\{s\}]) = 1$ since $v(p) \in \{[\{x\}] \mid x \in \mathbf{M}\}$ for every propositional variable p occurring in $T_1, \ldots, T_m$. Consequently, $v(Q(q_1, \ldots, q_m)) = 1$. $\square$

Theorem 4.6.7 (Khomich[21]). *Let $Q_1, \ldots, Q_n$ be $\mathcal{S}$-formulas. Every $Q = Q_1, \ldots, Q_n$ possesses the property $\mathfrak{C}_\mathcal{S}$ if and only if $\mathbf{H}_\mathcal{S} + Q_1 + \cdots + Q_n = (\mathbf{H} + Q_1 + \cdots + Q_n)_\mathcal{S}$.*

Proof. ($\Longleftarrow$) Let $Q \in \{Q_1, \ldots, Q_n\}$ and A is a conjunct of $\widetilde{Q}$. Thus, we obtain

$$\begin{aligned} & \mathbf{H} + Q_1 + \cdots + Q_n \vdash Q \\ \Longleftrightarrow & \mathbf{H} + Q_1 + \cdots + Q_n \vdash \widetilde{Q} \\ \Longrightarrow & \mathbf{H} + Q_1 + \cdots + Q_n \vdash A \\ \Longrightarrow & \mathbf{H}_\mathcal{S} + Q_1 + \cdots + Q_n \vdash A, \end{aligned}$$

since A is an $\mathcal{S}$-formula. Therefore, Q possesses the property $\mathfrak{C}_\mathcal{S}$.

($\Longrightarrow$) Let $\mathbf{M}$ be an $\mathbf{H}_\mathcal{S} + Q_1 + \cdots + Q_n$-algebra. We show that every $Q(q_1, \ldots, q_m) \in \{Q_1, \ldots, Q_n\}$ is valid on $C(\mathbf{M})$.

Let $T_1, \ldots, T_m$ be finite sets of propositional variables and v be a valuation on $C(\mathbf{M})$ satisfying that, for every propositional variable p occurring in $T_1, \ldots, T_m$, there exists $x \in \mathbf{M}$ such that $v(p) = [\{x\}]$. It is sufficient to show $v(Q(\bigwedge T_1, \ldots, \bigwedge T_m)) = 1$ by Lemma 4.6.6.

We have the formula $\widetilde{Q}$ from $Q(\bigwedge T_1, \ldots, \bigwedge T_m)$ which is constructed by the method of Definition 4.6.5. Let $\widetilde{Q} = A_1 \wedge \cdots \wedge A_r$, where $A_1, \ldots, A_r$ are $\mathcal{S}$-formulas. Thus $\mathbf{H}_\mathcal{S} + Q_1 + \cdots + Q_n \vdash A$ for every $A \in \{A_1, \ldots, A_r\}$ since $\mathbf{H} + Q_1 + \cdots + Q_n$ possesses the property

$\mathfrak{C}_{\mathcal{S}}$. Hence A is valid on $\mathbf{M}$ since $\mathbf{M}$ is an $\mathbf{H}_{\mathcal{S}} + Q_1 + \cdots + Q_n$-algebra. Thus A is valid also on $\mathcal{S}$-subalgebra $\{[\{x\}] \mid x \in \mathbf{M}\}$ of $C(\mathbf{M})$ since $\{[\{x\}] \mid x \in \mathbf{M}\}$ is $\mathcal{S}$-isomorphic to $\mathbf{M}$. Therefore, $v(A) = 1$ on $C(\mathbf{M})$ since $v(p) \in \{[\{x\}] \mid x \in \mathbf{M}\}$ for every propositional variable p occurring in A and A is an $\mathcal{S}$-formula. It implies

$$
\begin{aligned}
& v(Q(\bigwedge T_1, \ldots, \bigwedge T_m)) \\
&= v(\widetilde{Q}) = v(A_1 \wedge \cdots \wedge A_r) \\
&= v(A_1) \wedge \cdots \wedge v(A_r) \\
&= 1 \wedge \cdots \wedge 1 = 1.
\end{aligned}
$$

Consequently, $C(\mathbf{M})$ is an $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + Q_1 + \cdots + Q_n$-algebra. Therefore, we obtain $\mathbf{H}_{\mathcal{S}} + Q_1 + \cdots + Q_n = (\mathbf{H} + Q_1 + \cdots + Q_n)_{\mathcal{S}}$ by Theorem 4.5.4 (condition 1 $\Longleftrightarrow$ 7). $\square$

4.7 Conclusion

For given $\mathcal{S}$, $\mathcal{S}'$ and an axiomatization of an intermediate $\mathcal{S}$-logic $\mathbf{H}_{\mathcal{S}} + \Gamma$, we considered a general characterization for the conservativity condition (Theorem 4.5.1). Especially, for the case $\mathcal{S} \cup \{\wedge\} \subseteq \mathcal{S}'$, we proved that Horn's construction ([12]) helps proving the conservativity condition. In other words, the conservativity condition of $\mathbf{H}_{\mathcal{S}} + \Gamma$ (i.e, whether $\mathbf{H}_{\mathcal{S}'} + \Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}} + \Gamma$) depends on the class $\{C(\mathbf{M}) \mid \mathbf{M}$ is $\mathbf{H} + \Gamma$-algebra$\}$ of $\mathcal{S}$-algebras.

Also, as applications of this characterization, we gave algebraic proofs for some Khomich's theorems ([19, 21, 22]) (Theorem 4.6.2 and 4.6.7).

Question. *Let $\mathcal{S} \subseteq \mathcal{S}'$, $\wedge \notin \mathcal{S}'$ and Γ be a set of $\mathcal{S}$-formulas. What kind of class of algebras does the conservativity condition of $\mathbf{H}_{\mathcal{S}} + \Gamma$ (i.e, whether $\mathbf{H}_{\mathcal{S}'} + \Gamma$ is a conservative extension of $\mathbf{H}_{\mathcal{S}} + \Gamma$) depend on?*

Chapter 5

Separability of the Gabbay-de Jongh logics

In this chapter, we give a separable axiomatization of the Gabbay-de Jongh logics $\mathbf{D}_m$ ($m \geq 2$) [10], i.e., we show that the Gabbay-de Jongh logics satisfies the separable condition. The proofs are made as an application of the result of Chapter 4. In particular, the separable axiomatization which we give is constructed as a Jankov's characteristic formula. We also show that (a variant of) the standard axiomatization of the Gabbay-de Jongh logics is not separable. This chapter is based on [32].

5.1 Definitions and preliminaries

Gabbay-de Jongh logics $\mathbf{D}_m$ ($m \geq 2$) are the logics characterized by all Kripke frames each of that forms a finite tree whose points do not have more than m immediate successors (e.g., if $m = 2$, the logic is the set of formulas which is valid in all Kripke frames each of that forms a finite binary tree). They gave axiomatizations of $\mathbf{D}_m$ as follows.

Theorem 5.1.1 (Gabbay and de Jongh[10], Section 4). $\mathbf{H} + A_m$ *is an axiomatization of* $\mathbf{D}_m$ ($m \geq 2$)*, where*

$$A_m = \bigwedge_{i=0}^{m}((p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j) \to \bigvee_{i=0}^{m} p_i.$$

$\mathbf{H} + A_m$ is not separable since it is not normal. However, we can obtain a normal axiom A'_m by the well-known method (the definition is in Section 2), i.e.,

$$A'_m = \{(p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j \mid i = 0, \ldots, m\} \to \bigvee_{i=0}^{m} p_i.$$

Furthermore, the Khomich's result can not apply to $\mathbf{D}_m$ since disjunction occurs in A'_m.

5.2 Another axiomatization of Gabbay-de Jongh logics

Let $\mathbf{T}_m = \{1,\ \omega,\ a_0, \ldots, a_m\}$ be the $\{\rightarrow,\ \vee\}$-algebra defined by the following diagram:

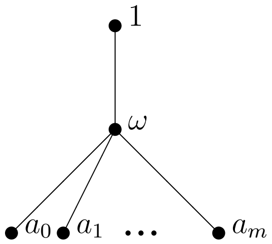


figure-1 $\mathbf{T}_m$

we define $x \rightarrow y = \begin{cases} 1 & (x \leq y) \\ y & (\text{otherwise}), \end{cases}$ $\quad x \vee y = \begin{cases} 1 & (x = 1 \text{ or } y = 1) \\ a_i & (x = y = a_i) \\ \omega & (\text{otherwise}). \end{cases}$

We show that $\mathbf{H} + X_{\mathbf{T}_m}^{\{\rightarrow,\ \vee\}}$ is an axiomatization of $\mathbf{D}_m$, i.e., $\mathbf{H} + X_{\mathbf{T}_m}^{\{\rightarrow,\ \vee\}} = \mathbf{H} + A'_m$.

Lemma 5.2.1. *Let $\mathcal{S} \supseteq \{\rightarrow,\ \wedge,\ \vee\}$. For every $\mathcal{S}$-algebra $\mathbf{M}$, the following are equivalent.*

1. *$X_{\mathbf{T}_m}^{\{\rightarrow,\ \vee\}}$ is refutable in $\mathbf{M}$.*

2. *$\mathbf{T}_m$ is $\{\rightarrow,\ \vee\}$-embeddable in $w(\mathbf{M})$, where $w(\mathbf{M})$ is an $\mathcal{S}$-homomorphic image of $\mathbf{M}$.*

3. *A'_m is refutable in $\mathbf{M}$.*

Proof. We posted the idea behind the proof of ($2 \Longleftrightarrow 3$) on remarks (the end of this section) since our proof is by long and boring calculations.

($1 \Longleftrightarrow 2$) Follows from Theorem 4.1.5.

($2 \Longrightarrow 3$) Let $h : \mathbf{T}_m \longrightarrow w(\mathbf{M})$ be an embedding. Define valuation v on $\mathbf{M}$ by $v(p_i) =$

$h(\bigwedge_{j\neq i} a_j)$. Then

$$
\begin{aligned}
& v(A'_m) \\
&= \{(v(p_i) \to \bigvee_{j\neq i} v(p_j)) \to \bigvee_{j\neq i} v(p_j) \mid i = 0, \ldots, m\} \to \bigvee_{i=0}^{m} v(p_i). \\
&= \{(h(\bigwedge_{j\neq i} a_j) \to \bigvee_{j\neq i} h(\bigwedge_{k\neq j} a_k)) \to \bigvee_{j\neq i} h(\bigwedge_{k\neq j} a_k) \mid i = 0, \ldots, m\} \to \bigvee_{i=0}^{m} h(\bigwedge_{j\neq i} a_j). \\
&= \{(h(\bigwedge_{j\neq i} a_j) \to h(\bigvee_{j\neq i}\bigwedge_{k\neq j} a_k)) \to h(\bigvee_{j\neq i}\bigwedge_{k\neq j} a_k) \mid i = 0, \ldots, m\} \to h(\bigvee_{i=0}^{m}\bigwedge_{j\neq i} a_j). \\
&= \{(h(\bigwedge_{j\neq i} a_j) \to h(a_i)) \to h(a_i) \mid i = 0, \ldots, m\} \to h(\omega). \\
&= \{(h(a_i) \to h(a_i) \mid i = 0, \ldots, m\} \to h(\omega). \\
&= h(\omega) \neq 1.
\end{aligned}
$$

Therefore, $\mathbf{M}$ also refutes A'_m.

($3 \Longrightarrow 2$) Assume that A'_m is refutable in $\mathbf{M}$. By Theorem 4.1.4, there exists a subdirectly irreducible $\mathcal{S}$-algebra $\mathbf{N}$ which is an $\mathcal{S}$-homomorphic image of $\mathbf{M}$ such that there exists a valuation h on $\mathbf{N}$ satisfying $h(A'_m) = \omega$. By Proposition 3.4.33, we have

$$
h((p_i \to \bigvee_{j\neq i} p_j) \to \bigvee_{j\neq i} p_j) = 1 \ (i = 0, \ldots, m);
$$
$$
h(\bigvee_{i=0}^{m} p_i) = \omega.
$$

Then we have

$$
h(p_i) \to \bigvee_{j\neq i} h(p_j) \leq \bigvee_{j\neq i} h(p_j) \ (i = 0, \ldots, m);
$$
$$
\bigvee_{i=0}^{m} h(p_i) = \omega.
$$

Furthermore, we have

$$
h(p_i) \to \bigvee_{j\neq i} h(p_j) = \bigvee_{j\neq i} h(p_j) \ (i = 0, \ldots, m),
$$

since $h(p_i) \to \bigvee_{j\neq i} h(p_j) \geq \bigvee_{j\neq i} h(p_j)$ always holds. Put $x_k = \bigvee_{j\neq k} h(p_j)$. We show that $\{x_k \mid k = 1, \ldots, m\} \cup \{\omega,\ 1\}$ is $\{\to,\ \vee\}$-isomorphic to $\mathbf{T}_m$. First, we have $x_k \to \omega = 1$ since $x_k = \bigvee_{j\neq k} h(p_j) \leq h(\bigvee_{i=0}^{m} p_i) = \omega$. Second, we obtain $\omega \to x_k = x_k$ by the following calculation:

$$\begin{aligned}
\omega \to x_k &= h(\bigvee_{i=0}^{m} p_i) \to \bigvee_{j\neq k} h(p_j) \\
&= (h(p_k) \vee \bigvee_{j\neq k} h(p_j)) \to \bigvee_{j\neq k} h(p_j) \\
&= h(p_k) \to \bigvee_{j\neq k} h(p_j) \\
&= \bigvee_{j\neq k} h(p_j) = x_k.
\end{aligned}$$

Third, $x_k \to x_l = x_l$ can be shown by similar calculations to above: $x_k \to x_l = h(p_l) \to \bigvee_{j\neq l} h(p_j) = \bigvee_{j\neq l} h(p_j) = x_l$. Finally, we have $x_k \vee x_l = \omega$ $(k \neq l)$ since $x_k \vee x_l = \bigvee_{j\neq k} h(p_j) \vee \bigvee_{j\neq l} h(p_j) = h(\bigvee_{i=0}^{m} p_i) = \omega$. Consequently, $\mathbf{T}_m$ is $\{\to,\ \vee\}$-embeddable in $\mathbf{N}$ ($\mathbf{N}$ is an $\mathcal{S}$-homomorphic image of $\mathbf{M}$). $\square$

Theorem 3.4.27 and Lemma 5.2.1 show the following theorem.

Theorem 5.2.2. $\mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$ *is an axiomatization of* $\mathbf{D}_m$.

Remark (The idea behind the proof of Lemma 5.2.1)**.** *Let* $\mathbf{2}$ *be the two-valued Boolean algebra and* $\mathbf{M}_m$ *be the* $\{\to,\ \vee\}$*-algebra defined by* $\mathbf{M}_m = (\mathbf{2}^{m+1} + \mathbf{2})_{\{\to,\ \vee\}} - \{0\}$*, where* $(\mathbf{2}^{m+1} + \mathbf{2})_{\{\to,\ \vee\}}$ *is the* $\{\to,\ \vee\}$*-reduct of* $\mathbf{2}^{m+1} + \mathbf{2}$ *(see figure-2).* $\mathbf{M}_m$ *refutes* $X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$ *since there exists a* $\{\to,\ \vee\}$*-embedding* $h : \mathbf{T}_m \longrightarrow \mathbf{M}_m$ *which is defined by* $h(a_i) = c_i$*, where* $c_0, \ldots, c_m$ *are the immediate predecessors of* ω*. On the other hand,* $\mathbf{M}_m$ *also refutes* A'_m *by the* ω*-refutation* v *of* A'_m *on* $\mathbf{M}_m$ *which is defined by* $v(p_i) = b_i$*, where* $b_0, \ldots, b_m$ *are the minimal elements of* $\mathbf{M}_m$*. Then we can easily verify* $b_i = \bigvee_{j\neq i} c_j$*. However, to represent* c_i *by* $\{b_0, \ldots, b_m\}$*, we need the conjunction (therefore, in the Lemma 5.2.1, we put the assumption* $\{\to,\ \wedge,\ \vee\} \subseteq \mathcal{S}$*). On the other hand, in the* $\{\to,\ \wedge,\ \vee\}$*-reduct of* $\mathbf{2}^{m+1} + \mathbf{2}$*, we have* $c_i = \bigwedge_{j\neq i} b_j$*. In the proof of* $((2) \Longleftrightarrow (3))$*, we verify that this relation between* b_i *and* c_i *holds on any* $\{\to,\ \wedge,\ \vee\}$*-algebra by calculation.*

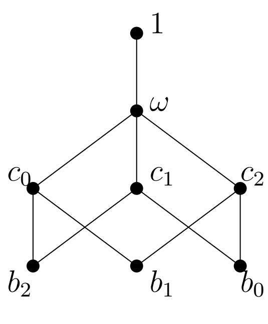


figure-2 $\mathbf{M}_2$

5.3 Separability of $\mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$

$\mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$ is normal since $X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$ is a $\{\to,\ \vee\}$-formula. Therefore, in this section, we show the $\mathcal{S}$-completeness, that is:

$$\mathbf{H}_{\mathcal{S}} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}} = (\mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}})_{\mathcal{S}} \text{ for every } \mathcal{S}.$$

First, we show for the case $\vee \in \mathcal{S}$, i.e., $\mathcal{S} = \{\to,\ \vee\},\ \{\to,\ \vee,\ \neg\}$. Our goal is:

$$\mathbf{H}_{\mathcal{S}} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}} = (\mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}})_{\mathcal{S}} \text{ if } \vee \in \mathcal{S}.$$

Let $\mathbf{M}$ be a finite subdirectly irreducible $\mathcal{S}$-algebra. We consider the following condition $(\phi^\star)$ for $\mathbf{M}$:

$(\phi^\star)$ for every $\mathcal{S}$-algebra $\mathbf{N}$ and $\mathcal{S}$-homomorphism h such that $h(\mathbf{N})$ is Gödelian, if there exists a $\mathcal{S}$-embedding $g : \mathbf{M} \longrightarrow C(h(\mathbf{N}))$ satisfying $g(\omega) = \omega$, $\mathbf{M}$ is $\mathcal{S}$-embeddable in $h(\mathbf{N})$.

We show that $(\phi^\star)$ implies $\mathbf{H}_{\mathcal{S}} + X_{\mathbf{M}}^{\mathcal{S}} = (\mathbf{H} + X_{\mathbf{M}}^{\mathcal{S}})_{\mathcal{S}}$ for every finite subdirectly irreducible $\mathcal{S}$-algebra $\mathbf{M}$ and $\mathbf{T}_m$ satisfies the condition $(\phi^\star)$.

First, we show $\mathbf{T}_m$ satisfies the condition $(\phi^\star)$.

Lemma 5.3.1. *Let $\mathbf{N}$ be a subdirectly irreducible $\mathcal{S}$-algebra. If there exists a $\{\to,\ \vee\}$-embedding $g : \mathbf{T}_m \longrightarrow C(\mathbf{N})$ satisfying $g(\omega) = \omega$, $\mathbf{T}_m$ is $\{\to,\ \vee\}$-embeddable in $\mathbf{N}$.*

Proof. By the assumption, we have

$$\begin{aligned} &g(1) = 1; \\ &g(\omega) = \omega; \\ &g(a_i) = x_0^i \wedge \cdots \wedge x_{n_i}^i \ (i = 0, \ldots, m), \end{aligned}$$

for some $x_0^i, \ldots, x_{n_i}^i \in \mathbf{N}$. We can assume $x_0^i, \ldots, x_{n_i}^i$ satisfy that for every $k = 0, \ldots, n_i$, $\bigwedge_{r=1}^{n_i} x_r^i \neq \bigwedge_{r \neq k} x_r^i$.

Now we put $x_i = \bigcup_{s \neq i}\{x_t^s \mid t = 0, \ldots, n_s\} \to x_0^i$ $(i = 0, \ldots, m)$. We have $x_i < \omega$ for every i since g is injective and Proposition 3.4.33. We show $\{1,\ \omega,\ x_0, \ldots, x_m\} \subseteq \mathbf{N}$ is $\{\to,\ \vee\}$-isomorphic to $\mathbf{T}_m$. Since $x_0^i \in \{x_t^s \mid s \neq j,\ t = 1, \ldots, n_s\}$, we have

$$\begin{aligned} &x_i \to x_j \\ = &(\bigcup_{s \neq i}\{x_t^s \mid t = 0, \ldots, n_s\} \to x_0^i) \to \bigcup_{s \neq j}\{x_t^s \mid t = 0, \ldots, n_s\} \to x_0^j \\ = &\bigcup_{s \neq j}\{x_t^s \mid t = 0, \ldots, n_s\} \to x_0^j \\ = &x_j. \end{aligned}$$

Next we show that $x_i \vee x_j = \omega$ if $i \neq j$. $x_i \vee x_j \geq \omega$ can be verified by the following calculation:

$$\begin{aligned} & x_i \vee x_j \\ \geq\ & x_0^i \vee x_0^j \\ \geq\ & g(a_i) \vee g(a_j) \\ =\ & g(a_i \vee a_j) \\ =\ & g(\omega) \\ =\ & \omega. \end{aligned}$$

Conversely, if $x_i = 1$, the following calculation holds on $C(\mathbf{N})$:

$$\begin{aligned} & x_0^i \wedge \cdots \wedge x_{n_i}^i \\ =\ & g(a_i) \\ =\ & \{g(a_j) \mid j \neq i\} \to g(a_i) \\ =\ & \bigcup_{s \neq i} \{x_t^s \mid t = 0, \ldots, n_s\} \to (x_0^i \wedge \cdots \wedge x_{n_i}^i) \\ =\ & (\bigcup_{s \neq i} \{x_t^s \mid t = 0, \ldots, n_s\} \to x_0^i) \wedge (\bigcup_{s \neq i} \{x_t^s \mid t = 0, \ldots, n_s\} \to (x_1^i \wedge \cdots \wedge x_{n_i}^i)) \\ =\ & x_i \wedge (\bigcup_{s \neq i} \{x_t^s \mid t = 0, \ldots, n_s\} \to (x_1^i \wedge \cdots \wedge x_{n_i}^i)) \\ =\ & \bigcup_{s \neq i} \{x_t^s \mid t = 0, \ldots, n_s\} \to (x_1^i \wedge \cdots \wedge x_{n_i}^i) \\ \geq\ & x_1^i \wedge \cdots \wedge x_{n_i}^i. \end{aligned}$$

Thus, we have $\bigwedge_{r=1}^{n_i} x_r^i = \bigwedge_{r \neq 1} x_r^i$ which contradicts with the incomparability as mentioned above. Consequently, we have $x_i \leq \omega$ which implies $x_i \vee x_j \leq \omega$. We proved that $x_i \vee x_j = \omega$.

Consequently, $\mathbf{T}_m$ is $\{\to,\ \vee\}$-isomorphic to $\{1,\ \omega,\ a_0, \ldots, a_m\} \subseteq \mathbf{N}$. Therefore, $\mathbf{T}_m$ is $\{\to,\ \vee\}$-embeddable in $\mathbf{N}$. □

Corollary 5.3.2. $\mathbf{T}_m$ *satisfies the condition* $(\phi^\star)$*.*

Next, we show that the condition $(\phi^\star)$ implies the following condition (ϕ) for every finite subdirectly irreducible $\mathcal{S}$-algebra $\mathbf{M}$:

(ϕ) for every $\mathcal{S}$-algebra $\mathbf{N}$ and $\mathcal{S} \cup \{\wedge\}$-homomorphism h, if $\mathbf{M}$ is $\mathcal{S}$-embeddable in $h(C(\mathbf{N}))$, $\mathbf{M}$ is also $\mathcal{S}$-embeddable in $h(\mathbf{N})$.

Theorem 5.3.3. *Let* $\mathbf{M}$ *be an* $\mathcal{S}$*-algebra,* $\mathbf{N}$ *be an* $\mathcal{S} \cup \{\wedge\}$*-algebra and* $h : C(\mathbf{M}) \longrightarrow \mathbf{N}$ *be an* $\mathcal{S} \cup \{\wedge\}$*-homomorphism. Then the* $\mathcal{S}$*-homomorphism* $\bar{h} : \mathbf{M} \longrightarrow \mathbf{N}$ *defined by* $\bar{h}(x) = h(x)$ *satisfies the following:*

1. $h(C(\mathbf{M}))$ *and* $C(\bar{h}(\mathbf{M}))$ *are* $\mathcal{S} \cup \{\wedge\}$*-algebras;*

2. $h(C(\mathbf{M}))$ *is* $\mathcal{S} \cup \{\wedge\}$*-isomorphic to* $C(\bar{h}(\mathbf{M}))$*.*

Proof. Put $\bar{\rightarrow}$, $\bar{\wedge}$, $\bar{\vee}$, $\bar{\neg}$ for the functions $\rightarrow$, $\wedge$, $\vee$, $\neg$ on $C(\bar{h}(\mathbf{M}))$ respectively. Let $x_1, \dots, x_m,\ y_1, \dots, y_n \in \mathbf{M}$. We show that the map $g : h(C(\mathbf{M})) \longrightarrow C(\bar{h}(\mathbf{M}))$ which is defined by $g(h(x_1 \wedge \cdots \wedge x_m)) = \bar{h}(x_1) \bar{\wedge} \cdots \bar{\wedge} \bar{h}(x_m)$ is the $\mathcal{S} \wedge \{\wedge\}$-isomorphism.

Assume $h(x_1 \wedge \cdots \wedge x_m) = h(y_1 \wedge \cdots \wedge y_n)$. Then, for every $i = 1, \dots, n$, we have

$$\begin{aligned}
&\bar{h}(x_1) \bar{\wedge} \cdots \bar{\wedge} \bar{h}(x_m) \bar{\rightarrow} \bar{h}(y_i) \\
&= \bar{h}(x_1) \bar{\rightarrow} \cdots \bar{\rightarrow} \bar{h}(x_m) \bar{\rightarrow} \bar{h}(y_i) \\
&= \bar{h}(x_1 \rightarrow \cdots \rightarrow x_m \rightarrow y_i) \\
&= h(x_1 \rightarrow \cdots \rightarrow x_m \rightarrow y_i) \\
&= h(x_1 \wedge \cdots \wedge x_m) \rightarrow h(y_i) \\
&\geq h(x_1 \wedge \cdots \wedge x_m) \rightarrow h(y_1 \wedge \cdots \wedge y_n) = 1.
\end{aligned}$$

Obviously, this calculation holds in the case of $\bar{h}(y_1) \bar{\wedge} \cdots \bar{\wedge} \bar{h}(y_n) \bar{\rightarrow} \bar{h}(x_j)$ $(j = 1, \dots, m)$. Therefore, $g(h(x_1 \wedge \cdots \wedge x_m)) = g(h(y_1 \wedge \cdots \wedge y_n))$ holds. We proved the well-definedness of g.

Similarly, we can prove that g is injective. For every $i = 1, \dots, n$, we have

$$\begin{aligned}
&\bar{h}(x_1) \wedge \cdots \wedge \bar{h}(x_m) = \bar{h}(y_1) \wedge \cdots \wedge \bar{h}(y_n) \\
\Longrightarrow\ &\bar{h}(x_1) \bar{\rightarrow} \cdots \bar{\rightarrow} \bar{h}(x_m) \bar{\rightarrow} \bar{h}(y_i) = 1_{\bar{h}(\mathbf{M})} \\
\Longleftrightarrow\ &\bar{h}(x_1 \rightarrow \cdots \rightarrow x_m \rightarrow y_i) = 1_{\bar{h}(\mathbf{M})} \\
\Longleftrightarrow\ &h(x_1 \rightarrow \cdots \rightarrow x_m \rightarrow y_i) = 1_{h(\mathbf{M})} = 1_{C(h(\mathbf{M}))} \\
\Longleftrightarrow\ &h(x_1) \rightarrow \cdots \rightarrow h(x_m) \rightarrow h(y_i) = 1_{C(h(\mathbf{M}))} \\
\Longleftrightarrow\ &h(x_1) \wedge \cdots \wedge h(x_m) \rightarrow h(y_i) = 1_{C(h(\mathbf{M}))}.
\end{aligned}$$

Therefore, we obtain $h(x_1) \wedge \cdots \wedge h(x_m) \leq h(y_1) \wedge \cdots \wedge h(y_n)$. Obviously, we also have $h(y_1) \wedge \cdots \wedge h(y_n) \leq h(x_1) \wedge \cdots \wedge h(x_m)$. Consequently, we have $h(x_1 \wedge \cdots \wedge x_m) = h(y_1 \wedge \cdots \wedge y_n)$. We proved g is injective.

By the definition of g, it is obvious that g is a surjection.

We can verify that g preserves $\rightarrow$ (g is an $\{\rightarrow\}$-homomorphism) by the similar calculations above.

We obtain that g preserves $\vee$ (if $\vee \in \mathcal{S}$) by the following calculation:

$$\begin{aligned}
&g(h(x_1 \wedge \cdots \wedge x_m) \vee h(y_1 \wedge \cdots \wedge y_n)) \\
&= g(h(\bigwedge_{1 \leq i \leq m,\ 1 \leq j \leq n} (x_i \vee y_j))) \\
&= \overline{\bigwedge}_{1 \leq i \leq m,\ 1 \leq j \leq n} \bar{h}(x_i \vee y_j) \\
&= \overline{\bigwedge}_{1 \leq i \leq m,\ 1 \leq j \leq n} (\bar{h}(x_i)\ \bar{\vee}\ \bar{h}(y_j)) \\
&= (\bar{h}(x_1) \bar{\wedge} \cdots \bar{\wedge} \bar{h}(x_m))\ \bar{\vee}\ (\bar{h}(y_1) \bar{\wedge} \cdots \bar{\wedge} \bar{h}(y_n)) \\
&= g(h(x_1 \wedge \cdots \wedge x_m))\ \bar{\vee}\ g(h(y_1 \wedge \cdots \wedge y_n)).
\end{aligned}$$

We can prove that g preserves $\wedge$ (if $\wedge \in \mathcal{S}$) similarly. g preserves $\neg$ (if $\neg \in \mathcal{S}$) since g preserves $\rightarrow$ and $g(0) = g(h(0)) = \bar{h}(0) = 0$. $\square$

Lemma 5.3.4. *Let $\mathbf{M}$ be finite subdirectly irreducible $\mathcal{S}$-algebra, $\mathbf{N}$ be an (arbitrary) $\mathcal{S}$-algebra and h be an $\mathcal{S} \cup \{\wedge\}$-homomorphism. If there exists an $\mathcal{S}$-embedding $f : \mathbf{M} \longrightarrow h(C(\mathbf{N}))$, there exist an $\mathcal{S}$-homomorphism h' and an $\mathcal{S}$-embedding $f' : \mathbf{M} \longrightarrow C(h'(\mathbf{N}))$ satisfying $f'(\omega) = \omega$.*

Proof. Let F be a maximal filter of $h(C(\mathbf{N}))$ such that $f(\omega) \notin F$ (such F is guaranteed to exist by Zorn's Lemma) and $\pi : h(C(\mathbf{N})) \longrightarrow \pi(h(C(\mathbf{N}))$ $(= h(C(\mathbf{N}))/F)$ be a canonical projection defined by F, i.e., π is defined by $\pi(x) = x/F$. By Proposition 3.4.34, $\pi \circ f : \mathbf{M} \longrightarrow \pi(h(C(\mathbf{N})))$ is an $\mathcal{S} \cup \{\wedge\}$-homomorphism satisfying that, $\pi(h(C(\mathbf{N}))$ is subdirectly irreducible and $\pi \circ f(\omega) = \omega$.

Therefore, we show $\pi \circ f$ is injective. Assume $x,\ y \in \mathbf{M}$ satisfy $x \neq y$ and $(\pi \circ f)(x) = (\pi \circ f)(y)$. Without loss of generality, we can assume $x \not\leq y$ which implies $x \rightarrow y \leq \omega$. By the definition of π, $(\pi \circ f)(x) = (\pi \circ f)(y)$ implies $f(x) \rightarrow f(y) \in F$. Furthermore, $x \rightarrow y \leq \omega$ implies $f(x \rightarrow y) \leq f(\omega)$. Therefore, we have $f(x) \rightarrow f(y) \leq f(\omega)$ which implies $f(\omega) \in F$, a contradiction. Consequently, $\pi \circ f$ is an embedding.

By Theorem 5.3.3, we have $\mathcal{S} \cup \{\wedge\}$-isomorphism $i : \pi(h(C(\mathbf{N}))) \longrightarrow C(\overline{\pi \circ h}(\mathbf{N}))$, where $\overline{\pi \circ h}$ is the $\mathcal{S}$-homomorphism defined by $\pi \circ h(x) = \overline{\pi \circ h}(x)$ for all $x \in \mathbf{N}$.

Therefore , we obtain the $\mathcal{S}$-embedding $i \circ \pi \circ f : \mathbf{M} \longrightarrow C(\overline{\pi \circ h}(\mathbf{N}))$ (i.e., $f' = i \circ \pi \circ f$, $h' = \overline{\pi \circ h}$). $\square$

Corollary 5.3.5. *The condition $(\phi^{\star})$ implies the condition (ϕ) for every finite subdirectly irreducible $\mathcal{S}$-algebra $\mathbf{M}$.*

Proof. Since Lemma 5.3.4 shows that the assumption of the condition (ϕ) implies the one of the condition $(\phi^{\star})$. $\square$

Finally, we show the condition (ϕ) implies $\mathbf{H}_{\mathcal{S}} + X^{\mathcal{S}}_{\mathbf{M}} = (\mathbf{H} + X^{\mathcal{S}}_{\mathbf{M}})_{\mathcal{S}}$ for every finite subdirectly irreducible $\mathcal{S}$-algebra $\mathbf{M}$.

Lemma 5.3.6. *The condition (ϕ) implies $\mathbf{H}_{\mathcal{S}} + X^{\mathcal{S}}_{\mathbf{M}} = (\mathbf{H} + X^{\mathcal{S}}_{\mathbf{M}})_{\mathcal{S}}$ for every finite subdirectly irreducible $\mathcal{S}$-algebra $\mathbf{M}$.*

Proof. By Theorem 4.1.5, (ϕ) implies the following condition (ψ) for every finite subdirectly irreducible $\mathcal{S}$-algebra $\mathbf{M}$ (consider the contraposition of (ψ)):

(ψ) if $\mathbf{N}$ is an $\mathbf{H}_{\mathcal{S}} + X^{\mathcal{S}}_{\mathbf{M}}$-algebra, $C(\mathbf{N})$ is an $\mathbf{H}_{\mathcal{S} \cup \{\wedge\}} + X^{\mathcal{S}}_{\mathbf{M}}$-algebra.

Therefore, by Theorem 4.5.3, the lemma is proved.

$\square$

By Corollary 5.3.2, 5.3.5 and Lemma 5.3.6, the $\mathcal{S}$-completeness for the case $\vee \in \mathcal{S}$ is proved.

Theorem 5.3.7. *If $\vee \in \mathcal{S}$, $\mathbf{H}_{\mathcal{S}} + X^{\{\rightarrow,\ \vee\}}_{\mathbf{T}_m} = (\mathbf{H} + X^{\{\rightarrow,\ \vee\}}_{\mathbf{T}_m})_{\mathcal{S}}$ holds.*

Next we show for the case $\vee \notin \mathcal{S}$. If $\vee \notin \mathcal{S}$, $X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$ is not an $\mathcal{S}$-formula but a $\{\to,\ \vee\}$-formula. Hence we have the following lemma.

Lemma 5.3.8. *If $\vee \notin \mathcal{S}$, $\mathbf{H}_\mathcal{S} + \{X_{\mathbf{T}_m}^{\{\to,\ \vee\}}\}_\mathcal{S} = \mathbf{H}_\mathcal{S}$ holds.*

On the other hand, Segerberg[28] proved the following theorem.

Theorem 5.3.9 (Segerberg[28])**.** *If $\vee \notin \mathcal{S}$, $(\mathbf{D}_m)_\mathcal{S} = \mathbf{H}_\mathcal{S}$ holds.*

Moreover, Theorem 5.2.2 shows $\mathbf{D}_m = \mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$. Consequently, the case $\vee \notin \mathcal{S}$ is proved.

Theorem 5.3.10. *If $\vee \notin \mathcal{S}$, $\mathbf{H}_\mathcal{S} + \{X_{\mathbf{T}_m}^{\{\to,\ \vee\}}\}_\mathcal{S} = (\mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}})_\mathcal{S}$ holds.*

Proof. $\mathbf{H}_\mathcal{S} + \{X_{\mathbf{T}_m}^{\{\to,\ \vee\}}\}_\mathcal{S} = \mathbf{H}_\mathcal{S} = (\mathbf{D}_m)_\mathcal{S} = (\mathbf{H} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}})_\mathcal{S}$. □

By Theorem 5.2.2, 5.3.7 and 5.3.10, the conclusion is proved.

Theorem 5.3.11. $\mathbf{H}_\mathcal{S} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$ *is a separable axiomatization of* $\mathbf{D}_m$.

Remark. *Let Γ is a set of Jankov's characteristic formulas of $\mathcal{S}$-algebras. The condition $(\phi^\star)$ (Lemma 5.3.1) implies $\mathcal{S}'(\supseteq \mathcal{S})$-completeness of $\mathbf{H} + \Gamma$. However, among $\mathcal{S} \supseteq \{\to,\ \vee\}$-algebras, we have not found any algebras which satisfy the assumption but $\mathbf{T}_m$ yet.*

5.4 Inseparability of the variant of the standard axiomatization of Gabbay-de Jongh logics

Finally, we will show that $\mathbf{H} + A'_m$ is *not* a separable axiomatization of $\mathbf{D}_m$.

Lemma 5.4.1. $\mathbf{T}_m$ *validates* A'_m *for every* m.

Proof. Suppose not; there is a refutation g of A'_m on $\mathbf{T}_m$, i.e.,

$$g(\{(p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j \mid i = 0, \ldots, m\} \to \bigvee_{i=0}^{m} p_i) \neq 1.$$

We have $g(p_k) \neq 1$ for any k since $g(A'_m) \leq g(p_k)$. Suppose $g(p_k) = \omega$ for some k. Then, there exists $l \neq k$ such that $g(p_l) \leq g(p_k) = \omega$ since $g(p_l) \neq 1$. Therefore, we have:

$$\begin{aligned}
& g(\{(p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j \mid i = 0, \ldots, m\} \to \bigvee_{i=0}^{m} p_i) \\
\geq\ & g(((p_l \to \bigvee_{j \neq l} p_j) \to \bigvee_{j \neq l} p_j) \to \bigvee_{i=0}^{m} p_i) \\
=\ & ((g(p_l) \to \omega) \to \omega) \to \omega \\
=\ & (1 \to \omega) \to \omega \\
=\ & 1,
\end{aligned}$$

this is a contradiction. Consequently, we have $g(p_k) \neq \omega,\ 1$ for any k.

Suppose $g(\bigvee_{i=0}^{m} p_i) = a_k$. Thus we have $g(p_i) = a_k$ for every i. Then we have

$$\begin{aligned} & g(\{(p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j \mid i = 0, \dots, m\} \to \bigvee_{i=0}^{m} p_i) \\ &= \{(a_k \to a_k) \to a_k \mid i = 0, \dots, m\} \to a_k \\ &= a_k \to a_k = 1. \end{aligned}$$

This is a contradiction. Therefore, we have $g(\bigvee_{i=0}^{m} p_i) = \omega$.

Consequently, there are $k,\ l \in \{0, \dots, m\}$ such that $k \neq l$ and $g(p_k) = a_s,\ g(p_l) = a_t$ $(t \neq s)$. Furthermore, there exists $u \neq \{0, \dots, m\} - \{k,\ l\}$ since $m \geq 2$. Therefore, we have

$$\begin{aligned} & g(\{(p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j \mid i = 0, \dots, m\} \to \bigvee_{i=0}^{m} p_i) \\ &\geq g(((p_u \to \bigvee_{j \neq u} p_j) \to \bigvee_{j \neq u} p_j) \to \bigvee_{i=0}^{m} p_i) \\ &= ((g(p_u) \to \omega) \to \omega) \to \omega \\ &= (1 \to \omega) \to \omega \\ &= 1. \end{aligned}$$

This is a contradiction with the assumption that g is a refutation. Consequently, A'_m is valid on $\mathbf{T}_m$. $\square$

On the other hand, by Theorem 4.1.5, a $\{\to,\ \vee\}$-algebra $\mathbf{T}_m$ refutes $X_{\mathbf{T}_m}^{\{\to,\ \vee\}}$. Therefore we have the following corollary.

Corollary 5.4.2. $(\mathbf{D}_m)_{\{\to,\ \vee\}} = \mathbf{H}_{\{\to,\ \vee\}} + X_{\mathbf{T}_m}^{\{\to,\ \vee\}} \supsetneq \mathbf{H}_{\{\to,\ \vee\}} + A'_m$.

Therefore $\mathbf{H}_{\{\to,\ \vee\}} + A'_m$ is not $\{\to,\ \vee\}$-complete.

Theorem 5.4.3. $\mathbf{H}_{\{\to,\ \vee\}} + A'_m$ *is not a separable axiomatization of* $\mathbf{D}_m$.

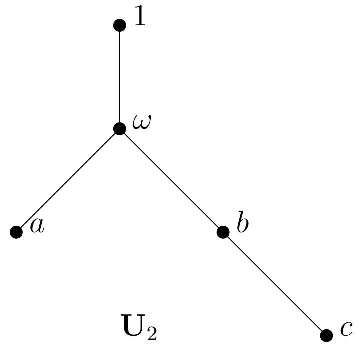
5.5 Conclusion

As an application of the result of Chapter 4, we gave an separable axiomatization of Gabbay-de Jongh logics $\mathbf{D}_m$ $(m \geq 2)$ by Jankov's characteristic formula. For the case $\vee \in \mathcal{S}$, $\mathcal{S}$-normality is proved as the corollary of the main theorem of Chapter 4 and theorem of Jankov's characteristic formulas. and the other case is proved by the Segerberg's theorem. In the former case depends on the structure of $\mathbf{T}_m$. In other words, it depends on the fact that Gabbay-de Jongh logics can be axiomatized by Jankov's characteristic formulas which the results of Chapter 4can be apply. Therefore, the following question remains.

Question. *Let $\mathcal{S} \subseteq \mathcal{S}'$. Is there a finite $\mathcal{S}'$-algebra $\mathbf{M}$ satisfying the following:*

1. $\mathbf{H} + X_{\mathbf{M}}^{\mathcal{S}}$ *is separable;*
2. $\mathbf{H}+X_{\mathbf{M}}^{\mathcal{S}}$ *is an axiomatization of an intermediate logic which is not known to be separable.*

However, we have not solved that the whether following simple $\{\to\}$-algebra $\mathbf{U}_2$ satisfies the above question yet.



$\mathbf{U}_2$

In $\mathbf{U}_2$, we define $x \to y = \begin{cases} 1 & (x \leq y) \\ y & (\text{otherwise}), \end{cases}$ $\quad x \vee y = \begin{cases} 1 & (x = 1 \text{ or } y = 1) \\ a_i & (x = y = a_i) \\ \omega & (\text{otherwise}). \end{cases}$

Chapter 6

Hypersequent calculi related Avron's GLCW

Avron[1] gives two hypersequent calculi **GLCW** and **GLC**, both of which are equivalent to the intermediate logic $\mathbf{H} + (p \to q) \vee (q \to p)$. Thus, $\mathbf{GLCW} \vdash H$ if and only if $\mathbf{GLC} \vdash H$ for any hypersequent H. However, $\mathbf{GLCW}_{\mathcal{S}}$ which is obtained by restricting logical symbols to $\mathcal{S}$ is strictly weaker than $\mathbf{GLC}_{\mathcal{S}}$ if $\wedge \notin \mathcal{S}$. In particular, Avron proved that $\mathbf{GLC}_{\{\to\}} \vdash p \Rightarrow q \mid q \Rightarrow p$ but $\mathbf{GLCW}_{\{\to,\vee,\neg\}} \nvdash p \Rightarrow q \mid q \Rightarrow p$, where $p \Rightarrow q \mid q \Rightarrow p$ is a hypersequent which is translated to $(p \to q) \vee (q \to p)$ itself. In this chapter, we revisit $\mathbf{GLCW}_{\mathcal{S}}$ from our point of view and give two results.

We correct Avron's cut-elimination theorem for $\mathbf{GLCW}_{\mathcal{S}}$ for the case that both of $\vee \in \mathcal{S}$ and $\wedge \notin \mathcal{S}$ hold.

We also give a new hypersequent calculus m-**GLCW** which is a generalization of **GLCW** for $m \geq 2$ (1-**GLCW** = **GLCW**). We show that same relation holds between m-**GLCW** and m-**GLC** as between **GLCW** and **GLC**, where m-**GLC** is a generalization of **GLC** defined by Ciabattoni and Ferrari[5].

6.1 The intuitionistic hypersequent calculus HLJ

We recall the definitions of hypersequent and **HLJ** in this section. A hypersequent is a natural generalization of a sequent. The intuitionistic hypersequent calculus **HLJ** is the hypersequent calculus obtained from **LJ**. We also recall the standard translation from a hypersequent into a formula.

Definition 6.1.1. *A hypersequent is a finite multiset of sequents written in a form* $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_m \Rightarrow \Delta_m$.

Definition 6.1.2. *The intuitionistic hypersequent calculus* **HLJ** *is the system defined by the axioms and the inference rules.*

The axioms $A \Rightarrow A$ *for any formula* A.

Inference rules *Let* A *and* B *be formulas,* γ *be a formula or emptyset,* Γ *and* Σ *are finite multisets of formulas and* G *and* H *be hypersequents. The inference rules are divided into* *two types, internal structural rules, external structural rules and rules for logical symbols.*

Internal structural rules

$$\frac{G \mid \Gamma \Rightarrow \gamma}{G \mid A, \Gamma \Rightarrow \gamma} \ (\mathbf{IW}\mathrm{L})$$

$$\frac{G \mid \Gamma \Rightarrow \emptyset}{G \mid \Gamma \Rightarrow A} \ (\mathbf{IW}\mathrm{R})$$

$$\frac{G \mid A, A, \Gamma \Rightarrow \gamma}{G \mid A, \Gamma \Rightarrow \gamma} \ (\mathbf{Con})$$

$$\frac{G \mid \Gamma \Rightarrow A \qquad G \mid A, \Sigma \Rightarrow \gamma}{G \mid \Gamma, \Sigma \Rightarrow \gamma} \ (\mathbf{Cut})$$

External structural rules

$$\frac{G}{G \mid H} \ (\mathbf{EW})$$

$$\frac{G \mid \Gamma \Rightarrow \gamma \mid \Gamma \Rightarrow \gamma}{G \mid \Gamma \Rightarrow \gamma} \ (\mathbf{EC})$$

Rules for logical symbols

$$\frac{G \mid \Gamma \Rightarrow A \qquad G \mid B, \Sigma \Rightarrow \gamma}{G \mid A \to B, \Gamma, \Sigma \Rightarrow \gamma} \ (\to \mathrm{L})$$

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \to B} \ (\to \mathrm{R})$$

$$\frac{G \mid A, \Gamma \Rightarrow \gamma}{G \mid A \wedge B, \Gamma \Rightarrow \gamma} \ (\wedge \mathrm{L})$$

$$\frac{G \mid B, \Gamma \Rightarrow \gamma}{G \mid A \wedge B, \Gamma \Rightarrow \gamma} \ (\wedge \mathrm{L})$$

$$\frac{G \mid \Gamma \Rightarrow A \qquad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B} \ (\wedge \mathrm{R})$$

$$\frac{G \mid A, \Gamma \Rightarrow \gamma \qquad G \mid B, \Gamma \Rightarrow \gamma}{G \mid A \vee B, \Gamma \Rightarrow \gamma} \ (\vee \mathrm{L})$$

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A \vee B}\ (\vee\text{R}-1)$$

$$\frac{G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \vee B}\ (\vee\text{R}-2)$$

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \neg A, \Gamma \Rightarrow \emptyset}\ (\neg\text{L})$$

$$\frac{G \mid A, \Gamma \Rightarrow \emptyset}{G \mid \Gamma \Rightarrow \neg A}\ (\neg\text{R})$$

The notation $\vdash_{\mathbf{HLJ}}$ *which defines provability in* **HLJ** *is defined as follows:*

1. $\vdash_{\mathbf{HLJ}} S$ *if* S *is an axiom sequent;*
2. $\vdash_{\mathbf{HLJ}}$ *is closed under the above inference rules.*

In the same way as the definition of $\mathcal{S}$-formulas, we define $\mathcal{S}$-sequents and $\mathcal{S}$-hypersequents. For example, $C \vee D \Rightarrow C \vee D \mid A,\ A \to B \Rightarrow B$ is an $\{\to,\ \vee\}$-hypersequent.

We define $\mathcal{S}$-subsystems of hypersequent calculi.

Definition 6.1.3. *Let* $\mathbf{G}$ *be a hypersequent calculus.* $\mathbf{G}_{\mathcal{S}}$ *is the hypersequent calculus obtained by adding the following restriction to* $\mathbf{G}$*:*

every proof of $\mathbf{G}_{\mathcal{S}}$ *consists of only* $\mathcal{S}$*-hypersequents.*

The symbol $\mid$ is translated to the disjunction $\vee$.

Theorem 6.1.4. *Let* $\Gamma_1 \cup \cdots \cup \Gamma_m \cup \{A_1, \ldots, A_m\}$ *is a set of formulas. The following are equivalent:*

1. $\vdash_{\mathbf{HLJ}} \Gamma_1 \Rightarrow A_1 \mid \cdots \mid \Gamma_m \Rightarrow A_m$*;*
2. $\vdash_{\mathbf{H}} (\Gamma_1 \to A_1) \vee \cdots \vee (\Gamma_m \to A_m)$.

Proof. Induction on the length of proof. □

By Theorem 6.1.4, we obtain a method of translation from a hypersequent into a formula. Let **HSeq** be the set of all hypersequents. We define a map $T : \mathbf{HSeq} \to \mathbf{Form}$ by

$$T(\Gamma_1 \Rightarrow \delta_1 \mid \cdots \mid \Gamma_m \Rightarrow \delta_m) = (T'(\Gamma_1) \to T'(\delta_1)) \vee \cdots \vee (T'(\Gamma_m) \to T'(\delta_m)),$$

where T' is defined by $T'(\Gamma_i) = \begin{cases} \Gamma_i & (\Gamma \neq \emptyset) \\ \top & (\Gamma_i = \emptyset) \end{cases}$ and $T'(\delta_i) = \begin{cases} \delta_i & (\delta_i \neq \emptyset) \\ \bot & (\delta_i = \emptyset,\ \neg \in \mathcal{S}) \\ r & (\delta_i = \emptyset,\ \neg \notin \mathcal{S}), \end{cases}$
where r is a variable which does not occur in $\Gamma_1 \Rightarrow \delta_1 \mid \cdots \mid \Gamma_m \Rightarrow \delta_m$.

If $\vee \notin \mathcal{S}$, the symbol "$\mid$" cannot be translated directly.

Lemma 6.1.5. *Let A and B be formulas and p be a propositional variable which does not occur in neither A nor B. Then, $\vdash_{\mathbf{H}} A \vee B$ if and only if $\vdash_{\mathbf{H}} (A \to p) \to (B \to p) \to p$.*

Proof. The only if part follows from the axiom ($\vee 3$) of **H**. The converse follows from:

$$\begin{aligned} &\vdash_{\mathbf{H}} (A \to p) \to (B \to p) \to p \\ \Longrightarrow &\vdash_{\mathbf{H}} (A \to A \vee B) \to (B \to A \vee B) \to A \vee B \\ \Longleftrightarrow &\vdash_{\mathbf{H}} A \vee B. \end{aligned}$$

□

For given formulas A and B, we abbreviate $(A \to p) \to (B \to p) \to p$ to $A \bar{\vee} B$, where p is a propositional variable which does occur in neither A nor B.

Corollary 6.1.6. *Let $\Gamma_1 \cup \cdots \cup \Gamma_m \cup \{A_1, \ldots, A_m\}$ is a set of formulas and p be a propositional symbol does not occurring in $\Gamma_1 \cup \cdots \cup \Gamma_m \cup \{A_1, \ldots, A_m\}$. The following are equivalent:*

1. $\vdash_{\mathbf{HLJ}} \Gamma_1 \Rightarrow A_1 \mid \cdots \mid \Gamma_m \Rightarrow A_m$;
2. $\vdash_{\mathbf{H}} (\Gamma_1 \to A_1) \bar{\vee} \cdots \bar{\vee} (\Gamma_m \to A_m)$.

6.2 Two hypersequent calculi GLCW and GLC

Definition 6.2.1 (Avron[1])**.** *Symbols γ, γ_1 and γ_2 mean a formula or $\emptyset$.*

1. **GLCW** *is the hypersequent calculus obtained by adding the following split rule to* **HLJ***:*

$$\frac{G \mid \Gamma, \Delta \Rightarrow \gamma}{G \mid \Gamma \Rightarrow \gamma \mid \Delta \Rightarrow \gamma} \ (\mathbf{sp})$$

2. **GLC** *is the hypersequent calculus obtained by adding the following communication rule to* **HLJ***:*

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \gamma_1 \qquad G \mid \Delta_1, \Delta_2 \Rightarrow \gamma_2}{G \mid \Gamma_1, \Delta_1 \Rightarrow \gamma_1 \mid \Gamma_2, \Delta_2 \Rightarrow \gamma_2} \ (\mathbf{com})$$

We note that (**sp**) is considered a special case of (**com**), i.e., (**sp**) and the case $A = B$ in (**com**) are equivalent rule.

There is the following simple relation between **GLCW** and **GLC**.

Theorem 6.2.2 (Avron[1])**.** *The following are equivalent for every hypersequent H:*

1. **GLCW** $\vdash H$;
2. **GLC** $\vdash H$;

3. $\mathbf{H} + (p \to q) \vee (q \to p) \vdash T(H)$.

Therefore, we consider $\mathbf{GLCW}_{\mathcal{S}}$ and $\mathbf{GLC}_{\mathcal{S}}$, $\mathcal{S}$-subsystems of **GLCW** and **GLC**, in the rest of the chapter.

It is known that $\mathbf{H}_{\mathcal{S}} + (p \to q) \vee (q \to p)$ is characterized by the class of totally ordered $\mathcal{S}$-algebras. Therefore, Theorem 6.2.2 implies that both of **GLCW** and **GLC** are characterized by the class of totally ordered Heyting algebras. However, if $\wedge \notin \mathcal{S}$, the class of $\mathcal{S}$-algebras characterizes $\mathbf{GLCW}_{\mathcal{S}}$ is different from the class of $\mathcal{S}$-algebras characterizes $\mathbf{GLC}_{\mathcal{S}}$

Let **G** be a hypersequent calculus and **M** be an $\mathcal{S}$-algebra. We say that **M** is **G**-algebra if **M** validates $T(H)$ for every hypersequent H satisfying $\mathbf{G} \vdash H$.

Proposition 6.2.3 (Avron[1])**.** *An $\mathcal{S}$-algebra* **M** *is* $\mathbf{GLCW}_{\mathcal{S}}$*-algebra if* **M** *satisfies the following condition:*

$$\textit{every } x \in \mathbf{M} \textit{ is indecomposable.}$$

Proposition 6.2.4 (Avron[1])**.** $\mathbf{GLC}_{\mathcal{S}}$ *is characterized by the class of totally ordered $\mathcal{S}$-algebras.*

Proposition 6.2.5 (Avron[1])**.**

1. *if $\wedge \in \mathcal{S}$, every* $\mathbf{GLCW}_{\mathcal{S}}$*-algebra is a* $\mathbf{GLC}_{\mathcal{S}}$*-algebra;*

2. *if $\wedge \notin \mathcal{S}$, there is a* $\mathbf{GLCW}_{\mathcal{S}}$*-algebra which is not a* $\mathbf{GLC}_{\mathcal{S}}$*-algebra.*

Proof. We show the contraposition of the case $\wedge \in \mathcal{S}$. Let an $\mathcal{S}$-algebra **M** is not a $\mathbf{GLC}_{\mathcal{S}}$-algebra. Then we have $x, y \in \mathbf{M}$ such that x and y are incomparable. Therefore we obtain $x \to x \wedge y = x \to y \geq y > x \wedge y$, which implies that **M** is not a $\mathbf{GLCW}_{\mathcal{S}}$-algebra.

The case $\wedge \notin \mathcal{S}$ follows from the $\mathbf{GLCW}_{\mathcal{S}}$-algebra defined by Figure 9.

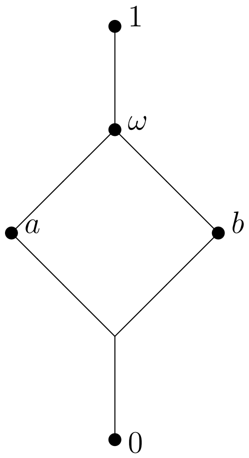


Figure 9: a $\mathbf{GLCW}_{\mathcal{S}}$-algebra but not a $\mathbf{GLC}_{\mathcal{S}}$-algebra

□

Theorem 6.2.6 (Avron[1])**.** $\mathbf{GLC}_{\mathcal{S}} = \mathbf{GLCW}_{\mathcal{S}}$ *if and only if $\wedge \in \mathcal{S}$*

Avron also gives Hilbert-style formulations of $\mathbf{GLCW}_{\mathcal{S}}$.

Theorem 6.2.7 (Avron[1]). *The following are equivalent:*

1. $\mathbf{GLCW}_{\mathcal{S}} \vdash H$;
2. $\mathbf{H}_{\mathcal{S}} + ((p \to q) \to q)\bar{\vee}(p \to q) \vdash T(H)$.

6.3 Cut-elimination theorem of $\mathbf{GLCW}_{\mathcal{S}}$

In Avron[1], he claims the following cut-elimination theorem.

Claim 6.3.1 (Avron[1] Theorem 1 and 5). $\mathbf{GLCW}_{\mathcal{S}}$ *admits the cut-elimination theorem if* $\wedge \notin \mathcal{S}$.

We discovered a mistake in the proof for the case $\vee \in \mathcal{S}$. Since the proof still correct for the the case $\vee \notin \mathcal{S}$, we corrected his claim.

Theorem 6.3.2 (c.f., Avron[1] Theorem 1 and 5). *If* $\mathcal{S} \subseteq \{\to, \neg\}$*, the cut-elimination theorem holds on* $\mathbf{GLCW}_{\mathcal{S}}$.

We give a proof of it in detail since the original proof in Avron[1] is written in very simple and this proof is used in the cut-elimination theorem for the generalized case in Section 6.5.

In this section, we assume $\mathcal{S} \subseteq \{\to, \neg\}$ and δ or it with subscript means an $\mathcal{S}$-formula or empty. $\mathbf{GLCW}^{cf}_{\mathcal{S}}$ is the $\mathcal{S}$-hypersequent calculus obtained by removing (**cut**) from $\mathbf{GLCW}_{\mathcal{S}}$. $\mathbf{GLCW}^{s}_{\mathcal{S}}$ is the $\mathcal{S}$-hypersequent calculus obtained by replacing (**cut**) by the following (**spmix**) rule from $\mathbf{GLCW}_{\mathcal{S}}$.

$$\frac{G_1 \mid \Gamma_1, \Gamma'_1 \Rightarrow A \mid \cdots \mid \Gamma_n, \Gamma'_n \Rightarrow A \qquad G_2 \mid \Delta_1 \Rightarrow \delta_1 \mid \cdots \mid \Delta_k \Rightarrow \delta_k}{G_1 \mid G_2 \mid \Gamma'_1 \Rightarrow A \mid \cdots \mid \Gamma'_n \Rightarrow A \mid \Gamma, \Delta^A_1 \Rightarrow \delta_1 \mid \cdots \mid \Gamma, \Delta^A_k \Rightarrow \delta_k}\ (\mathbf{spmix})$$

where $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$, a component $\Gamma'_i \Rightarrow \delta$ is actually exist only if $\Gamma'_i \neq \emptyset$ and Δ^{δ}_i ($i = 1, \ldots, k$) is the multiset obtained by removing all δ from Δ_i.

Lemma 6.3.3. $\mathbf{GLCW}_{\mathcal{S}} \vdash G$ *if and only if* $\mathbf{GLCW}^{s}_{\mathcal{S}} \vdash G$ *for every* $\mathcal{S}$*-hypersequent* G.

Proof. For simplicity, we assume that $n = k = 2$. We first have the following proof by split-rule.

$$\frac{\dfrac{\Gamma'_1, \Gamma_1 \Rightarrow A \mid \Gamma_2, \Gamma'_2 \Rightarrow A}{\Gamma'_1 \Rightarrow A \mid \Gamma'_2 \Rightarrow A \mid \Gamma_1 \Rightarrow A \mid \Gamma_2 \Rightarrow A}}{\Gamma'_1 \Rightarrow A \mid \Gamma'_2 \Rightarrow A \mid \Gamma \Rightarrow A}$$

Then we obtain (**spmix**) by the following proof.

$$
\dfrac{\dfrac{\Gamma_1' \Rightarrow A \mid \Gamma_2' \Rightarrow A \mid \Gamma \Rightarrow A \qquad \dfrac{\Gamma_1' \Rightarrow A \mid \Gamma_2' \Rightarrow A \mid \Gamma \Rightarrow A \qquad \dfrac{\Delta_1 \Rightarrow \delta_1 \mid \Delta_2 \Rightarrow \delta_2}{A, \Delta_1^A \Rightarrow \delta_1 \mid A, \Delta_2^A \Rightarrow \delta_2}}{\Gamma_1' \Rightarrow A \mid \Gamma_2' \Rightarrow A \mid \Gamma, \Delta_1^A \Rightarrow \delta_1 \mid A, \Delta_2^A \Rightarrow \delta_2}}{\Gamma_1' \Rightarrow A \mid \Gamma_2' \Rightarrow A \mid \Gamma_1' \Rightarrow A \mid \Gamma_2' \Rightarrow A \mid \Gamma, \Delta_1^A \Rightarrow \delta_1 \mid \Gamma, \Delta_2^A \Rightarrow \delta_2}}{\Gamma_1' \Rightarrow A \mid \Gamma_2' \Rightarrow A \mid \Gamma, \Delta_1^A \Rightarrow \delta_1 \mid \Gamma, \Delta_2^A \Rightarrow \delta_2}
$$

□

Lemma 6.3.4 (Inversion lemma)**.** *Let* $\Gamma_1, \dots, \Gamma_n$ *be multisets of* $\mathcal{S}$*-formulas and* $A_1, \dots, A_n, B_1, \dots, B_n$ *be formulas. The following are equivalent:*

1. $\mathbf{GLCW}_{\mathcal{S}}^{cf} \vdash G \mid \Gamma_1 \Rightarrow A_1 \to B_1 \mid \cdots \mid \Gamma_n \Rightarrow A_n \to B_n$.
2. $\mathbf{GLCW}_{\mathcal{S}}^{cf} \vdash G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_n, A_n \Rightarrow B_n$

Proof. (*1* $\Longrightarrow$ *2*) Induction on the length of the proof. We give the proof for the case that (**EC**) and (**sp**). since the rest cases can be shown similarly to Theorem 2.2.5.

We show the case (**EC**), we can assume that the proof is the following form:

$$
\dfrac{G \mid \Gamma_1 \Rightarrow A_1 \to B_1 \mid \cdots \mid \Gamma_n \Rightarrow A_n \to B_n \mid \Gamma_n \Rightarrow A_n \to B_n}{G \mid \Gamma_1 \Rightarrow A_1 \to B_1 \mid \cdots \mid \Gamma_n \Rightarrow A_n \to B_n}
$$

By I.H., we have

$$
\mathbf{GLCW}_{\mathcal{S}}^{cf} \vdash G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_n, A_n \Rightarrow B_n \mid \Gamma_n, A_n \Rightarrow B_n.
$$

Therefore we obtain

$$
\dfrac{G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_n, A_n \Rightarrow B_n \mid \Gamma_n, A_n \Rightarrow B_n}{G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_n, A_n \Rightarrow B_n}
$$

We show the case (**sp**), there are two proofs we need to consider. One of them is

$$
\dfrac{G \mid \Gamma_1 \Rightarrow A_1 \to B_1 \mid \cdots \mid \Gamma_{n-1}, \Gamma_n \Rightarrow A_n \to B_n}{G \mid \Gamma_1 \Rightarrow A_1 \to B_1 \mid \cdots \mid \Gamma_n \Rightarrow A_n \to B_n}
$$

where $A_{n-1} \to B_{n-1} = A_n \to B_n$. By I.H., we have

$$
\mathbf{GLCW}_{\mathcal{S}}^{cf} \vdash G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_{n-1}, \Gamma_n, A_n \Rightarrow B_n.
$$

Therefore we obtain

$$
\dfrac{\dfrac{G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_{n-1}, \Gamma_n, A_n \Rightarrow B_n}{G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_{n-1}, \Gamma_n, A_n, A_n \Rightarrow B_n}}{G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_n, A_n \Rightarrow B_n}
$$

The other is

$$
\dfrac{G^- \mid \Gamma_1 \Rightarrow A_1 \to B_1 \mid \cdots \mid \Gamma_{n-1} \Rightarrow A_{n-1} \to B_{n-1} \mid \Sigma, \Gamma_n \Rightarrow A_n \to B_n}{G^- \mid \Sigma \Rightarrow A_n \to B_n \mid \Gamma_1 \Rightarrow A_1 \to B_1 \mid \cdots \mid \Gamma_n \Rightarrow A_n \to B_n}
$$

where $G = G^- \mid \Sigma \Rightarrow A_n \to B_n$. By I.H., we have

$$\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G^- \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_{n-1}, A_{n-1} \Rightarrow B_{n-1} \mid \Sigma, \Gamma_n, A_n \Rightarrow B_n.$$

Therefore we obtain

$$\frac{G^- \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_{n-1}, A_{n-1} \Rightarrow B_{n-1} \mid \Sigma, \Gamma_n, A_n \Rightarrow B_n}{G \mid \Gamma_1, A_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_n, A_n \Rightarrow B_n}$$

The converse is obtained by applying ($\to$R) n times. □

Lemma 6.3.5. *Let $\Gamma_1, \ldots, \Gamma_n, \Sigma$ be multisets of $\mathcal{S}$-formulas and A and B be formulas. The following are equivalent:*

1. $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid \Gamma_1 \Rightarrow A \to B \mid \cdots \mid \Gamma_n \Rightarrow A \to B \mid \Sigma \Rightarrow A \to B$;
2. $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid \Gamma_1 \Rightarrow B \mid \cdots \mid \Gamma_n \Rightarrow B \mid \Sigma, A \Rightarrow B$.

Proof.

$$\begin{aligned}
&\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid \Gamma_1 \Rightarrow A \to B \mid \cdots \mid \Gamma_n \Rightarrow A \to B \mid \Sigma \Rightarrow A \to B\\
\Longleftrightarrow\ &\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid \Gamma_1, \ldots, \Gamma_n, \Sigma \Rightarrow A \to B\\
\Longleftrightarrow\ &\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid \Gamma_1, \ldots, \Gamma_n, \Sigma, A \Rightarrow B\\
\Longleftrightarrow\ &\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid \Gamma_1 \Rightarrow B \mid \cdots \mid \Gamma_n \Rightarrow B \mid \Sigma, A \Rightarrow B.
\end{aligned}$$

□

Lemma 6.3.6. *Let G be a hypersequent, Γ be a set of formulas and A and B be formulas.* $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid \Gamma \Rightarrow \delta \mid \cdots \mid \Gamma \Rightarrow \delta$ *implies* $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash G \mid B, \Gamma^{A \to B} \Rightarrow \delta$.

Proof. Induction on the length of the proof. we verify the case ($\to$L) and (**EC**).

In the case ($\to$L), we can assume that the proof is the following form:

$$\frac{G \mid \Delta \Rightarrow D \qquad G \mid E, \Sigma \Rightarrow \delta}{G \mid D \to E, \Delta, \Sigma \Rightarrow \delta}$$

If $B \neq E$ holds, $D \to E \neq A \to B$ also holds. Hence we have

$$\frac{\overset{\text{(I.H.)}}{G \mid B, \Delta^{A \to B} \Rightarrow D} \qquad \overset{\text{(I.H.)}}{G \mid B, E, \Sigma^{A \to B} \Rightarrow \delta}}{G \mid B, D \to E, \Delta^{A \to B}, \Sigma^{A \to B} \Rightarrow \delta}$$

If $B = E$, we have

$$\cfrac{\cfrac{\overset{\text{(I.H.)}}{G \mid B, B, \Sigma^{A \to B} \Rightarrow \delta}}{G \mid B, \Sigma^{A \to B} \Rightarrow \delta}}{G \mid B, D \to E, \Delta^{A \to B}, \Sigma^{A \to B} \Rightarrow \delta}$$

In the case (**EC**), we can assume that the proof is the following form:

$$\frac{G \mid \Gamma \Rightarrow \delta \mid \Gamma \Rightarrow \delta}{G \mid \Gamma \Rightarrow \delta}$$

Therefore we obtain $G \mid B, \Gamma^{A \to B} \Rightarrow \delta$ by the induction hypothesis. $\square$

Now we show the cut-elimination theorem for $\mathbf{GLCW}_{\mathcal{S}}$ for the case $\mathcal{S} \subseteq \{\to, \neg\}$. It is sufficient to prove that the rule (**spmix**) can be eliminated on $\mathbf{GLCW}_{\mathcal{S}}^{s}$, i.e.,

$$\mathbf{GLCW}_{\mathcal{S}}^{cf} \vdash G_1 \mid \Gamma_1, \Gamma'_1 \Rightarrow \gamma \mid \cdots \mid \Gamma_n, \Gamma'_n \Rightarrow \gamma$$

(we call it the *left premise*) and

$$\mathbf{GLCW}_{\mathcal{S}}^{cf} \vdash G_2 \mid \Delta_1 \Rightarrow \delta_1 \mid \cdots \mid \Delta_k \Rightarrow \delta_k$$

(we call it the *right premise*) implies

$$\mathbf{GLCW}_{\mathcal{S}}^{cf} \vdash G_1 \mid G_2 \mid \Gamma'_1 \Rightarrow \gamma \mid \cdots \mid \Gamma'_n \Rightarrow \gamma \mid \Gamma, \Delta_1^A \Rightarrow \delta_1 \mid \cdots \mid \Gamma, \Delta_k^A \Rightarrow \delta_k$$

(we call it the *conclusion*), where $\gamma, \delta_1, \ldots, \delta_k$ are a formula or $\emptyset$ (in the proof, we use γ, δ and these with a subscript for a formula or $\emptyset$). We show it by double induction on the complexity of the cut formula and the sum of the length of the proof of the premises. For simplicity, we omit the side sequents that are not involved in the "action" in our proof.

If the left premise is an axiom $A \Rightarrow A$, we need to prove the following two cases:

$$\frac{A \Rightarrow A \qquad \overset{\mathcal{R}}{\Delta \Rightarrow \delta}}{A \Rightarrow A \mid \Delta^A \Rightarrow \delta}$$

and

$$\frac{A \Rightarrow A \qquad \overset{\mathcal{R}}{\Delta \Rightarrow \delta}}{A, \Delta^A \Rightarrow \delta}$$

They can be solved by the following proofs respectively:

$$\frac{A \Rightarrow A}{A \Rightarrow A \mid \Delta^A \Rightarrow \delta}$$

and

$$\frac{\overset{\mathcal{R}}{\Delta \Rightarrow \delta}}{A, \Delta^A \Rightarrow \delta}\text{ some }(\mathbf{IC})\text{s or }(\mathbf{IW})$$

If the left premise is obtained by applying a (internal or external) structural rule finally, we verify the case (**EW**) and (**EC**) (the other cases can be proved similarly). In the case (**EW**), we can assume that the proof is the following form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{L}\\ \Gamma_1,\Gamma_1'\Rightarrow\delta\end{matrix}}{\Gamma_1,\Gamma_1'\Rightarrow\delta\mid\Gamma_2,\Gamma_2'\Rightarrow\delta}\qquad\begin{matrix}\mathcal{R}\\ \Delta\Rightarrow\delta_1\end{matrix}}{\Gamma_1'\Rightarrow A\mid\Gamma_2'\Rightarrow A\mid\Gamma_1,\Gamma_2,\Delta^{\delta}\Rightarrow\delta_1}\,{\scriptstyle (c,l)}$$

where c and l are the complexity of the cut formula and the sum of the length of the proof of the premises respectively. This proof can be replaced with:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{L}\\ \Gamma_1,\Gamma_1'\Rightarrow\delta\end{matrix}\qquad\begin{matrix}\mathcal{R}\\ \Delta\Rightarrow\delta_1\end{matrix}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Delta^{\delta}\Rightarrow\delta_1}\,{\scriptstyle (c,l-1)}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Gamma_2,\Delta^{\delta}\Rightarrow\delta_1}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_2'\Rightarrow\delta\mid\Gamma_1,\Gamma_2,\Delta^{\delta}\Rightarrow\delta_1}$$

The (**spmix**) rule in this deduction can be eliminated by induction hypothesis.

In the case (**EC**), we can assume that the proof is the following form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{L}\\ \Gamma_1,\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Gamma_1'\Rightarrow\delta\end{matrix}}{\Gamma_1,\Gamma_1'\Rightarrow\delta}\qquad\begin{matrix}\mathcal{R}\\ \Delta\Rightarrow\delta_1\end{matrix}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Delta^{\delta}\Rightarrow\delta_1}\,{\scriptstyle (c,l)}$$

This proof can be replaced with:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{L}\\ \Gamma_1,\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Gamma_1'\Rightarrow\delta\end{matrix}\qquad\begin{matrix}\mathcal{R}\\ \Delta\Rightarrow\delta_1\end{matrix}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Gamma_1\Delta^{\delta}\Rightarrow\delta_1}\,{\scriptstyle (c,l-1)}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1'\Rightarrow\delta\mid\Gamma_1\Delta^{\delta}\Rightarrow\delta_1}}{\Gamma_1'\Rightarrow\delta,\Gamma_1\mid\Delta^{\delta}\Rightarrow\delta_1}$$

The (**spmix**) rule in this deduction can be eliminated by induction hypothesis.

If the left premise is obtained by applying (**sp**) finally, we need to prove two cases. In the first case, we assume that the proof is the following form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{L}\\ \Gamma_1,\Gamma_1',\Gamma_2,\Gamma_2'\Rightarrow\delta\end{matrix}}{\Gamma_1,\Gamma_1'\Rightarrow\delta\mid\Gamma_2,\Gamma_2'\Rightarrow\delta}\qquad\begin{matrix}\mathcal{R}\\ \Delta\Rightarrow\delta_1\end{matrix}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_2'\Rightarrow\delta\mid\Gamma_1,\Gamma_2,\Delta^{\delta}\Rightarrow\delta_1}\,{\scriptstyle (c,l)}$$

This proof can be replaced with

$$\dfrac{\dfrac{\begin{matrix}\mathcal{L}\\ \Gamma_1,\Gamma_1',\Gamma_2,\Gamma_2'\Rightarrow\delta\end{matrix}\qquad\begin{matrix}\mathcal{R}\\ \Delta\Rightarrow\delta_1\end{matrix}}{\Gamma_1',\Gamma_2'\Rightarrow\delta\mid\Gamma_1,\Gamma_2,\Delta^{\delta}\Rightarrow\delta_1}\,{\scriptstyle (c,l-1)}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_2'\Rightarrow\delta\mid\Gamma_1,\Gamma_2,\Delta^{\delta}\Rightarrow\delta_1}$$

The (**spmix**) rule in this deduction can be eliminated by induction hypothesis. In the other case, we assume that the proof is the following form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{L}\\ \Gamma_1, \Gamma_1', \Sigma \Rightarrow \delta\end{array}}{\Gamma_1, \Gamma_1' \Rightarrow \delta \mid \Sigma \Rightarrow \delta} \qquad \begin{array}{c}\mathcal{R}\\ \Delta \Rightarrow \delta_1\end{array}}{\Sigma \Rightarrow \delta \mid \Gamma_1' \Rightarrow \delta \mid \Gamma_1, \Delta^\delta \Rightarrow \delta_1}\,{\scriptstyle (c,l)}$$

This proof can be replaced with

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{L}\\ \Gamma_1,\Gamma_1',\Sigma\Rightarrow\delta\end{array}\qquad\begin{array}{c}\mathcal{R}\\ \Delta\Rightarrow\delta_1\end{array}}{\Gamma_1',\Sigma\Rightarrow\delta\mid\Gamma_1,\Delta^\delta\Rightarrow\delta_1}\,{\scriptstyle (c,l-1)}}{\Sigma\Rightarrow\delta\mid\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Delta^\delta\Rightarrow\delta_1}$$

The (**spmix**) rule in this deduction can be eliminated by induction hypothesis.

If the left premise is obtained by applying (→R) finally and the right premise is obtained by applying (→L) finally, we need to prove two cases. If the principal formula is the cut formula, the proof is on Avron[1]. If not, we can assume that the proof is the following form:

$$\dfrac{\begin{array}{c}\mathcal{L}\\ \Gamma_1,\Gamma_1'\Rightarrow\delta\end{array}\qquad\dfrac{\begin{array}{c}\mathcal{R}_1\\ \Sigma_1\Rightarrow C\end{array}\qquad\begin{array}{c}\mathcal{R}_2\\ D,\Sigma_2\Rightarrow\delta_1\end{array}}{\Sigma_1,\Sigma_2,C\to D\Rightarrow\delta_1}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Sigma_1^\delta,\Sigma_2^\delta,C\to D\Rightarrow\delta_1}\,{\scriptstyle (c,l)}$$

This proof can be replaced with

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{L}\\ \Gamma_1,\Gamma_1'\Rightarrow\delta\end{array}\qquad\begin{array}{c}\mathcal{R}_1\\ \Sigma_1\Rightarrow C\end{array}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Sigma_1^\delta\Rightarrow C}\,{\scriptstyle (c,<l)}\qquad\dfrac{\begin{array}{c}\mathcal{L}\\ \Gamma_1,\Gamma_1'\Rightarrow\delta\end{array}\qquad\begin{array}{c}\mathcal{R}_2\\ D,\Sigma_2\Rightarrow\delta_1\end{array}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Sigma_2^\delta,D\Rightarrow\delta_1}\,{\scriptstyle (c,<l)}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Gamma_1,\Sigma_1^\delta,\Sigma_2^\delta,C\to D\Rightarrow\delta_1}}{\Gamma_1'\Rightarrow\delta\mid\Gamma_1,\Sigma_1^\delta,\Sigma_2^\delta,C\to D\Rightarrow\delta_1}$$

The (**spmix**) rule in this deduction can be eliminated by induction hypothesis.

The rest case is that both premises are obtained by applying (→L) finally. We need to show two cases. The first case is that the principal formula in the right premise is not the cut formula. Then we can assume that the proof is the following form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{L}_1\\ \Sigma_1,\Sigma_1'\Rightarrow C\end{array}\qquad\begin{array}{c}\mathcal{L}_2\\ D,\Sigma_2,\Sigma_2'\Rightarrow\delta\end{array}}{\Sigma_1,\Sigma_2,\Sigma_1',\Sigma_2',C\to D\Rightarrow\delta}\qquad\dfrac{\begin{array}{c}\mathcal{R}_1\\ \Pi_1\Rightarrow E\end{array}\qquad\begin{array}{c}\mathcal{R}_2\\ F,\Pi_2\Rightarrow\delta_1\end{array}}{E\to F,\Pi_1,\Pi_2\Rightarrow\delta_1}}{\Sigma_1',\Sigma_2'\Rightarrow\delta\mid\Sigma_1,\Sigma_2,\Pi_1^\delta,\Pi_2^\delta,C\to D,E\to F\Rightarrow\delta_1}\,{\scriptstyle (c,l)}$$

This proof can be replaced with

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{L}_1\\ \Sigma_1,\Sigma_1'\Rightarrow C\end{array}\qquad\begin{array}{c}\mathcal{L}_2\\ D,\Sigma_2,\Sigma_2'\Rightarrow\delta\end{array}}{\Sigma_1,\Sigma_2,\Sigma_1',\Sigma_2',C\to D\Rightarrow\delta}\qquad\begin{array}{c}\mathcal{R}_1\\ \Pi_1\Rightarrow E\end{array}}{\Sigma_1',\Sigma_2'\Rightarrow\delta\mid\Sigma_1,\Sigma_2,\Pi_1^{\delta},C\to D\Rightarrow E}\,(c,<l)\qquad\dfrac{\dfrac{\begin{array}{c}\mathcal{L}_1\\ \Sigma_1,\Sigma_1'\Rightarrow C\end{array}\qquad\begin{array}{c}\mathcal{L}_2\\ D,\Sigma_2,\Sigma_2'\Rightarrow\delta\end{array}}{\Sigma_1,\Sigma_2,\Sigma_1',\Sigma_2',C\to D\Rightarrow\delta}\qquad\begin{array}{c}\mathcal{R}_2\\ F,\Pi_2\Rightarrow\delta_1\end{array}}{\Sigma_1',\Sigma_2'\Rightarrow\delta\mid\Sigma_1,\Sigma_2,\Pi_2^{\delta},C\to D,F\Rightarrow\delta_1}\,(c,<l)}{\Sigma_1',\Sigma_2'\Rightarrow\delta\mid\Sigma_1,\Sigma_2,\Pi_1^{\delta},\Pi_2^{\delta},C\to D,E\to F\Rightarrow\delta_1}$$

The (**spmix**) rule in this deduction can be eliminated by induction hypothesis.

The other case is that the principal formula of the right sequent is actually the cut formula. Then we can assume that the proof is the following form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{L}_1\\ \Sigma_1,\Sigma_1'\Rightarrow C\end{array}\qquad\begin{array}{c}\mathcal{L}_2\\ D,\Sigma_2,\Sigma_2'\Rightarrow E\to F\end{array}}{\Sigma_1,\Sigma_2,\Sigma_1',\Sigma_2',C\to D\Rightarrow E\to F}\qquad\dfrac{\begin{array}{c}\mathcal{R}_1\\ \Pi_1\Rightarrow E\end{array}\qquad\begin{array}{c}\mathcal{R}_2\\ F,\Pi_2\Rightarrow\delta_1\end{array}}{E\to F,\Pi_1,\Pi_2\Rightarrow\delta_1}}{(\star)\Sigma_1',\Sigma_2'\Rightarrow E\to F\mid\Sigma_1,\Sigma_2,\Pi_1^{E\to F},\Pi_2^{E\to F},C\to D\Rightarrow\delta_1}\,(c,l)$$

This proof implies:

$$\mathbf{GLCW}_{\mathcal{S}}^{cf}\vdash\Sigma_1,\Sigma_2,\Sigma_1',\Sigma_2',C\to D\Rightarrow E\to F;$$
$$\mathbf{GLCW}_{\mathcal{S}}^{cf}\vdash F,\Pi_2\Rightarrow\delta_1.$$

By Lemma 6.3.4 and 6.3.6, we obtain:

$$\mathbf{GLCW}_{\mathcal{S}}^{cf}\vdash\Sigma_1',\Sigma_2',E\Rightarrow F\mid\Sigma_1,\Sigma_2,C\to D\Rightarrow F;$$
$$\mathbf{GLCW}_{\mathcal{S}}^{cf}\vdash F,\Pi_2^{E\to F}\Rightarrow\delta_1.$$

Therefore we obtain the following proof:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{L}'\\ \Sigma_1',\Sigma_2',E\Rightarrow F\mid\Sigma_1,\Sigma_2,C\to D\Rightarrow F\end{array}\qquad\begin{array}{c}\mathcal{R}_2'\\ F,\Pi_2^{E\to F}\Rightarrow\delta_1\end{array}}{\Sigma_1',\Sigma_2',E\Rightarrow F\mid\Sigma_1,\Sigma_2,C\to D,\Pi_2^{E\to F}\Rightarrow\delta_1}\,(c-1)}{\Sigma_1',\Sigma_2'\Rightarrow E\to F\mid\Sigma_1,\Sigma_2,\Pi_1^{E\to F},\Pi_2^{E\to F},C\to D\Rightarrow\delta_1}$$

The (**spmix**) rule in this deduction can be eliminated by induction hypothesis. We note that the same proof works if the conclusion $(\star)$ is

$$\Sigma_1',\Sigma_2',C\to D\Rightarrow E\to F\mid\Gamma_1,\Sigma_1,\Sigma_2,\Pi_1^{E\to F},\Pi_2^{E\to F}\Rightarrow\delta_1.$$

We omit the proof for the following similar cases:

- the right premise is obtained by applying a (internal or external) structural rule finally;
- the right premise is obtained by applying $(\to\mathrm{R})$ finally;
- the right or left premise are obtained by applying a $(\neg\mathrm{L})$ or $(\neg\mathrm{R})$ finally.

$\square$

We next give a counterexample for Avron's cut-elimination theorem for the case $\vee \in \mathcal{S}$.

Let a hypersequent $\mathcal{M} = \Gamma_a \Rightarrow a \mid \Gamma_b \Rightarrow b \mid \Gamma_c \Rightarrow c \mid \Gamma_d \Rightarrow d \mid \Gamma_{b\vee c} \Rightarrow b \vee c \mid \Gamma_{a\to b\vee c} \Rightarrow a \to b \vee c$, where

$$\begin{aligned}
\Gamma_a &= \{b,\ c,\ d,\ b \to d,\ c \to d,\ b \vee c,\ a \to b \vee c\};\\
\Gamma_b &= \{a,\ c,\ d,\ b \to d,\ c \to d,\ b \vee c,\ a \to b \vee c\};\\
\Gamma_c &= \{a,\ b,\ d,\ b \to d,\ c \to d,\ b \vee c,\ a \to b \vee c\};\\
\Gamma_d &= \{b \to d,\ c \to d,\ a \to b \vee c\};\\
\Gamma_{b\vee c} = \Gamma_{a\to b\vee c} &= \{a,\ d,\ b \to d,\ c \to d\}.
\end{aligned}$$

We note that $\Gamma_d \subseteq \Gamma_a,\ \Gamma_b,\ \Gamma_c$ and $\Gamma_{b\vee c} = \Gamma_{a\to b\vee c} \subseteq \Gamma_b,\ \Gamma_c$. Let $\mathbf{Sub}(\mathcal{M}) = \{a,\ b,\ c,\ d,\ b \vee c,\ b \to c,\ c \to d,\ a \to b \vee c\}$, i.e., $\mathbf{Sub}(\mathcal{M}) = \{A \mid A$ is a subformula of some B occurring in $\mathcal{M}\}$.

Lemma 6.3.7. *Let $\vee \in \mathcal{S}$. If a hypersequent H consists of formulas in $\mathbf{Sub}(\mathcal{M})$ satisfies $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash H$, $\mathcal{M}$ cannot be provable from H by applying only structural rules.*

Proof. Induction on length l of the proof of $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash H$.

The case $l = 0$, $H = A \Rightarrow A$ for a $\mathcal{S}$-formula A. Thus $\mathcal{M}$ cannot be provable from H by applying only structural rules since any hypersequent which can be obtained by applying only structural rules from $A \Rightarrow A$ must contain $A \Rightarrow A$ itself.

If H is obtained by applying (internal or external) structural rules finally, we can assume that the proof is the following form:

$$\begin{array}{c}\vdots\\ \dfrac{G}{H}\ \text{(a structural rule)}\end{array}$$

Therefore, if $\mathcal{M}$ is provable from H by applying only structural rules, $\mathcal{M}$ is provable from G by applying only structural rules too. It contradicts with induction hypothesis.

If H is obtained by applying ($\vee$ R) finally, we can assume that $H = G \mid \Gamma \Rightarrow b \vee c$ and the proof of $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash H$ is the following form:

$$\begin{array}{c}\vdots\\ \dfrac{G \mid \Gamma \Rightarrow x}{G \mid \Gamma \Rightarrow b \vee c}\end{array}$$

where x is b or c. Therefore, if $\mathcal{M}$ is provable from H by applying only structural rules, we have $\Gamma \subseteq \Gamma_{b\vee c}$ [1]. Therefore, $\Gamma \subseteq \Gamma_x$ which implies that the sequent $G \mid \Gamma_x \Rightarrow x$ is provable from $G \mid \Gamma \Rightarrow x$ by applying only structural rules. Consequently, $\mathcal{M}$ is provable from $G \mid \Gamma \Rightarrow x$ by applying only structural rules too. It contradicts with induction hypothesis.

If H is obtained by applying ($\vee$ L) finally, we can assume that $H = G \mid \Gamma, b \vee c \Rightarrow y$ and the proof of $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash H$ is the following form:

[1] We Ignore duplication of formulas in sequents. For example, $\{a,\ a,\ b,\ b,\ b\} \subseteq \{a,\ b,\ c\}$.

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ G \mid \Gamma, b \Rightarrow y & & G \mid \Gamma, c \Rightarrow y\end{matrix}}{G \mid \Gamma, b \vee c \Rightarrow y}$$

Therefore, if $\mathcal{M}$ is provable from H by applying only structural rules, we have three cases. (Case 1: $\Gamma \subseteq \Gamma_a$ and $y = a$) we have $\Gamma \cup \{b\} \subseteq \Gamma_a$. Therefore $\Gamma_a \Rightarrow a$ can be obtained from $\Gamma, b \Rightarrow y$ by applying only structural rules. It contradicts with induction hypothesis.
(Case 2: $\Gamma \subseteq \Gamma_b$ and $y = b$) we have $\Gamma \cup \{b\} \subseteq \Gamma_b$. Therefore $\Gamma_b \Rightarrow b$ can be obtained from $\Gamma, b \Rightarrow y$ by applying only structural rules. It contradicts with induction hypothesis.
(Case 3: $\Gamma \subseteq \Gamma_c$ and $y = c$) similarly to the case 2.

If H is obtained by applying ($\to$ R) finally, we can assume that $H = G \mid \Gamma \Rightarrow A \to B$. $\mathcal{M}$ cannot be obtained from it by applying only structural rules.

If H is obtained by applying ($\to$ L) finally, we can assume that $H = G \mid v \to w, \Gamma, \Delta \Rightarrow z$ and the proof of $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash H$ is the following form:

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ G \mid \Gamma \Rightarrow v & & G \mid w, \Delta \Rightarrow z\end{matrix}}{G \mid v \to w, \Gamma, \Delta \Rightarrow z}$$

Therefore, if $\mathcal{M}$ is provable from H by applying only structural rules, we have some cases. (Case 1: $z = b \vee c$ or $z = a \to b \vee c$) we have $v \to w = x \to d$ and $\Gamma \cup \Delta \cup \{x \to d\} \subseteq \Gamma_{b \vee c}$, where $x = b$ or c. Since $\Gamma_{b\vee c} \subseteq \Gamma_b$, Γ_c holds, $\Gamma_x \Rightarrow x$ is provable from $\Gamma \Rightarrow v$ by applying only structural rules. It contradicts with induction hypothesis.
(Case 2: $z = d$) Similarly to the case 1. (Case 3-1: $z = a$, $x \to y = a \to b \vee c$) we have $\Gamma \cup \Delta \cup \{a \to b \vee c\} \subseteq \Gamma_a$. Therefore, $\Gamma_a \Rightarrow a$ is provable from $\Gamma \Rightarrow v$ by applying only structural rules. It contradicts with induction hypothesis.
(Case 3-2: $z = a$, $v \to w = x \to d$, where $x = b$ or c) we have $\Gamma \cup \Delta \cup \{x \to d\} \subseteq \Gamma_a$. Therefore, $\Gamma_a \Rightarrow a$ is provable from $w, \Delta \Rightarrow z$ by applying only structural rules since $\Delta \cup \{d\} \subseteq \Gamma_a$. It contradicts with induction hypothesis.

(Case 4-1: $z = x$, $v \to w = a \to b \vee c$, where $x = b$ or c) we have $\Gamma \cup \Delta \cup \{a \to b \vee c\} \subseteq \Gamma_x$. Therefore, $\Gamma_x \Rightarrow x$ is provable from $\Gamma \Rightarrow v$ by applying only structural rules. It contradicts with induction hypothesis.
(Case 4-2: $z = x$, $v \to w = x \to d$, where $x = b$ or c) we have $\Gamma \cup \Delta \cup \{x \to d\} \subseteq \Gamma_x$. Therefore, $\Gamma_x \Rightarrow x$ is provable from $w, \Delta \Rightarrow z$ by applying only structural rules since $\Delta \cup \{d\} \subseteq \Gamma_x$. It contradicts with induction hypothesis.

Since $\mathbf{GLCW}^{cf}_{\mathcal{S}}$ has the subformula property, There are no applications of rules ($\wedge$L), ($\wedge$R), ($\neg$L) and ($\neg$R) in the proof of $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash H$.

We verified all cases. $\square$

Corollary 6.3.8. *Let $\vee \in \mathcal{S}$. The following hold.*

1. $\mathbf{GLCW}^{cf}_{\mathcal{S}} \vdash a \Rightarrow b \vee c \mid a \to b \vee c, b \to d, c \to d \Rightarrow d$;

2. $\mathbf{GLCW}_{\mathcal{S}} \vdash a \Rightarrow b \vee c \mid a \to b \vee c, b \to d, c \to d \Rightarrow d$.

Proof. The former follows from Lemma 6.3.7. The latter follows from the proof below:

$$\dfrac{\dfrac{\dfrac{\vdots}{a, a \to b \vee c \Rightarrow b \vee c}}{a \Rightarrow b \vee c \mid a \to b \vee c \Rightarrow b \vee c} \qquad \dfrac{\vdots}{b \vee c, b \to d, c \to d \Rightarrow d}}{a \Rightarrow b \vee c \mid a \to b \vee c, b \to d, c \to d \Rightarrow d.}$$

□

Theorem 6.3.9. $\mathbf{GLCW}_{\mathcal{S}}$ *does not admit the cut-elimination theorem if* $\vee \in \mathcal{S}$.

Theorem 6.3.10 (Avron[1]). *The cut-elimination theorem does not hold on* $\mathbf{GLCW}_{\mathcal{S}}$ *if* $\wedge \in \mathcal{S}$.

Proof. The following proof shows $\mathbf{GLCW}_{\mathcal{S}} \vdash p \Rightarrow q \mid q \Rightarrow p$:

$$\dfrac{\dfrac{\dfrac{p,\ q \Rightarrow p \wedge q}{p \Rightarrow p \wedge q \mid q \Rightarrow p \wedge q} \qquad p \wedge q \Rightarrow p}{p \Rightarrow p \wedge q \mid q \Rightarrow p} \qquad p \wedge q \Rightarrow q}{p \Rightarrow q \mid q \Rightarrow p}$$

However, $\mathbf{GLCW}^{cf}_{\mathcal{S}} \not\vdash p \Rightarrow q \mid q \Rightarrow p$ since, any hypersequent which does not consist any logical symbols is provable without cuts only if one of its components is of the form $\Gamma \Rightarrow p$, where $p \in \Gamma$. □

In conclusion, the following theorem is obtained.

Theorem 6.3.11. $\mathbf{GLCW}_{\mathcal{S}}$ *has the cut-elimination theorem if and only if* $\mathcal{S} \subseteq \{\to, \neg\}$ *(i.e.,* $\mathcal{S} = \{\to\},\ \{\to,\ \neg\}$*).*

6.4 Conservativity results for $\mathbf{GLCW}_{\mathcal{S}}$

In **GLCW**, the cut-elimination theorem implies the conservativity.

Let **G** be a hypersequent calculus and $\mathcal{S} \subseteq \mathcal{S}'$. We say $\mathbf{G}_{\mathcal{S}'}$ is a conservative extension of $\mathbf{G}_{\mathcal{S}}$ if $\mathbf{G}_{\mathcal{S}'} \vdash H$ implies $\mathbf{G}_{\mathcal{S}} \vdash H$ for every $\mathcal{S}$-hypersequent H.

Proposition 6.4.1. $\mathbf{GLCW}_{\{\to,\neg\}}$ *is a conservative extension of* $\mathbf{GLCW}_{\{\to\}}$.

Proof. If a $\{\to\}$-hypersequent H satisfies $\mathbf{GLCW}_{\{\to,\neg\}} \vdash H$, we have a cut-free proof $\mathcal{P}$ of $\mathbf{L}_{\mathcal{S}} \vdash \mathbf{GLCW}_{\{\to,\neg\}} \vdash H$. Since $\mathbf{GLCW}^{cf}_{\{\to,\neg\}}$ has the subformula property, $\mathcal{P}$ is a proof also of $\mathbf{GLCW}^{cf}_{\{\to\}} \vdash G$. Therefore, $\mathbf{GLCW}_{\{\to,\neg\}}$ is a conservative extension of $\mathbf{GLCW}_{\{\to\}}$. □

We give an alternative proof for Theorem 6.3.10 by using our result for conservativity. Recall that $\mathbf{GLCW}_{\mathcal{S}}$ and $\mathbf{H}_{\mathcal{S}} + (p \to q)\bar{\vee}((p \to q) \to q)$ are equivalent.

Lemma 6.4.2. $\mathbf{H}_{\{\to,\wedge\}} + (p \to q)\bar{\vee}((p \to q) \to q)$ *is not a conservative extension of* $\mathbf{H}_{\{\to\}} + (p \to q)\bar{\vee}((p \to q) \to q)$.

Proof. The $\{\to\}$-reduct of $\mathbf{M}_3$ (defined in pp.46, Figure 8) is an $\mathbf{H}_{\{\to\}} + (p \to q)\bar{\vee}((p \to q) \to q)$-algebra. However, $C(\mathbf{M}_3)$ is the $\{\to, \wedge\}$-reduct of $\mathbf{N}_3$ (defined in pp.46, figure-7). Therefore, $C(\mathbf{M}_3)$ is not an $\mathbf{H}_{\{\to,\wedge\}} + (p \to q)\bar{\vee}((p \to q) \to q)$. By Theorem 4.5.3, the lemma is proved. □

Therefore. by considering the contraposition of Proposition 6.4.1, we obtain the fact that the cut-elimination theorem does not hold on $\mathbf{GLCW}_{\mathcal{S}}$ if $\wedge \in \mathcal{S}$.

However, the converse does not necessarily hold. Precisely, the fact that

$$\mathbf{GLCW}_{\mathcal{S}} \text{ does not admit the cut-elimination theorem}$$

does not necessarily implies

$$\mathbf{GLCW}_{\mathcal{S}} \text{ is not a conservative extension of } \mathbf{GLCW}_{\{\to\}}.$$

Avron gave direct proofs for the conservativity ([1]) for the case $\mathcal{S} \subseteq \mathcal{S}' \subseteq \{\to, \vee, \neg\}$.

Theorem 6.4.3 (Avron[1], Theorem 4 and Corollary 1)**.** *Let* $\mathcal{S} \subseteq \mathcal{S}' \subseteq \{\to, \vee, \neg\}$. *then* $\mathbf{GLCW}_{\mathcal{S}'}$ *is a conservative extension of* $\mathbf{GLCW}_{\mathcal{S}}$.

6.5 Generalized splitting

Ciabattoni and Ferrari[5] gave a generalization of **GLC**. For $m \geq 1$, m-**GLC** is obtained by adding the following rule to **HLJ**:

$$\frac{G \mid \Gamma_0, \Gamma_1 \Rightarrow A_0 \ldots G \mid \Gamma_0, \Gamma_m \Rightarrow A_0 \ldots G \mid \Gamma_m, \Gamma_0 \Rightarrow A_m \ldots G \mid \Gamma_m, \Gamma_{m-1} \Rightarrow A_m}{G \mid \Gamma_0 \Rightarrow A_0 \mid \cdots \mid \Gamma_m \Rightarrow A_m} \,(Bw_m)$$

Theorem 6.5.1 (Ciabattoni and Ferrari[5])**.** *Let H be an $\mathcal{S}$-hypersequent. The following are equivalent:*

1. m-$\mathbf{GLC}_{\mathcal{S}} \vdash H$;
2. $\mathbf{HJ}_{\mathcal{S}} + \bigvee_{i=0}^{m}(p_i \to \bigvee_{j \neq i} p_j) \vdash T(H)$;
3. *every $\mathcal{S}$-algebra whose width is m or less validates $T(H)$.*

We define a generalization of **GLCW**. For $m \geq 1$, let m-**GLCW** be the hypersequent calculus obtained by adding the generalized splitting rule (**sp**-m) to $\mathbf{HLJ}_{\mathcal{S}}$ as follows:

$$\frac{\Gamma_i, \Gamma_j \Rightarrow A \ (i,\ j = 0, \ldots, m,\ i < j)}{\Gamma_1 \Rightarrow A \mid \cdots \mid \Gamma_m \Rightarrow A} \,(\text{sp-}m)$$

For example, (**sp**-2) is:

$$\frac{\Gamma_0, \Gamma_1 \Rightarrow A \qquad \Gamma_1, \Gamma_2 \Rightarrow A \qquad \Gamma_2, \Gamma_0 \Rightarrow A}{\Gamma_0 \Rightarrow A \mid \Gamma_1 \Rightarrow A \mid \Gamma_2 \Rightarrow A}\ (\mathbf{sp}\text{-}2)$$

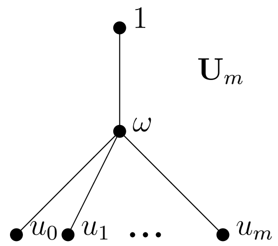
Similar to the case $m = 1$, ($\mathbf{sp}$-m) is considered a special case of (Bw_m), i.e., ($\mathbf{sp}$-m) and the case $A_0 = \cdots = A_m$ in (Bw_m) are equivalent rule.

$$\mathbf{CI}_m = \bar{\bigvee}_{k=1}^{m} \{((r_1 \to \cdots \to r_{k-1} \to (r_k \to s) \to s)\} \bar{\vee} (r_1 \to \cdots \to r_m \to s).$$

We have 1-$\mathbf{GLCW}$ = $\mathbf{GLCW}$ and $(p \to q) \vee ((p \to q) \to q) = \mathbf{CI}_1$. Notice that $\bar{\vee}$ in $\mathbf{CI}_m$ can be replaced with $\vee$ if $\vee \in \mathcal{S}$.

We show that m-$\mathbf{GLCW}$ and $\mathbf{H}_{\mathcal{S}} + \mathbf{CI}_m$ are equivalent.

Let $\mathbf{U}_m = \{1, \omega\} \cup \{u_0, \ldots, u_m\}$ be a $\{\to\}$-algebra defined by the following figure:

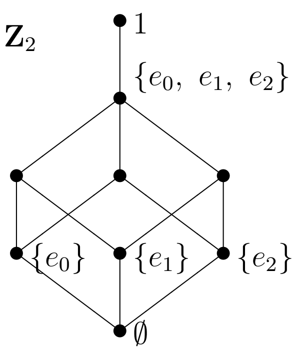


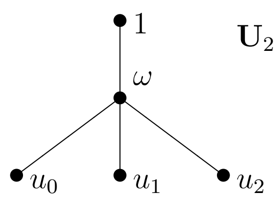
On $\mathbf{U}_m$, we define $x \to y = \begin{cases} 1 & (x \leq y) \\ y & (\text{otherwise}). \end{cases}$

Let $\mathbf{Z}_m = \{1\} \cup \mathcal{P}(\{e_0, \ldots, e_m\})$ be a $\{\to\}$-algebra defined by

$$x \to y = \begin{cases} 1 & (y = 1 \text{ or } x \subseteq y) \\ y & (x = 1) \\ x^c \cup y & (\text{otherwise}), \end{cases}$$

where $\{e_0, \ldots, e_m\}$ is an arbitrary set satisfying $|\{e_0, \ldots, e_m\}| = m + 1$.





Proposition 6.5.2. $\mathbf{H}_{\mathcal{S}} + X^{\{\to\}}_{\mathbf{U}_m}$ *is characterized by the class of* $\mathcal{S}$*-algebras* $\{\mathbf{M} \mid \mathbf{M}$ *is an* $\mathcal{S}$*-algebra such that* $(\text{width of } \mathbf{M}) \leq m\}$.

Proof. Let $\mathbf{M}$ be an $\mathcal{S}$-algebra satisfying (width of $\mathbf{M}$) $> m$. Then we have mutually incomparable elements $a_0, \ldots, a_m \in \mathbf{M}$. Therefore we have an embedding $h : \mathbf{Z}_m \to \mathbf{M}$ defined by $h(1) = 1$, $h(\omega) = \omega$ and $h(u_i) = a_i$. Consequently, $\mathbf{M}$ refutes $X^{\{\to\}}_{\mathbf{U}_m}$ by Theorem 4.1.5.

The converse follows from the fact that (width of $\mathbf{U}_m$) $= m + 1$. □

We show that $\mathbf{GLCW}_{\mathcal{S}}$ and $\mathbf{H}_{\mathcal{S}} + X^{\{\to\}}_{\mathbf{Z}_m}$ are equivalent if $\vee \notin \mathcal{S}$.

First, we construct an intermediate logic equivalent to $\mathbf{GLCW}_{\mathcal{S}}$ by a simple method. We obtain the formula $T^1_m = \{p_i \to p_j \to q \mid i, j = 0, \ldots, m,\ i < j\} \to \bar{\bigvee}^m_{k=0}(p_k \to q)$ by the inference rule (**sp**-m). However, the rule (**sp**-m) allows substituting a multiset of formulas in Γ_k, whereas T^1_m above only applies substituting a single formula in each p_k. We note that the case $\wedge \notin \mathcal{S}$ causes this problem since a multiset of formulas can be substituted in p_k if $\wedge \in \mathcal{S}$. For example, Let σ be the substitution defined by $\sigma(p_i) = B_1 \wedge B_2$ and $\sigma(p_j) = C_1 \wedge C_2$. Then we have $\sigma(p_i \to p_j \to q) = B_1 \wedge B_2 \to C_1 \wedge C_2 \to q = B_1 \to B_2 \to C_1 \to C_2 \to q$. Therefore, we define

$$T^a_m = \{p^1_i \to \cdots \to p^a_i \to p^1_j \to \cdots \to p^a_j \to q \mid i, j = 0, \ldots, m,\ i < j\} \to \bar{\bigvee}^m_{k=0}(p^1_k \to \cdots p^a_k \to q)$$

for every $a \geq 1$.

Lemma 6.5.3. *The following logics are equivalent:*

1. m-$\mathbf{GLCW}_{\mathcal{S}}$;

2. $\mathbf{H}_{\mathcal{S}} + \{T^a_m \mid a \geq 1\}$.

Proof. ($1 \implies 2$) is obvious. We show the converse. Let $\Gamma_i = \{p^1_i, \ldots, p^a_i\} \cup \{p^1_j \to \cdots \to p^a_j \to q \mid j \neq i\}$. Then we have m-$\mathbf{GLCW}_{\mathcal{S}} \vdash \Gamma_i,\ \Gamma_j \Rightarrow q$ if $i \neq j$. Thus, by applying (**sp**-m), we have m-$\mathbf{GLCW}_{\mathcal{S}} \vdash \Gamma_i \Rightarrow q$ for every $i = 0, \ldots, m$. So, for each i, we have:

$$\cfrac{\cfrac{\cfrac{G \mid \Gamma_i \Rightarrow q}{G \mid p^1_i, \ldots, p^a_i, \{p^1_i \to \cdots \to p^a_i \to p^1_j \to \cdots \to p^a_j \to q \mid j \neq i\} \Rightarrow q}}{G \mid \{p^1_j \to \cdots \to p^a_j \to q \mid j \neq i\} \Rightarrow p^1_i \to \cdots \to p^a_i \to q}}{G \mid \emptyset \Rightarrow T^a_m}$$

Therefore, we obtain m-$\mathbf{GLCW}_{\mathcal{S}} \vdash T^a_m$. The converse is proved. □

Lemma 6.5.4. *Let* $\mathbf{M}$ *be a finite* $\mathcal{S}$*-algebra and* g *be a refutation of* T^a_m *on* $\mathbf{M}$*. Then, there is an* $\mathcal{S}$*-homomorphism* h *and* $x \in h(\mathbf{M})$ *satisfying* $r(x) \geq m+1$ *(the definition of* $r(x)$ *is on Section 3.5).*

Proof. By Proposition 3.4.34, we have an $\mathcal{S}$-homomorphism h such that $h(\mathbf{M})$ is subdirectly irreducible and $h \circ g(T^a_m) = \omega$. Thus, by Proposition 3.4.33, we have

$$h \circ g(p^1_i \to \cdots \to p^a_i \to p^1_j \to \cdots \to p^a_j \to q) = 1$$

for every $i,\ j = 0, \dots, m$ $(i < j)$ and

$$h \circ g(p_k^1 \to \cdots p_k^a \to q) \neq 1$$

for every $k = 0, \dots, m$. We show that $r(h \circ g(q)) \geq m+1$. If not, we can assume $r(h \circ g(q)) = \{c_1, \dots, c_m\}$. By Theorem 3.5.3, we have

$$\begin{aligned}
&r(h \circ g(p_k^1 \to \cdots p_k^a \to q)) \\
&= r(h \circ g(p_k^1) \to \cdots h \circ g(p_k^a) \to h \circ g(q)) \\
&= r(h \circ g(q)) - (I(h \circ g(p_k^1) \cup \cdots \cup I(h \circ g(p_k^a)).
\end{aligned}$$

Therefore, for every $k = 0, \dots, m$, we have $f(k) \in \{1, \dots, m\}$ satisfying $c_{f(k)} \notin I(h \circ g(p_k^1) \cup \cdots \cup I(h \circ g(p_k^a)$. However, $f : \{0, \dots, m\} \longrightarrow \{1, \dots, m\}$ cannot be injective. Therefore, there are $s, t \in \{0, \dots, m\}$ satisfying $f(s) = f(t)$. Consequently, we obtain that

$$\begin{aligned}
&r(h \circ g(p_s^1 \to \cdots \to p_s^a \to p_t^1 \to \cdots \to p_t^a \to q)) \\
&= r(q) - (I(h \circ g(p_s^1) \cup \cdots \cup I(h \circ g(p_s^a)) \cup (I(h \circ g(p_t^1) \cup \cdots \cup I(h \circ g(p_t^a)) \\
&\ni c_{f(s)}.
\end{aligned}$$

Therefore, we obtain $h \circ g(p_s^1 \to \cdots \to p_s^a \to p_t^1 \to \cdots \to p_t^a \to q) \neq 1$, contradiction. □

Lemma 6.5.5. *If $\vee \notin \mathcal{S}$, the following logics are equivalent for every m:*

1. $\mathbf{H}_{\mathcal{S}} + X_{\mathbf{Z}_m}^{\{\to\}}$
2. $\mathbf{H}_{\mathcal{S}} + \{T_m^a \mid a \geq 1\}$
3. $\mathbf{H}_{\mathcal{S}} + T_m^1$

Proof. $(1 \subseteq 3)$ By Proposition 3.4.11, it is sufficient to show $V(\mathbf{H}_{\mathcal{S}} + T_m^1) \subseteq V(X_{\mathbf{Z}_m}^{\{\to\}})$ (recall that $V(\mathbf{L})$ is the class of all $\mathbf{L}$-algebras for a given $\mathcal{S}$-logic $\mathbf{L}$). Let $\mathbf{M}$ be an $\mathcal{S}$-algebra satisfying $\mathbf{M} \notin V(X_{\mathbf{Z}_m}^{\{\to\}})$. Then we have an $\mathcal{S}$-homomorphism f and a $\{\to\}$-embedding $h : \mathbf{Z}_m \longrightarrow f(\mathbf{M})$ by Theorem 4.1.5. Let v be a valuation on $f(\mathbf{M})$ defined by $v(p_i^1) = h(\{e_i\})$ and $v(q) = h(\emptyset)$. Then, we can easily verify $v(T_m^1) = h(\{e_0, \dots, e_m\}) \neq 1$. Therefore, $f(\mathbf{M})$ also refutes T_m^1 and it implies $\mathbf{M}$ refutes T_m^1 by Theorem 3.3.13. We proved the contraposition.

$(3 \subseteq 2)$ is trivial.

$(2 \subseteq 1)$ By Proposition 3.4.11, it is sufficient to show $V(X_{\mathbf{Z}_m}^{\{\to\}}) \subseteq V(\mathbf{H}_{\mathcal{S}} + T_m^a)$. Let $\mathbf{M}$ be an $\mathcal{S}$-algebra satisfying $\mathbf{M} \notin V(\mathbf{H}_{\mathcal{S}} + T_m^a)$. Then there is a valuation g on $\mathbf{M}$ such that $g(T_m^b) \neq 1$ for some $b \geq 1$. Since T_m^b is a $\{\to\}$-formula, we obtain a finite $\{\to\}$-subalgebra $\mathbf{N} \subseteq \mathbf{M}$ which refutes T_m^b by Theorem 4.4.2. Therefore, by Lemma 6.5.4, we have $\{\to\}$-homomorphism $h : \mathbf{N} \longrightarrow h(\mathbf{N})$ and $x \in h(\mathbf{N})$ satisfying $r(x) \geq m+1$. Therefore we can assume that $r(x) = \{c_0, \dots, c_m\} \cup D$ for $D - \{c_0, \dots, c_m\} \neq \emptyset$. Then, the element $y = D \to x$ satisfies that $r(y) = \{c_0, \dots, c_m\}$. Moreover, for each subset $\Gamma \subseteq \{c_0, \dots, c_m\}$, we obtain the element $e(\Gamma) = (\{c_0, \dots, c_m\} - \Gamma) \to y$ such that $r(e(\Gamma)) = \Gamma$ by Theorem 3.5.3. Similarly, by Theorem 3.5.3, we have

$$r(e(\Gamma) \to e(\Sigma)) = \begin{cases} \Sigma - \Gamma & (\Sigma \not\subseteq \Gamma) \\ \{1\} & (\Sigma \subseteq \Gamma). \end{cases}$$

Therefore, we obtain

$$e(\Gamma) \to e(\Sigma) = \begin{cases} e(\Sigma - \Gamma) & (\Sigma \not\subseteq \Gamma) \\ 1 & (\Sigma \subseteq \Gamma). \end{cases}$$

by Proposition 3.5.5. Therefore, $\{e(\Gamma) \mid \Gamma \subseteq \{c_0, \dots, c_m\}\} \cup \{1\}$ is $\{\to\}$-isomorphic to $\mathbf{Z}_m$. Consequently, by Theorem 4.1.5, $\mathbf{N}$ refutes $X^{\{\to\}}_{\mathbf{Z}_m}$. Thus, $\mathbf{M}$ also refutes $X^{\{\to\}}_{\mathbf{Z}_m}$. We proved the contraposition. □

Lemma 6.5.6. *If $\vee \notin \mathcal{S}$, the following logics are equivalent:*

1. $\mathbf{H}_\mathcal{S} + X^{\{\to\}}_{\mathbf{Z}_m}$;
2. $\mathbf{H}_\mathcal{S} + \mathbf{CI}_m$;
3. $\mathbf{H}_\mathcal{S} + T^1_m$.

Proof. ($1 \subseteq 2$) By Proposition 3.4.11, it is sufficient to show that $V(\mathbf{H}_\mathcal{S} + \mathbf{CI}_m) \subseteq V(\mathbf{H}_\mathcal{S} + X^{\{\to\}}_{\mathbf{Z}_m})$. Let $\mathbf{M}$ be an $\mathcal{S}$-algebra satisfying $\mathbf{M} \notin V(\mathbf{H}_\mathcal{S} + X^{\{\to\}}_{\mathbf{Z}_m})$. Then we have there is an $\mathcal{S}$-homomorphism f and a $\{\to\}$-embedding $h : \mathbf{Z}_m \longrightarrow f(\mathbf{M})$ by Theorem 4.1.5. Let v be a valuation on $f(\mathbf{M})$ defined by $v(p_i) = h(\{e_0, \dots, e_m\} - \{e_i\})$ and $v(q) = h(\emptyset)$. Then we obtain $v(\mathbf{CI}_m) = h(\{e_0, \dots, e_m\}) \neq 1$. Therefore, $f(\mathbf{M})$ refutes $\mathbf{CI}_m$. Thus $\mathbf{M}$ also refutes $\mathbf{CI}_m$. We proved the contraposition.

($2 \subseteq 3$) It follows from the fact $\mathbf{H}_\mathcal{S} + T^1_m \vdash \mathbf{CI}_m$, which is obtained by a substitution σ defined by the following (σ satisfies $\sigma(\mathbf{CI}_m) = T^m_1$):

$$\begin{aligned} \sigma(r_i) &= \{((p_1 \to \dots \to p_{k-1} \to (p_k \to q) \to q)\} \ (i = 1, \dots, m); \\ \sigma(r_0) &= p_1 \to \dots \to p_m \to q; \\ \sigma(s) &= q. \end{aligned}$$

($3 \subseteq 1$) is already shown in Lemma 6.5.5. □

Corollary 6.5.7. *The following a hypersequent calculi and logics are equivalent for every $\mathcal{S}$ (i.e., even if $\vee \in \mathcal{S}$):*

1. $\mathbf{H}_\mathcal{S} + X^{\{\to\}}_{\mathbf{Z}_m}$;
2. $\mathbf{H}_\mathcal{S} + \{T^a_m \mid a \geq 1\}$;
3. $\mathbf{H}_\mathcal{S} + T^1_m$;

4. $\mathbf{H}_\mathcal{S} + \mathbf{CI}_m$.

Proof. Lemma 6.5.5 and 6.5.6 shows that these logics are equivalent if $\vee \notin \mathcal{S}$, especially, the case $\mathcal{S} = \{\to\}$ (notice that the each of the above four logics is axiomatized by a $\{\to\}$-formula). Thus, these formula, $X^{\{\to\}}_{\mathbf{Z}_m}$, $\{T^a_m \mid a \geq 1\}$, T^1_m and $\mathbf{CI}_m$ are provable in the all above four logics for every $\mathcal{S}$. Therefore, we obtain the corollary. □

The above corollary and Lemma 6.5.3 implies the following conclusion.

Theorem 6.5.8. *For every $\mathcal{S}$, The following a hypersequent calculi and logics are equivalent.*

1. m-$\mathbf{GLCW}_\mathcal{S}$;
2. $\mathbf{H}_\mathcal{S} + \mathbf{CI}_m$;
3. $\mathbf{H}_\mathcal{S} + T^1_m$;
4. $\mathbf{H}_\mathcal{S} + X^{\{\to\}}_{\mathbf{Z}_m}$.

We obtain cut-elimination theorem m-$\mathbf{GLCW}_\mathcal{S}$ in the same way as the case $m = 1$.

Theorem 6.5.9. m-$\mathbf{GLCW}_\mathcal{S}$ *admits the cut-elimination theorem if* $\mathcal{S} \subseteq \{\to, \neg\}$.

In the case $\vee \in \mathcal{S}$, the formula $\mathbf{CI}_m$ can be applied in the method in Lemma 6.3.7. Let $\Gamma_0 = \{a_1, \ldots, a_m\}$ and $\Gamma_i = \{a_1, \ldots, a_{i-1}, a_i \to b \vee c\}$ for $i = 1, \ldots, m$. Then, we obtain the following theorem for every m in the same way as the case $m = 1$.

Lemma 6.5.10. *If $\vee \in \mathcal{S}$, the following hypersequent is provable in m-$\mathbf{GLCW}_\mathcal{S}$ with the* (**cut**) *rule. However, without the* (**cut**) *rule, the following hypersequent is not provable:*

$$\Gamma_0 \Rightarrow b \vee c \mid \Gamma_2 \Rightarrow b \vee c \mid \cdots \mid \Gamma_m \Rightarrow b \vee c \mid a_1 \to b \vee c, b \to d, c \to d \Rightarrow d.$$

Theorem 6.5.11. m-$\mathbf{GLCW}_\mathcal{S}$ *does not admit the cut-elimination theorem if* $\vee \in \mathcal{S}$.

6.6 Conclusion

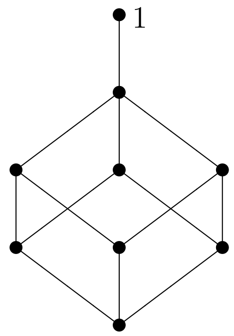
On $\mathbf{GLCW}_\mathcal{S}$, the $\mathcal{S}$-reduct of the hypersequent calculus characterized by all linear Kripke frames, we revised Avron's cut-elimination theorem for $\mathbf{GLCW}_{\{\to,\ \vee\}}$ and $\mathbf{GLCW}_{\{\to,\ \vee,\ \neg\}}$. Also, for $m \geq 2$, we defined m-$\mathbf{GLCW}_\mathcal{S}$, which are generalizations of $\mathbf{GLCW}_\mathcal{S}$ and proved that Avron's almost results still hold on m-$\mathbf{GLCW}_\mathcal{S}$ for any m.

Our idea is based on the fact that, a finite $\mathcal{S}$-algebra $\mathbf{M}$ is linear if and only if every element $\mathbf{M}$ is indecomposable. So, we define m-$\mathbf{GLCW}_\mathcal{S}$ by the concept of that, a finite $\mathcal{S}$-algebra $\mathbf{N}$ is an m-$\mathbf{GLCW}_\mathcal{S}$-algebra if and only if every element of $\mathbf{N}$ can be represented by m indecomposable elements, i.e., every $x \in \mathbf{M}$ satisfies $|r(x)| \leq m$ (see Section 3.5). Therefore, we cannot generalization Avron's result on the part of conservativity problem (Section 6.4) since the fact $\mathbf{M}$ is m-$\mathbf{GLCW}_\mathcal{S}$-algebra may not imply the fact $\mathbf{M}^{\{\vee\}}$ is m-$\mathbf{GLCW}_{\mathcal{S}\cup\{\vee\}}$-algebra.

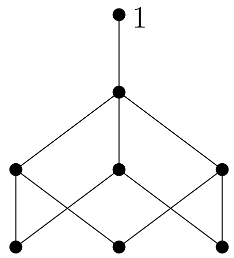
Thus, we have the following question.

Question. *Let $\mathcal{S} = \{\to\}$ or $\{\to, \neg\}$ and $m \geq 2$. Is m-$\mathbf{GLCW}_{\mathcal{S}\cup\{\vee\}}$ conservative extension of m-$\mathbf{GLCW}_{\mathcal{S}}$?*

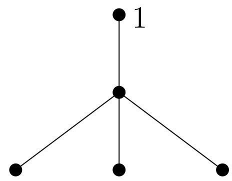
Also, in Chapter 5 and 6, we obtain two intermediate logics, m-$\mathbf{GLCW}_{\mathcal{S}}$ and Gabbay-de Jongh logic $\mathbf{D}_m$. We proved that they are equivalent to $\mathbf{H} + X_{\mathbf{Z}_m}^{\to}$ and $\mathbf{H} + \mathbf{T}_m$ respectively. It is trivial that $\mathbf{T}_m$ can be embeddable in $\mathbf{Z}_m$, which implies m-$\mathbf{GLCW}_{\mathcal{S}} \subseteq \mathbf{D}_m$. Therefore we can consider $\mathbf{M}_m$, middle algebra between $\mathbf{Z}_m$ and $\mathbf{T}_m$, whose diagram is on the figure below.



$\mathbf{Z}_2$



$\mathbf{M}_2$



$\mathbf{T}_2$

Question. *Let $m \geq 2$. Can we apply our technique and obtain result in the middle algebra $\mathbf{M}_m$. Precisely:*

1. *Is there a set of structural rules $\mathbf{R}$ such that a hypersequent calculus $\mathbf{HLJ} + \mathbf{R}$ is equivalent to $\mathbf{H} + X_{\mathbf{M}_m}^{\{\to\}}$?*

2. *Is there good definition of the set of $\mathbf{H} + X_{\mathbf{M}_m}^{\{\to\}}$-algebras?*

3. *Does conservativity condition holds on $\mathbf{H} + X_{\mathbf{M}_m}^{\{\to\}}$ between $\mathcal{S}$ and $\mathcal{S}'$ such that $\mathcal{S} \subseteq \mathcal{S}'$?*

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