Multiple-brane Solutions and Singular Gauge Transformations in Open String Field Theory

(開弦の場の理論における多重ブレーン解と特異なゲージ変換)

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Chapter 1

Introduction

The ultimate goal of the elementary particle physics is to understand the fundamental dynamical variables and the laws of physics governing their dynamics. According to the present understanding, except for the gravitational interaction, the standard model describes particle physics by using the framework of the quantum field theory. However, in this framework, fundamental particles are basically regarded as points, i.e., zero-dimensional objects, and there exist divergences coming from the quantum effects in the ultra-violet region. In the case of the gravitational interaction, because of the serious divergences, quantization based on the standard quantum field theory is not available. Hence, constructing the framework of quantum gravity is one of the most important theme in the research area of the elementary particle physics.

String theory is a candidate for the theory including the quantum gravity. This theory avoids the above-mentioned divergences by treating "particles" as strings, i.e., one-dimensional objects. However, at present, we have not yet understood any satisfactory formulation of string theory which does not rely on the perturbation theory. String field theory (SFT) is a candidate for such a non-perturbative formulation of string theory. In 1980's, two types of the covariant open bosonic SFT actions were proposed. The first action was constructed in [1], and the second one was constructed in [2]. These two types of the actions adopted different definitions of the interaction of strings. We only consider the latter type of the interaction in this thesis, i.e., the midpoint interaction. This action was extended to the supersymmetric case in [3]. However, this supersymmetric action is problematic concerning the existence of the picture changing operators (PCOs). Then, in order to avoid the problem, a modified version of the action was proposed in [4–6], and the theory defined by this action is called the modified cubic superstring field theory. However, this action again is problematic with respect to the gauge fixing because of the PCO.[1] In 1990's, another action for string fields in the Neveu–Schwarz (NS) sector was constructed by Berkovits in [8, 9]. This action is free from the problem of the PCOs because it does not use any PCOs.

SFT has been applied to analyses of the phenomenon called the tachyon condensation. It is a transition from an unstable vacuum to another stable vacuum, and it requires non-perturbative analyses. The vacuum in which the tachyon field has condensed is

[1] A recent development related to this issue is discussed in [7].

called the tachyon vacuum, and it is a non-trivial classical solution of the equation of motion (EOM) of SFT. Sen conjectured [10–12] that the value of the potential energy at the tachyon vacuum is lower than the trivial vacuum by the value of the tension of an unstable D-brane. After this conjecture was proposed, the calculations in studying the tachyon condensation had been investigated by using the numerical technique [13] known as the level truncation, and also analytic classical solutions had been searched in the bosonic cubic SFT. Finally, in 2005, Schnabl found the analytic tachyon vacuum solution [14]. Although the first form of the solution is complicated, now we have a simpler form of the solution written in terms of the so-called KBc algebra which was introduced by Okawa [15]. While the tachyon vacuum solution reproduces the energy of the vacuum without any D25-brane, Murata and Schnabl proposed a multiple-brane solution, and claimed that it reproduces the energy of the vacuum with n D25-branes [16, 17]. This solution is constructed by using a singular gauge transformation [18–20], and hence in general, the solution includes singular string fields. Therefore, it requires regularization in order for the solution to be defined properly [16, 17, 21, 22]. After the appropriate regularization, the multiple-brane solution is valid only when $n = 0, 1, 2$.[2] More recently, Erler and Maccaferri proposed another type of multiple-brane solution (EM solution) [25], which indeed can describe solutions with arbitrary number of D-branes. The EM solution is an extension of the solution by Kiermaier, Okawa and Soler (KOS solution) [26], which is also an extension of the solution based on the marginal deformation. Both of the EM solution and the KOS solution are constructed by using the boundary condition changing operators (BCCOs).

In this thesis, we discuss new multiple-brane solutions, by using singular gauge transformations in three different theories. First, we discuss string fields which are constructed by using singular gauge transformations for the EM solution in the bosonic cubic SFT. We will give a support for the expectation that the singular gauge transformation creates a D25-brane. Second, we discuss string fields which are constructed by using the singular gauge transformation for the half-brane solution constructed by Erler [27] in the modified cubic SFT. The energy of the half-brane solution is known to coincide with one half the tension of a D9-brane. Although this solution might not have any physical significance, the fact that the solution uses the extension of the KBc algebra is interesting. Third, we discuss a string field which is constructed by using the singular gauge transformation for the tachyon vacuum solution found by Erler [28] in the Berkovits' SFT.

This thesis is organized as follows. In chapter 2, we review the bosonic cubic SFT. We introduce the action for this theory, the KBc algebra and the pure-gauge-form solution to construct the classical analytic solution of the EOM. Then, we further review the tachyon vacuum solution, the EM solution and the multiple-brane solution. In chapter 3, we discuss candidates for the new solutions by performing singular gauge transformations for the EM solution. Here, the gauge parameter is the same as the one in the pure-gauge form of the "simple" tachyon vacuum solution [29]. We evaluate the energy of the solution and also the gauge invariant observable (GIO) [30]. Then, we study a concrete example of

[2] In this thesis we consider the singularity from $K = 0$, and $n \geqslant 0$. Other attempts can be found in [23, 24].

the solution describing a D24-brane placed on a D25-brane. In chapter 4, we review the modified cubic SFT. We consider the non-GSO-projected action so that the trivial vacuum can correspond to an unstable D9-brane. We introduce an algebra in the modified cubic SFT which includes string fields whose definition is based on the superconformal ghost γ and the supercurrent G. We review some known solutions in the modified cubic SFT, i.e., the tachyon vacuum solution and the half-brane solution. In chapter 5, we discuss a solution obtained by performing a singular gauge transformation whose gauge parameter is taken to be the same as the one in the pure-gauge form of the half-brane solution. Since the solution includes a singular string field, we introduce a G_ϵ-regularization as the K_ϵ-regularization. We check the EOMS and evaluate the energy. In chapter 6, we review the Berkovits' SFT and the tachyon vacuum solution in this theory. In chapter 7, we discuss a candidate for the solution by performing a singular gauge transformation whose gauge parameter is appeared in the tachyon vacuum solution in the bosonic cubic SFT, the modified cubic SFT and the Berkovits' SFT. We try to evaluate the energy and derive its integral form. However, since the integral is rather complicated we do not reach the result. Then alternatively, we try to evaluate the GIO. The chapter 8 is devoted to the conclusion. Some detailed derivations of correlators and algebras, and also the detailed calculations of energies and EOMS are given in the appendices.

Chapter 2

Review of the Bosonic Cubic String Field Theory

2.1 Cubic Action

We review the bosonic cubic SFT. Since the bosonic theory does not include the fermion, it cannot describe our universe, i.e., it should be regarded as a kind of a toy model. However, such a toy model is important in order to understand essential physics and to develop methods of analyses of more realistic theories including fermions.

First, we review structure of the action, and next, we give definitions of building blocks of the action by using conformal field theory (CFT). See e.g., [31–35], as textbooks and pedagogiacal reviews.

2.1.1 Cubic Action

Let us consider the physical state condition:

$$Q\Psi = 0, \tag{2.1.1}$$

where Ψ is a ghost number one string field which is a state in the Hilbert space $\mathcal{H}$ of CFT:

$$\Psi \in \mathcal{H}, \tag{2.1.2}$$

and Q is a Grassmann-odd operator called the BRST (Becchi–Rouet–Stora–Tyutin) operator:

$$Q : \mathcal{H} \to \mathcal{H}. \tag{2.1.3}$$

Since the BRST operator Q is nilpotent:

$$Q^2 = 0, \tag{2.1.4}$$

the physical state condition is invariant under the following gauge transformation:

$$\delta_\Lambda \Psi = Q\Lambda, \tag{2.1.5}$$

where the gauge parameter Λ is a ghost-number-zero string field. We use the BPZ (Belavin–Polyakov–Zamolodchikov) inner product:

$$\langle \bullet, \bullet \rangle : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}, \tag{2.1.6}$$

which satisfies the following properties

$$\langle \varphi_1, \varphi_2 \rangle = (-)^{\epsilon(\varphi_1)\epsilon(\varphi_2)} \langle \varphi_2, \varphi_1 \rangle, \tag{2.1.7}$$

$$\langle Q\varphi_1, \varphi_2 \rangle + (-)^{\epsilon(\varphi_1)} \langle \varphi_1, Q\varphi_2 \rangle = 0. \tag{2.1.8}$$

Here, $\epsilon(\varphi)$ is equal to 0 for a Grassmann-even φ and equal to 1 for a Grassmann-odd φ.

We can construct the free action by using the BRST operator and the BPZ inner product:

$$S_{\text{free}}(\Psi) = -\frac{1}{2} \langle \Psi, Q\Psi \rangle, \tag{2.1.9}$$

which is invariant under (2.1.5):

$$\delta_\Lambda S_{\text{free}}(\Psi) = -\frac{1}{2} \langle Q\Lambda, Q\Psi \rangle - \frac{1}{2} \langle \Psi, Q^2 \Lambda \rangle = 0,$$

where we use the nilpotency of the BRST operator (2.1.4) and the property of the BPZ inner product (2.1.8). By taking the variation, we obtain the equation of motion (EOM) which reproduces the physical state condition (2.1.1).

Next, we introduce interactions by using a product between string fields. The product is called the star product:

$$* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \tag{2.1.10}$$

and the gauge transformation is extended as

$$\delta_\Lambda \Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi. \tag{2.1.11}$$

Witten constructed an action [2]:

$$S(\Psi) = -\frac{1}{2} \langle \Psi, Q\Psi \rangle - \frac{1}{3} \langle \Psi, \Psi * \Psi \rangle, \tag{2.1.12}$$

which is invariant under the gauge transformation (2.1.11). Necessary condtions for the invariance are the cyclicity of the BPZ inner product, the associativity of the $*$ product and the Leibniz rule of the BRST operator Q:

$$\langle \varphi_1, \varphi_2 * \varphi_3 \rangle = \langle \varphi_1 * \varphi_2, \varphi_3 \rangle, \tag{2.1.13}$$

$$(\varphi_1 * \varphi_2) * \varphi_3 = \varphi_1 * (\varphi_2 * \varphi_3) = \varphi_1 * \varphi_2 * \varphi_3 \tag{2.1.14}$$

$$Q(\varphi_1 * \varphi_2) = (Q\varphi_1) * \varphi_2 + (-)^{\epsilon(\varphi_1)} \varphi_1 * Q\varphi_2, \tag{2.1.15}$$

in addition to the properties (2.1.4), (2.1.7) and (2.1.8):

$$\begin{aligned}\delta_\Lambda S(\Psi) &= -\langle \delta_\Lambda \Psi, Q\Psi\rangle - \langle \delta_\Lambda \Psi, \Psi * \Psi\rangle \\ &= -\langle Q\Lambda, Q\Psi\rangle - \langle \Psi * \Lambda, Q\Psi\rangle + \langle \Lambda * \Psi, Q\Psi\rangle \\ &\quad - \langle Q\Lambda, \Psi * \Psi\rangle - \langle \Psi * \Lambda, \Psi * \Psi\rangle + \langle \Lambda * \Psi, \Psi * \Psi\rangle \\ &= -\langle \Lambda, Q\Psi * \Psi\rangle + \langle \Lambda, \Psi * Q\Psi\rangle - \langle Q\Lambda, \Psi * \Psi\rangle \\ &\quad - \langle \Lambda, \Psi * \Psi * \Psi\rangle + \langle \Lambda, \Psi * \Psi * \Psi\rangle \\ &= 0. \end{aligned} \tag{2.1.16}$$

Here, from the cyclicity (2.1.7) and (2.1.13), we can derive the following relation:

$$\langle \varphi_1, \varphi_2 * \varphi_3\rangle = (-)^{\epsilon(\varphi_3)(\epsilon(\varphi_1)+\epsilon(\varphi_2))}\langle \varphi_3, \varphi_1 * \varphi_2\rangle = (-)^{\epsilon(\varphi_1)(\epsilon(\varphi_2)+\epsilon(\varphi_3))}\langle \varphi_2, \varphi_3 * \varphi_1\rangle, \tag{2.1.17}$$

and we used the equations. We also consider a finite gauge trasformation:

$$\begin{aligned}\Psi^{\mathrm{g}} := \sum_{n=0}^{\infty} \frac{1}{n!}{\delta_\Lambda}^n \Psi &= \left(1 - \Lambda + \frac{1}{2!}\Lambda * \Lambda - \cdots\right) * \left(Q\Lambda + \frac{1}{2!}Q(\Lambda * \Lambda) + \cdots\right) \\ &\quad + \left(1 - \Lambda + \frac{1}{2!}\Lambda * \Lambda - \cdots\right) * \Psi * \left(1 + \Lambda + \frac{1}{2!}\Lambda * \Lambda + \cdots\right) \\ &= e^{-\Lambda} * Qe^{\Lambda} + e^{-\Lambda} * \Psi * e^{\Lambda}, \end{aligned} \tag{2.1.18}$$

where $e^{s\Lambda} := 1 + s\Lambda + \frac{s^2}{2!}\Lambda * \Lambda + \cdots$, $(s \in \mathbb{C})$. When we define $u := e^{\Lambda}$ and $u^{-1} = e^{-\Lambda}$, we can rewrite the finite gauge transformation as

$$\Psi \xrightarrow{\;u\;} \Psi^{\mathrm{g}} = u^{-1}(Q + \Psi)u, \tag{2.1.19}$$

where $u^{-1}(Q+\Psi)u$ is an abbreviation for $u^{-1} * Qu + u^{-1} * \Psi * u$, and the arrow with u represents the gauge transformation whose gauge parameter is u. For the finite gauge transformation, the action becomes

$$S(\Psi^{\mathrm{g}}) = S(\Psi) + S(u^{-1} * Qu). \tag{2.1.20}$$

Let us show this. We define the kinetic term of the action S_{kin} and the interaction term S_{int}:

$$S_{\mathrm{kin}}(\Psi) := -\frac{1}{2}\langle \Psi, Q\Psi\rangle, \quad S_{\mathrm{int}}(\Psi) := -\frac{1}{3}\langle \Psi, \Psi * \Psi\rangle. \tag{2.1.21}$$

First, we perform the finite gauge transformation (2.1.19) for the kinetic term:

$$\begin{aligned}S_{\mathrm{kin}}(\Psi^{\mathrm{g}}) &= -\frac{1}{2}\langle u^{-1}(Q+\Psi)u, Q(u^{-1}(Q+\Psi)u)\rangle \\ &= S_{\mathrm{kin}}(u^{-1} * Qu) + S_{\mathrm{kin}}(\Psi) \\ &\quad - \langle Qu * Qu^{-1}, \Psi\rangle - \langle u * Qu^{-1}, \Psi * \Psi\rangle. \end{aligned} \tag{2.1.22}$$

Here we used

$$Qu^{-1} * u = -u^{-1} * Qu, \qquad \because Q(u^{-1} * u) = 0. \tag{2.1.23}$$

Second, we perform the finite gauge transformation (2.1.19) for the interaction term:

$$\begin{aligned} S_{\text{int}}(\Psi^{\text{g}}) &= -\frac{1}{3}\langle u^{-1}(Q+\Psi)u, u^{-1}(Q+\Psi)u * u^{-1}(Q+\Psi)u\rangle \\ &= S_{\text{int}}(u^{-1} * Qu) + S_{\text{int}}(\Psi) \\ &\quad + \langle Qu * Qu^{-1}, \Psi\rangle - \langle Qu * u^{-1}, \Psi * \Psi\rangle. \end{aligned} \tag{2.1.24}$$

Here we used

$$\begin{aligned} &\langle \varphi_1 + \varphi_2, (\varphi_1 + \varphi_2) * (\varphi_1 + \varphi_2)\rangle \\ &= \langle \varphi_1, \varphi_1 * \varphi_1\rangle + 3\langle \varphi_1 * \varphi_1, \varphi_2\rangle + 3\langle \varphi_1, \varphi_2 * \varphi_2\rangle + \langle \varphi_2, \varphi_2 * \varphi_2\rangle. \end{aligned} \tag{2.1.25}$$

Therefore, combining the two terms, we obtain

$$\begin{aligned} S(\Psi^{\text{g}}) &= S_{\text{kin}}(\Psi^{\text{g}}) + S_{\text{int}}(\Psi^{\text{g}}) \\ &= S_{\text{kin}}(u^{-1} * Qu) + S_{\text{kin}}(\Psi) - \langle Qu * Qu^{-1}, \Psi\rangle - \langle u * Qu^{-1}, \Psi * \Psi\rangle \\ &\quad + S_{\text{int}}(u^{-1} * Qu) + S_{\text{int}}(\Psi) + \langle Qu * Qu^{-1}, \Psi\rangle - \langle Qu * u^{-1}, \Psi * \Psi\rangle \\ &= S(\Psi) + S(u^{-1} * Qu). \end{aligned} \tag{2.1.26}$$

Let us show that $S(u^{-1} * Qu)$ vanishes. First, the action can be written as

$$S(u^{-1} * Qu) = -\frac{1}{6}\langle u^{-1} * Qu, Qu^{-1} * Qu\rangle. \tag{2.1.27}$$

Next, we introduce u_τ s.t. $u_0 = 1$ and $u_1 = u$:

$$u_\tau := e^{\tau\Lambda}. \tag{2.1.28}$$

We consider the following quantity:

$$C(\tau) = \langle {u_\tau}^{-1} * Qu_\tau, Q{u_\tau}^{-1} * Qu_\tau\rangle. \tag{2.1.29}$$

By considering $\partial_\tau c(\tau)$, we find

$$\begin{aligned} \partial_\tau C(\tau) &= \langle -\Lambda * {u_\tau}^{-1} * Qu_\tau, Q{u_\tau}^{-1} * Qu_\tau\rangle + \langle {u_\tau}^{-1} * Q(\Lambda * u_\tau), Q{u_\tau}^{-1} * Qu_\tau\rangle \\ &\quad + \langle {u_\tau}^{-1} * Qu_\tau, Q(-\Lambda * {u_\tau}^{-1}) * Qu_\tau\rangle + \langle {u_\tau}^{-1} * Qu_\tau, Q{u_\tau}^{-1} * Q(\Lambda u_\tau)\rangle \\ &= -\langle \Lambda * {u_\tau}^{-1} * Qu_\tau, Q{u_\tau}^{-1} * Qu_\tau\rangle \\ &\quad + \langle {u_\tau}^{-1} * Q\Lambda * u_\tau, Q{u_\tau}^{-1} * Qu_\tau\rangle + \langle {u_\tau}^{-1} * \Lambda * Qu_\tau, Q{u_\tau}^{-1} * Qu_\tau\rangle \\ &\quad - \langle {u_\tau}^{-1} * Qu_\tau, Q\Lambda * {u_\tau}^{-1} * Qu_\tau\rangle - \langle {u_\tau}^{-1} * Qu_\tau, \Lambda * Q{u_\tau}^{-1} * Qu_\tau\rangle \\ &\quad + \langle u_\tau^{-1} * Qu_\tau, Q{u_\tau}^{-1} * Q\Lambda * u_\tau\rangle + \langle u_\tau^{-1} * Qu_\tau, Q{u_\tau}^{-1} * \Lambda * Qu_\tau\rangle \\ &= -3\langle \Lambda, Q^2(u_\tau * Q{u_\tau}^{-1})\rangle \\ &= 0, \end{aligned} \tag{2.1.30}$$

where we used the property of the BPZ inner product and Q, $[Q, \partial_\tau] = 0$ and $u_\tau * \Lambda - \Lambda * u_\tau = {u_\tau}^{-1} * \Lambda - \Lambda * {u_\tau}^{-1} = 0$. Because $C(0) = 0$, we have

$$C(1) = \langle u^{-1}Qu, Qu^{-1} * Qu\rangle = 0. \tag{2.1.31}$$

Therefore, we obtain

$$S(u^{-1} * Qu) = 0. \tag{2.1.32}$$

By taking a variation of the action, we obtain the EOM:

$$\begin{aligned}
\delta_\Psi S(\Psi) &= -\frac{1}{2}\langle\delta\Psi, Q\Psi\rangle - \frac{1}{2}\langle\Psi, Q\delta\Psi\rangle \\
&\quad - \frac{1}{3}\langle\delta\Psi, \Psi * \Psi\rangle - \frac{1}{3}\langle\Psi, \delta\Psi * \Psi\rangle - \frac{1}{3}\langle\Psi, \Psi * \delta\Psi\rangle \\
&= -\langle\delta\Psi, Q\Psi\rangle - \langle\delta\Psi, \Psi * \Psi\rangle \\
&= 0
\end{aligned} \tag{2.1.33}$$

$$\Leftrightarrow \quad \forall\delta\Psi, \quad Q\Psi + \Psi * \Psi = 0. \tag{2.1.34}$$

2.1.2 Definitions by Using CFT

We give definitions of building blocks of the action of SFT by using CFT. See e.g., [36–38], for textbooks on CFT.

The action of the string world sheet[1] is given by:

$$S_{\mathrm{CFT}} = \frac{1}{2\pi}\int d^2z \left(\partial X^\mu \bar{\partial} X_\mu(z,\bar{z}) + b\bar{\partial}c(z,\bar{z}) + \tilde{b}\partial\tilde{c}(z,\bar{z})\right). \tag{2.1.35}$$

where $\mu = 0, \cdots, 25$. This dimension is decided by demanding the nilpotency of the BRST operator, $Q^2 = 0$. A string field φ is defined as the sum of the states of CFT:

$$\begin{aligned}
\varphi := \int \frac{d^{26}k}{(2\pi)^{26}} \Big(& T(k)c_1|0;k\rangle + A_\mu(k)\alpha^\mu_{-1}c_1|0;k\rangle + \frac{i}{\sqrt{2}}B(k)c_0|0;k\rangle \\
& + \cdots + C(k)b_{-2}c_0c_1|0;k\rangle + \cdots \Big),
\end{aligned} \tag{2.1.36}$$

where α_n, c_n and b_n are defined as[2]

$$\alpha^\mu_n := \sqrt{-2}\oint \frac{dz}{2\pi i z} z^{n+1}\partial X^\mu(z), \tag{2.1.37}$$

$$c_n := \oint \frac{dz}{2\pi i z} z^{n-1}c(z), \tag{2.1.38}$$

$$b_n := \oint \frac{dz}{2\pi i z} z^{n+2}b(z). \tag{2.1.39}$$

Here, the conformal weight of $\partial X(z)$, $c(z)$ and $b(z)$ is 1, -1 and 2, respectively. These

[1] In this thesis, we use $\alpha' = 1$.
[2] We use the doubling trick to define the contour integrals.

operators satisfy the following (anti-)commutation relations:

$$[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{2.1.40}$$

$$\{b_m, c_n\} = \delta_{m+n,0}. \tag{2.1.41}$$

The state $|0;k\rangle$ is defined by $|0;k\rangle = e^{ik_\mu X^\mu}|0\rangle$, where $|0\rangle$ is the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum:

$$\alpha_n|0\rangle = 0, \quad n \geqslant 0, \tag{2.1.42}$$

$$c_n|0\rangle = 0, \quad n \geqslant 2, \tag{2.1.43}$$

$$b_n|0\rangle = 0, \quad n \geqslant -1. \tag{2.1.44}$$

$T(k)$ and $A_\mu(k)$, etc. are the component fields; $T(k)$ is a tachyon field and $A_\mu(k)$ is a gauge field.

A conformal transformation of a weight h primary field is defined as

$$f \circ \Phi(z) := \left(\frac{df(z)}{dz}\right)^h \Phi\big(f(z)\big). \tag{2.1.45}$$

A correlator (N point function) of the CFT is defined as

$$\langle\, \varphi_1(\xi_1)\cdots\varphi_N(\xi_N)\,\rangle_\Sigma := \int D\varphi \prod_{n=1}^{N} \varphi_n(\xi_n) e^{-S_{\mathrm{CFT}}}, \tag{2.1.46}$$

where Σ represents a two dimensional surface. The conformal invariance of the correlator is described as

$$\langle\, \varphi_1(\xi_1)\cdots\varphi_N(\xi_N)\,\rangle_\Sigma = \langle\, f\circ\varphi_1(\xi_1)\cdots f\circ\varphi_N(\xi_N)\,\rangle_{f(\Sigma)}, \tag{2.1.47}$$

where $f(\Sigma)$ represents a transformed two dimensional surface.

By using the state-operator correspondence, the BPZ inner product is defined as follows:

$$\langle\varphi_1, \varphi_2\rangle := \langle\, I\circ\varphi_1(0)\varphi_2(0)\,\rangle_{\mathrm{UHP}}. \tag{2.1.48}$$

Here, the subscript UHP represents the upper half plane, and a conformal transformation $I(\xi)$ is the inversion:

$$I : \mathrm{UHP} \to \mathrm{UHP}, \tag{2.1.49}$$

$$I(\xi) = -\frac{1}{\xi}. \tag{2.1.50}$$

The UHP coordinate z is related to the strip coordinate (τ, σ) through the conformal transformation:

$$z = e^{\tau + i\sigma}. \tag{2.1.51}$$

We note that I is an involution:

$$I \circ I = \mathrm{id}\ , \tag{2.1.52}$$

then, the cyclicity of the BPZ inner product is shown as

$$\begin{aligned}\langle\, \varphi_1, \varphi_2 \,\rangle &= \langle\, I \circ \varphi_1(0)\varphi_2(0) \,\rangle_{\text{UHP}} \\ &= \langle\, I^2 \circ \varphi_1(0) I \circ \varphi_2(0) \,\rangle_{\text{UHP}} \\ &= (-)^{\epsilon(\varphi_1)\epsilon(\varphi_2)} \langle\, I \circ \varphi_2(0)\varphi_1(0) \,\rangle_{\text{UHP}} \\ &= (-)^{\epsilon(\varphi_1)\epsilon(\varphi_2)} \langle \varphi_2, \varphi_1 \rangle, \end{aligned} \tag{2.1.53}$$

where in the second line we used the conformal invariance of the correlator and in the third line we used the fact that I is an involution.

Next, we give the definition of the BRST operator:

$$Q := \oint \frac{dz}{2\pi i} j_{\text{B}}(z), \tag{2.1.54}$$

$$j_{\text{B}}(z) := -cT^{\text{m}}(z) + bc\partial c(z) + \frac{3}{2}\partial^2 c(z), \tag{2.1.55}$$

where $T^{\text{m}}(z)$ is the matter energy-momentum tensor:

$$T^{\text{m}}(z) := -\partial X^\mu \partial X_\mu(z). \tag{2.1.56}$$

The total energy-momentum tensor $T(z)$ is given by $T(z) = T^{\text{m}}(z) + T^{\text{g}}(z)$, where the ghost energy-momentum tensor $T^{\text{g}}(z)$ is

$$T^{\text{g}}(z) := \partial b \cdot c(z) - 2\partial(bc)(z). \tag{2.1.57}$$

The energy-momentum tensior $T(z)$ is the generator of the conformal transformation. The BRST operator Q is nilpotent iff the spacetime dimension is equal to 26. We can show the nilpotency by using the following operator product expansions (OPEs):

$$\partial X^\mu(z)\partial X^\nu(0) \sim \frac{-\frac{1}{2}\eta^{\mu\nu}}{z^2}, \tag{2.1.58}$$

$$b(z)c(0) \sim \frac{1}{z}, \tag{2.1.59}$$

where $\eta^{\mu\nu}$ is the Lorentz metric. The property (2.1.8) can be derived by using CFT:

$$\begin{aligned}\langle Q\varphi_1, \varphi_2 \rangle &= \langle\, I \circ \left(\oint_0 \frac{dz}{2\pi i} j_{\text{B}}(z)\varphi_1(0) \right) \varphi_2(0) \,\rangle_{\text{UHP}} \\ &= -(-)^{\epsilon(\varphi_1)} \langle\, I \circ \varphi_1(0) \left(\oint_0 \frac{dI(z)}{2\pi i} j_{\text{B}}(I(z))\varphi_2(0) \right) \rangle_{\text{UHP}} \\ &= -(-)^{\epsilon(\varphi_1)} \langle \varphi_1, Q\varphi_2 \rangle, \end{aligned} \tag{2.1.60}$$

where the subscript z of $\oint_z$ represents that the contour encircles the point z.

We give the definition of the star product:

$$\langle \varphi_1, \varphi_2 * \varphi_3 \rangle := \langle\, f_1 \circ \varphi_1(0) f_2 \circ \varphi_2(0) f_3 \circ \varphi_3(0) \,\rangle_{\text{UHP}}, \tag{2.1.61}$$

where $f_j(\xi) := \tan(\frac{2}{3}(\arctan\xi + c_j))$, $c_j = \frac{\pi}{2}(2-j)$. Note that each string midpoint in the UHP, $\xi = i$, maps to itself:

$$f_j(i) = \tan\left(\frac{2}{3}(\arctan i + c_j)\right) = i. \tag{2.1.62}$$

We also consider the following conformal transformation:

$$\tilde{f}(\xi) := \tan\left(\arctan z + \frac{\pi}{3}\right). \tag{2.1.63}$$

This conformal transformation satisfies the following equations:

$$\tilde{f}\circ f_1(\xi) = f_2(\xi), \quad \tilde{f}\circ f_2(\xi) = f_3(\xi), \quad \tilde{f}\circ f_3(\xi) = f_1(\xi). \tag{2.1.64}$$

By using these equations, we can show the cyclicity:

$$\begin{aligned}\langle\varphi_1, \varphi_2 * \varphi_3\rangle &= \langle\, \tilde{f}\circ f_1\circ\varphi_1(0)\tilde{f}\circ f_2\circ\varphi_2(0)\tilde{f}\circ f_3\circ\varphi_3(0)\,\rangle_{\mathrm{UHP}} \\ &= (-)^{\epsilon(\varphi_3)(\epsilon(\varphi_1)+\epsilon(\varphi_2))}\langle\, f_1\circ\varphi_3(0)f_2\circ\varphi_1(0)f_3\circ\varphi_2(0)\,\rangle_{\mathrm{UHP}} \\ &= (-)^{\epsilon(\varphi_3)(\epsilon(\varphi_1)+\epsilon(\varphi_2))}\langle\varphi_3, \varphi_1 * \varphi_2\rangle \\ &= \langle\varphi_1 * \varphi_2, \varphi_3\rangle.\end{aligned} \tag{2.1.65}$$

By using the deformation of the contour in the BRST operator Q, we can show that Q is a derivative:

$$\begin{aligned}\langle\varphi_1, Q(\varphi_2 * \varphi_3)\rangle &= \langle\, f_1\circ\varphi_1(0)\oint_{\mathrm{C}_{2,3}}\frac{dz}{2\pi i}\Big(j_{\mathrm{B}}(z)f_2\circ\varphi_2(0)f_3\circ\varphi_3(0)\Big)\,\rangle_{\mathrm{UHP}} \\ &= \langle\, f_1\circ\varphi_1(0)\left(\oint_{f_2(0)}\frac{dz}{2\pi i}j_{\mathrm{B}}(z)f_2\circ\varphi_2(0)\right)f_3\circ\varphi_3(0)\,\rangle_{\mathrm{UHP}} \\ &\quad + (-)^{\epsilon(\varphi_2)}\langle\, f_1\circ\varphi_1(0)f_2\circ\varphi_2(0)\left(\oint_{f_3(0)}\frac{dz}{2\pi i}j_{\mathrm{B}}(z)f_3\circ\varphi_3(0)\right)\,\rangle_{\mathrm{UHP}} \\ &= \langle\varphi_1, Q\varphi_2 * \varphi_3\rangle + (-)^{\epsilon(\varphi_2)}\langle\varphi_1, \varphi_2 * Q\varphi_3\rangle,\end{aligned} \tag{2.1.66}$$

where the contour $\mathrm{C}_{2,3}$ encloses $f_2(0)$ and $f_3(0)$.

We have demonstrated the properties of the BRST operator, the BPZ inner product and the star product which are needed for the gauge invariance of the action by using CFT.

2.2 KBc Algebra

The EOM of the bosonic cubic SFT:

$$Q\Psi + \Psi * \Psi = 0,$$

consists of the BRST operator Q and the star product $*$. To construct solutions of the EOM, we define a special set of string fields closed under Q and $*$. Okawa found the set of string fields [15], which is defined in a useful coordinate of CFT [39].

2.2.1 Sliver Frame

We introduce the sliver frame [39] which is a useful coordinate system to describe the star product. The sliver frame is defined by the following conformal transformation:

$$f_{\mathrm{s}} : \mathrm{UHP} \to \mathrm{sliver},$$
$$f_{\mathrm{s}}(\xi) := \frac{2}{\pi} \arctan \xi. \tag{2.2.1}$$

Note that the midpoint of the string in the sliver frame is given by

$$f_{\mathrm{s}}(i) = i\infty, \tag{2.2.2}$$

and the upper unit semi-circle $\xi = e^{i\theta}$, $\theta \in [0, \pi]\backslash\{\frac{\pi}{2}\}$ in the UHP is mapped to the semi-infinite vertical lines:

$$f_{\mathrm{s}}(e^{i\theta}) = \begin{cases} \frac{1}{2} + is, \; s \geqslant 0 & \theta \in [0, \frac{\pi}{2}), \\ -\frac{1}{2} + is, \; s \geqslant 0 & \theta \in (\frac{\pi}{2}, \pi], \end{cases} \tag{2.2.3}$$

and the origin maps to the origin

$$f_{\mathrm{s}}(0) = 0. \tag{2.2.4}$$

The upper half unit disk in the UHP maps to a "sliver."[3]

The star product can be realized by placing "slivers" side by side. In the correlator, the left edge of the "sliver" and the right edge is glued, then it becomes a semi-infinite cylinder. We have the identification $z \simeq z + L$ where L is the circumference of the sliver in the correlator.

Next, we introduce the wedge state. First we consider the following correlator:

$$\langle \varphi_{\mathrm{test}}, \, |0\rangle \rangle = \langle \, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) 1 \, \rangle_{\mathrm{C}_2} = \langle \, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \, \rangle_{\mathrm{C}_2}, \tag{2.2.5}$$

where the subscript C_2 denotes that the coordinate system is the sliver frame and the circumference of the cylinder is equal to 2. The state $|0\rangle$ in the left-hand side is the $\mathrm{SL}(2, \mathbb{R})$ invariant vacuum and by applying the state-operator correspondence, it corresponds to the operator 1 in the right-hand side.

Let us increase the number of the state $|0\rangle$ by one in the left-hand side:

$$\langle \varphi_{\mathrm{test}}, \, |0\rangle * |0\rangle \rangle = \langle \, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \, \rangle_{\mathrm{C}_3}, \tag{2.2.6}$$

then the circumference is increased by 1 because the state $|0\rangle$ has the width 1 in the sliver frame. Therefore,

$$\langle \varphi_{\mathrm{test}}, \, |0\rangle * |0\rangle * \cdots * |0\rangle \rangle = \langle \, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \, \rangle_{\mathrm{C}_{n+1}}, \tag{2.2.7}$$

[3] Sliver means a slender fragment.

where $n \in \mathbb{Z}_{>0}$ is the number of $|0\rangle$ in the left-hand side. Next we extend the result as

$$\langle \varphi_{\text{test}}, \Omega^{\alpha} \rangle := \langle\, f_{\text{s}} \circ \varphi_{\text{test}}(0) \,\rangle_{\text{C}_{\alpha+1}}, \tag{2.2.8}$$

where

$$\alpha \in \mathbb{R}_{\geqslant 0}. \tag{2.2.9}$$

The string field Ω^{α} is called the wedge state. The wedge state satisfies the following equation:

$$\langle \varphi_{\text{test}}, \Omega^{\alpha} * \Omega^{\beta} \rangle = \langle\, f_{\text{s}} \circ \varphi_{\text{test}}(0) \,\rangle_{\text{C}_{\alpha+\beta+1}} = \langle \varphi_{\text{test}}, \Omega^{\alpha+\beta} \rangle. \tag{2.2.10}$$

It is known that the limit Ω^{∞} is finite and this string field is called the sliver state. The string field Ω^{0} is the identity string field under the star product:

$$\langle \varphi_{\text{test}}, \Omega^{0} * \varphi \rangle = \langle \varphi_{\text{test}}, \varphi * \Omega^{0} \rangle = \langle \varphi_{\text{test}}, \varphi \rangle = \langle\, t \circ f_{\text{s}} \circ \varphi_{\text{test}}(0) f_{\text{s}} \circ \varphi(0) \,\rangle_{\text{C}_{2}}, \tag{2.2.11}$$

where the conformal transformation t is the translation $z \to z+1$ in the sliver frame. This gives a precise definition of the identity string field 1, which appeared in (2.1.18). We introduce a Tr:

$$\begin{aligned} \text{Tr} &: \mathcal{H} \to \mathbb{C}, \\ \text{Tr}[\bullet] &:= \langle \Omega^{0}, \bullet \rangle. \end{aligned} \tag{2.2.12}$$

If the star product of the two string fields is input in Tr, it is same as the BPZ inner product.

$$\text{Tr}[\varphi_1 * \varphi_2] = \langle \Omega^{0}, \varphi_1 * \varphi_2 \rangle = \langle \Omega^{0} * \varphi_1, \varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle = \langle\, t \circ f_{\text{s}} \circ \varphi_1(0) f_{\text{s}} \circ \varphi_2(0) \,\rangle_{\text{C}_2} \tag{2.2.13}$$

2.2.2 Definition of String Fields K, B, c

We introduce the string fields K, B, and c [15] defined in the sliver frame. These string fields are closed under Q and $*$. Because of the form of the EOM, these string fields are useful to construct solutions of the EOM. First we define the string field K:

$$\text{Tr}[\varphi_{\text{test}} * K] := \langle\, f_{\text{s}} \circ \varphi_{\text{test}}(0) \int_{\downarrow \frac{1}{2}} \frac{dz}{2\pi i} T(z) \,\rangle_{\text{C}_1}, \tag{2.2.14}$$

where we define

$$\int_{\downarrow x} := \int_{x+i\infty}^{x-i\infty}, \quad \int_{\uparrow x} := \int_{x-i\infty}^{x+i\infty}, \quad x \in \mathbb{R}. \tag{2.2.15}$$

We can find that $\Omega^\alpha = e^{\alpha K}$ from (2.2.10), (2.2.11) and the following equation:

$$
\begin{aligned}
&\partial_\alpha \mathrm{Tr}[\varphi_{\mathrm{test}} * \Omega^\alpha]|_{\alpha\to 0} \\
&= \partial_\alpha \langle f_{\alpha+1\to 1} \circ f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \rangle_{\mathrm{C}_1}|_{\alpha\to 0} \\
&= \partial_\alpha \Big\{ \langle f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \rangle_{\mathrm{C}_1} - \alpha \langle \oint_0 \frac{dz}{2\pi i} z T(z) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \rangle_{\mathrm{C}_1} + \mathrm{O}(\alpha^2) \Big\}\Big|_{\alpha\to 0} \\
&= -\langle f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow 1-0} \frac{dz}{2\pi i}(z-1)T(z) \rangle_{\mathrm{C}_1} + \langle f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow +0} \frac{dz}{2\pi i} z T(z) \rangle_{\mathrm{C}_1} \\
&= \langle f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow +\frac{1}{2}} \frac{dz}{2\pi i} T(z) \rangle_{\mathrm{C}_1} \\
&= \mathrm{Tr}[\varphi_{\mathrm{test}} * K].
\end{aligned} \tag{2.2.16}
$$

Here, we used the conformal transformation of the scaling:

$$
f_{\alpha\to\beta}(z) := \frac{\beta}{\alpha} z, \tag{2.2.17}
$$

in the form

$$
f_{\alpha+1\to 1}(z) = \frac{1}{\alpha+1} z = z - \alpha z + \mathrm{O}(\alpha^2), \tag{2.2.18}
$$

and the periodicity $z \simeq z+1$ in the $\langle \bullet \rangle_{\mathrm{C}_1}$. Next, we define the string field B:

$$
\mathrm{Tr}[\varphi_{\mathrm{test}} * B] := \langle f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}} \frac{dz}{2\pi i} b(z) \rangle_{\mathrm{C}_1}. \tag{2.2.19}
$$

In the end, we define the string field c:

$$
\mathrm{Tr}[\varphi_{\mathrm{test}} * c] := \langle f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) c(\tfrac{1}{2}) \rangle_{\mathrm{C}_1}. \tag{2.2.20}
$$

In the next subsection, we demonstrate the algebras.

2.2.3 Algebra

Operations of the BRST operator Q for the string fields K, B and c are

$$
QB = K, \quad QK = 0, \quad Qc = c * \partial c = c * K * c. \tag{2.2.21}
$$

Here, we define

$$
\partial c := K * c - c * K. \tag{2.2.22}
$$

Algebras among K, B and c are as follows:

$$
[K, B] = 0, \quad \{B, B\} = \{c, c\} = 0, \quad \{B, c\} = 1, \tag{2.2.23}
$$

where we define

$$
[\varphi_1, \varphi_2] := \varphi_1 * \varphi_2 - \varphi_2 * \varphi_1, \quad \{\varphi_1, \varphi_2\} := \varphi_1 * \varphi_2 + \varphi_2 * \varphi_1. \tag{2.2.24}
$$

We demonstrate the above equations (2.2.21) and (2.2.23) by using the OPEs. First, we list the OPE with the BRST current j_{B}:

$$j_{\mathrm{B}}(z)j_{\mathrm{B}}(0) \sim 0 \times \frac{1}{z}c\partial^3 c + \cdots, \quad j_{\mathrm{B}}(z)b(0) \sim \frac{T(0)}{z}, \quad j_{\mathrm{B}}(z)c(0) \sim \frac{c\partial c(0)}{z}. \tag{2.2.25}$$

By using these, we have the operation of Q for K, B, and c:

$$\begin{aligned}\mathrm{Tr}[\varphi_{\mathrm{test}} * QB] &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \oint_w \frac{dz}{2\pi i} \int_{\downarrow \frac{1}{2}} \frac{dw}{2\pi i} j_{\mathrm{B}}(z) b(w) \,\rangle_{\mathrm{C}_1} \\ &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}} \frac{dw}{2\pi i} T(w) \,\rangle_{\mathrm{C}_1} \\ &= \mathrm{Tr}[\varphi_{\mathrm{test}} * K],\end{aligned} \tag{2.2.26}$$

$$\mathrm{Tr}[\varphi_{\mathrm{test}} * QK] = \mathrm{Tr}[\varphi_{\mathrm{test}} * Q^2 B] = 0, \tag{2.2.27}$$

$$\begin{aligned}&\mathrm{Tr}[\varphi_{\mathrm{test}} * Qc] \\ &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \oint_{\frac{1}{2}} \frac{dz}{2\pi i} j_{\mathrm{B}}(z) c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \\ &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) c\partial c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \\ &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) c(\tfrac{1}{2}) \oint_{\frac{1}{2}} \frac{dz}{2\pi i} T(z) c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \\ &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) c(\tfrac{1}{2}) \left(\int_{\downarrow \frac{1}{2}-0} + \int_{\uparrow \frac{1}{2}+0} \right) \frac{dz}{2\pi i} T(z) c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \\ &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) c(\tfrac{1}{2}) \int_{\downarrow \frac{1}{2}-0} \frac{dz}{2\pi i} T(z) c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} - \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \left(c(\tfrac{1}{2})\right)^2 \int_{\downarrow \frac{1}{2}+0} \frac{dz}{2\pi i} T(z) \,\rangle_{\mathrm{C}_1} \\ &= \mathrm{Tr}[\varphi_{\mathrm{test}} * c * K * c].\end{aligned} \tag{2.2.28}$$

By using the following the OPEs:

$$b(z)b(0) \sim \mathrm{O}(z), \quad c(z)c(0) \sim \mathrm{O}(z), \quad b(z)c(0) \sim \frac{1}{z}, \tag{2.2.29}$$

and the Leibniz rule of Q, the remaining relations of the algebra are derived as

$$\begin{aligned}
\mathrm{Tr}[\varphi_{\mathrm{test}} * \{B,B\}] &= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}-0} \frac{dz}{2\pi i} \int_{\downarrow \frac{1}{2}} \frac{dw}{2\pi i} b(z) b(w) \,\rangle_{\mathrm{C}_1} \\
&\quad + \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}} \frac{dw}{2\pi i} \int_{\downarrow \frac{1}{2}+0} \frac{dz}{2\pi i} b(w) b(z) \,\rangle_{\mathrm{C}_1} \\
&= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}-0} \frac{dz}{2\pi i} \int_{\downarrow \frac{1}{2}} \frac{dw}{2\pi i} b(z) b(w) \,\rangle_{\mathrm{C}_1} \\
&\quad - (-)^{\epsilon(b)\epsilon(b)} \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}} \frac{dz}{2\pi i} \int_{\uparrow \frac{1}{2}+0} \frac{dw}{2\pi i} b(z) b(w) \,\rangle_{\mathrm{C}_1} \\
&= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}} \frac{dw}{2\pi i} \oint_{\frac{1}{2}} \frac{dz}{2\pi i} b(z) b(w) \,\rangle_{\mathrm{C}_1} \\
&= 0, \qquad (2.2.30)\\
\mathrm{Tr}[\varphi_{\mathrm{test}} * \{c,c\}] &= 2 \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) (c(\tfrac{1}{2}))^2 \,\rangle_{\mathrm{C}_1} = 0, \qquad (2.2.31)
\end{aligned}$$

$$0 = \mathrm{Tr}[\varphi_{\mathrm{test}} * Q(B * B)] = \mathrm{Tr}[\varphi_{\mathrm{test}} * (K * B - B * K)], \qquad (2.2.32)$$

$$\begin{aligned}
&\mathrm{Tr}[\varphi_{\mathrm{test}} * (B * c + c * B)] \\
&= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \int_{\downarrow \frac{1}{2}-0} \frac{dz}{2\pi i} b(z) c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} + \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) c(\tfrac{1}{2}) \int_{\downarrow \frac{1}{2}+0} \frac{dz}{2\pi i} b(z) \,\rangle_{\mathrm{C}_1} \\
&= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \left(\int_{\downarrow \frac{1}{2}-0} - \int_{\downarrow \frac{1}{2}+0} \right) \frac{dz}{2\pi i} b(z) c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \\
&= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \left(\int_{\downarrow \frac{1}{2}-0} + \int_{\uparrow \frac{1}{2}+0} \right) \frac{dz}{2\pi i} b(z) c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \\
&= \left\langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \oint_{\frac{1}{2}} \frac{dz}{2\pi i} \frac{1}{z - \frac{1}{2}} \,\right\rangle_{\mathrm{C}_1} \\
&= \langle\, f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) 1 \,\rangle_{\mathrm{C}_1} \\
&= \mathrm{Tr}[\varphi_{\mathrm{test}} * I]. \qquad (2.2.33)
\end{aligned}$$

Therefore, we obtain the KBc algebra:

$$\begin{aligned}
[K,B] = 0, \quad [K,c] = \partial c, \quad \{B,c\} = 1 \\
QB = K, \quad QK = 0, \quad Qc = c\partial c. \qquad (2.2.34)
\end{aligned}$$

2.3 Pure-gauge-form Solution

Let us consider the EOM:

$$Q\Psi + \Psi^2 = 0,$$

where we omit the star symbol $*$ hereafter. This equation is non-linear in terms of the string field Ψ, and a string field is a superposition of any state of CFT with ghost number 1. Therefore, it is difficult to find an analytic solution. However there exists a solution trivially satisfying the EOM, i.e.,

$$\Psi = 0. \tag{2.3.1}$$

We consider a regular gauge transformation u for this trivial solution, namely a trivial pure-gauge solution:

$$0 \xrightarrow{u} u^{-1}Qu =: \Psi_{\rm p}. \tag{2.3.2}$$

The trivial pure-gauge solution $\Psi_{\rm p}$ always satisfies the EOM algebraically:

$$\begin{aligned} Q\Psi_{\rm p} + {\Psi_{\rm p}}^2 &= Q(u^{-1}Qu) + u^{-1}Qu \cdot u^{-1}Qu \\ &= Qu^{-1} \cdot Qu - Qu^{-1} \cdot Qu \\ &= 0. \end{aligned} \tag{2.3.3}$$

Conversely, a solution $\Psi_{\rm s}$, i.e., $Q\Psi_{\rm s} + {\Psi_{\rm s}}^2 = 0$, can be always written as "pure-gauge form" formally [18, 40, 41], by using the homotopy operator A_1 s.t. $QA_1 = 1$. By taking the gauge parameter as

$$U_{\rm s} := 1 + A_1\Psi_{\rm s}, \tag{2.3.4}$$

we can construct the pure-gauge-form solution:

$$\begin{aligned} \because\ QU_{\rm s} = Q(A_1\Psi_{\rm s}) &= \Psi_{\rm s} - A_1 Q\Psi_{\rm s} \\ &= \Psi_{\rm s} + A_1{\Psi_{\rm s}}^2 \\ &= U_{\rm s}\Psi_{\rm s}, \end{aligned}$$

$$\Psi_{\rm s} = {U_{\rm s}}^{-1}QU_{\rm s}. \tag{2.3.5}$$

When a solution is pure-gauge form but is not pure gauge, the gauge transformation should be singular. We call such a gauge transformation the singular gauge transformation. For a singular gauge transformation U, we have

$$S(U^{-1}QU) \neq 0. \tag{2.3.6}$$

Indeed, as we will see, the tachyon vacuum solution can be written as the pure-gauge form, $U^{-1}QU$. However, the tachyon vacuum solution is not pure gauge. Namely, it is not gauge equivalent to the trivial solution 0 because of the energy or other property of the solution.

Let us express the pure-gauge-form solution by using the KBc algebra [15]. We choose the gauge parameter as

$$U(g) := Bc + cBg(K), \tag{2.3.7}$$

where $g(K)$ is the function of the string field K and the definition in the sliver frame will be discussed later. A product between these gauge parameters is as follows:

$$\begin{aligned} U(g)U(g') &= (Bc + cBg(K))(Bc + cBg'(K)) \\ &= BcBc + Bc^2Bg'(K) + cBg(K)Bc + cBg(K)cBg'(K) \\ &= Bc + cBg(K)g'(K) \\ &= U(gg'). \end{aligned} \tag{2.3.8}$$

Here, we use the following equations, which we will use frequently in the rest of the thesis:

$$BcB = B(1 - Bc) = B - B^2c = B, \tag{2.3.9}$$

$$cBc = c(1 - cB) = c - c^2B = c, \tag{2.3.10}$$

where the equations $\{B, c\} = 1$ and $B^2 = c^2 = 0$ are used, and we assume $[B, g(K)] = 0$, since $[B, K] = 0$. We can find the inverse of $U(g)$ formally:

$$(U(g))^{-1} = U(g^{-1}) = Bc + cBg(K)^{-1}, \tag{2.3.11}$$

$$\because \quad U(g)U(g^{-1}) = U(gg^{-1}) = U(1) = Bc + cB = 1. \tag{2.3.12}$$

Then, by using the explicit forms of $U(g)$ and $U(g)^{-1}$, the pure-gauge-form solution $\Psi(g) := U(g)^{-1}QU(g)$ can be written formally as follows:

$$\begin{aligned} \Psi(g) &= U(g)^{-1}Q\big(1 + cB(g(K) - 1)\big) \\ &= (Bc + cBg(K)^{-1})(-cBKc)(g(K) - 1) \\ &= -cBKg(K)^{-1}c(g(K) - 1). \end{aligned} \tag{2.3.13}$$

This is the pure-gauge-form solution using K, B, c which we study.

2.4 Tachyon Vacuum Solution

We review the tachyon vacuum solution written in the pure-gauge form found in [29], which is the "simple" solution, though Schnabl first gave another form of the analytic tachyon vacuum solution in [14]. The tachyon vacuum solution is the vacuum which is the result of the tachyon condensation, i.e., the phenomenon of the disappearance of an unstable D-brane. Since this phenomenon requires non-perturbative analyses, using SFT is essential for studies of this phenomenon. In general, the form of the solution and computations are simplest in the case of the tachyon vacuum solution among other known solutions.

2.4.1 Solution

The simple solution can be written in the pure-gauge form formally by choosing the gauge parameter as:

$$U_1 := Bc + cBG_1, \tag{2.4.1}$$

$$G_1 := \frac{-K}{1-K}. \tag{2.4.2}$$

We give the relation between the trivial vacuum $0 =: \Psi_1$ and the simple solution Ψ_0, and also the explicit form of the simple solution:

$$\begin{aligned} \Psi_1 \xleftarrow{U_1{}^{-1}} \Psi_0 &:= U_1{}^{-1} Q U_1 \\ &= -cB(1-K)c\frac{1}{1-K} \\ &= (Q(Bc) - c)\frac{1}{1-K}. \end{aligned} \tag{2.4.3}$$

Here, the arrow with $U_1{}^{-1}$ represents the gauge transformation whose gauge parameter is $U_1{}^{-1}$. The string field $\frac{1}{1-K}$, which is the function of the string field K, is defined by the Laplace transformation:

$$\frac{1}{1-K} := \int_0^\infty dx\, e^{-x}\Omega^x. \tag{2.4.4}$$

This means that the string field $\frac{1}{1-K}$ is the superposition ($\int_0^\infty dx$) of the wedge state (Ω^x) with the weight (e^{-x}).

In $U_1{}^{-1}$, there exist the string field $1/K$ and we just assume that this string field is the inverse of the string field K and that it is Q-closed. However, in the explicit form of the solution (2.4.3), the string field $1/K$ does not exist.

2.4.2 Energy

Let us calculate the energy of the tachyon vacuum solution. The energy of a solution is given by :

$$E(\Psi) := -S(\Psi) = \mathrm{Tr}\left[\frac{1}{2}\Psi Q \Psi + \frac{1}{3}\Psi^3\right]. \tag{2.4.5}$$

To compute the energy, we will use the following formulae:

$$\begin{aligned} \mathrm{Bcccc}[t_1,t_2,t_3,t_4] &:= \mathrm{Tr}[Bc\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}c\Omega^{t_4}] \\ &= -\frac{L^2}{4\pi^3}(t_3\sin 2\theta_{t_1} - (t_2+t_3)\sin 2\theta_{t_1+t_2} + t_2 \sin 2\theta_{t_1+t_2+t_3} \\ &\qquad + t_1 \sin 2\theta_{t_3} - (t_1+t_2)\sin 2\theta_{t_2+t_3} + (t_1+t_2+t_3)\sin 2\theta_{t_2}), \end{aligned} \tag{2.4.6}$$

$$\begin{aligned} \mathrm{Bcddd}[t_1,t_2,t_3] &:= \mathrm{Tr}[Bc\partial c\Omega^{t_1}\partial c\Omega^{t_2}\partial c\Omega^{t_3}] \\ &= -\frac{1}{\pi}(\sin 2\theta_{t_2} + \sin 2\theta_{t_3} - \sin 2\theta_{t_2+t_3}), \end{aligned} \tag{2.4.7}$$

where

$$\theta_x := \frac{\pi x}{L}, \tag{2.4.8}$$

and L is the circumference of the cylinder in the sliver frame. We will give the derivations of these in appendix A.

First, we calculate the kinetic term:

$$\begin{aligned}
-S_{\text{kin}}(\Psi_0) &= \frac{1}{2}\text{Tr}\left[\left(Q(Bc)-c\right)\frac{1}{1-K}Q\left(\left(Q(Bc)-c\right)\frac{1}{1-K}\right)\right] \\
&= \frac{1}{2}\text{Tr}\left[c\frac{1}{1-K}cKc\frac{1}{1-K}\right] \\
&= \frac{1}{2}\lim_{y\to 0}\partial_y \iint_0^\infty dx_1 dx_2\, e^{-(x_1+x_2)}\,\text{Bcccc}[x_1,y,x_2,0] \\
&= -\frac{3}{2\pi^2}.
\end{aligned} \tag{2.4.9}$$

Second, we calculate the interaction term:

$$\begin{aligned}
-S_{\text{int}}(\Psi_0) &= -\frac{1}{3}\text{Tr}\left[cB(1-K)c\frac{1}{1-K}cB(1-K)c\frac{1}{1-K}cB(1-K)c\frac{1}{1-K}\right] \\
&= -\frac{1}{3}\text{Tr}\left[Bc\partial c\frac{1}{1-K}\partial c\frac{1}{1-K}\partial c\frac{1}{1-K}\right] \\
&= -\frac{1}{3}\iiint_0^\infty dx_1 dx_2 dx_3\, e^{-(x_1+x_2+x_3)}\,\text{Bcddd}[x_1,x_2,x_3] \\
&= \frac{1}{\pi^2}.
\end{aligned} \tag{2.4.10}$$

Therefore, the energy of the tachyon vacuum solution Ψ_0 is

$$E(\Psi_0) = -S_{\text{kin}}(\Psi_0) - S_{\text{int}}(\Psi_0) = -\frac{1}{2\pi^2} = -T_{25}, \tag{2.4.11}$$

where T_{25} is the tension of the D25-brane [12]. The energy of the tachyon vacuum solution is lower than the perturbative vacuum by the tension of the D25-brane. Hence, Sen's conjecture has been proven.

2.4.3 Trivial Cohomology

Let us show another Sen's conjecture, i.e., there is no physical excitation around the tachyon vacuum solution [42]. We consider the homotopy operator A_0 and the shifted kinetic operator Q_0 around the tachyon vacuum solution Ψ_0:

$$\begin{aligned}
A_0 &:= -\frac{B}{1-K}, \\
Q_0\varphi &:= Q\varphi + \Psi_0 * \varphi - (-)^{\epsilon(\varphi)}\varphi * \Psi_0.
\end{aligned} \tag{2.4.12}$$

Since A_0 does not have any singularity, it is well-defined in the sliver frame. The homotopy operator becomes the identity string field with the operation of the shifted kinetic operator:

$$Q_0 A_0 = 1. \tag{2.4.13}$$

We can show that any Q_0-closed state φ can be written in the Q_0-exact form:

$$Q_0(A_0\varphi) = (Q_0A_0)\varphi - A_0Q_0\varphi = \varphi. \tag{2.4.14}$$

Since any Q_0-closed state around the tachyon vacuum solution is Q_0-exact, there is no physical excitation around the tachyon vacuum solution.

2.4.4 Gauge Invariant Observable

Let us discuss another gauge invariant quantity (not the action). It is called the Ellwood invariant or the gauge invariant observable (GIO) [30]. The definition of GIO is

$$\begin{aligned} W(\varphi, \mathcal{V}) &:= \langle\, \mathcal{V}(i) f_{\mathrm{E}} \circ \varphi(0)\, \rangle_{\mathrm{UHP}} \\ &= \langle\, \mathcal{V}(i\infty) f_{\mathrm{s}} \circ \varphi(0)\, \rangle_{\mathrm{C}_1}, \end{aligned} \tag{2.4.15}$$

where $\mathcal{V}$ is an on-shell closed string vertex operator $\mathcal{V} = c\tilde{c}V^{(1,1)}$, $V^{(1,1)}$ is a matter $(1,1)$ primary operator, and

$$\begin{aligned} f_{\mathrm{E}} &: \mathrm{UHP} \to \mathrm{UHP}, \\ f_{\mathrm{E}}(\xi) &= \frac{2\xi}{1-\xi^2}. \end{aligned} \tag{2.4.16}$$

We can show that the GIO $W(\varphi, \mathcal{V})$ is gauge invariant. First, since $\mathcal{V}$ is on-shell,

$$\begin{aligned} W(Q\Lambda, \mathcal{V}) &= \langle \oint_0 \frac{dz}{2\pi i} \mathcal{V}(i)\big(j_{\mathrm{BRS}}(z) f_{\mathrm{E}} \circ \mathcal{O}_\Lambda(0)\big)\, \rangle_{\mathrm{UHP}} \\ &= -\langle \oint_i \frac{dz}{2\pi i} \big(\mathcal{V}(i) j_{\mathrm{BRS}}(z)\big) f_{\mathrm{E}} \circ \mathcal{O}_\Lambda(0)\, \rangle_{\mathrm{UHP}} \\ &= 0, \end{aligned} \tag{2.4.17}$$

where $\mathcal{O}_\Lambda(0)$ is the operator corresponding to the state Λ. Second, $W(\varphi_1 * \varphi_2, \mathcal{V})$ has the cyclicity:

$$\begin{aligned} W(\varphi_1 * \varphi_2, \mathcal{V}) &= \langle\, \mathcal{V}(i) I \circ \varphi_1(0)\varphi_2(0)\, \rangle_{\mathrm{UHP}} \\ &= \langle\, I \circ \mathcal{V}(i) I \circ I \circ \varphi_1(0) I \circ \varphi_2(0)\, \rangle_{\mathrm{UHP}} \\ &= (-)^{\epsilon(\varphi_1)\epsilon(\varphi_2)} \langle\, \mathcal{V}(i) I \circ \varphi_2(0)\varphi_1(0)\, \rangle_{\mathrm{UHP}} \\ &= (-)^{\epsilon(\varphi_1)\epsilon(\varphi_2)} W(\varphi_2 * \varphi_1, \mathcal{V}), \end{aligned} \tag{2.4.18}$$

then we have

$$\begin{aligned} W([\varphi, \Lambda], \mathcal{V}) &= W(\varphi\Lambda, \mathcal{V}) - W(\varphi\Lambda, \mathcal{V}) \\ &= 0. \end{aligned} \tag{2.4.19}$$

Therefore, the GIO is gauge invariant:

$$\begin{aligned} W(\delta_\Lambda \varphi, \mathcal{V}) &= W(Q\Lambda + [\varphi, \Lambda], \mathcal{V}) \\ &= 0. \end{aligned} \tag{2.4.20}$$

Ellwood conjectured that the GIO for the solution Ψ satisfies the following equation:

$$W(\Psi, \mathcal{V}) = \mathcal{A}_*(\mathcal{V}) - \mathcal{A}_0(\mathcal{V}), \tag{2.4.21}$$

where $\mathcal{A}_i(\mathcal{V})$ is a closed string one-point function on the disk:

$$\begin{aligned} \mathcal{A}_0(\mathcal{V}) &:= \frac{1}{2\pi i}\langle\, \mathcal{V}(0)c(1)\,\rangle_{\rm disk}, \\ \mathcal{A}_*(\mathcal{V}) &:= \frac{1}{2\pi i}\langle\, \mathcal{V}(0)c(1)\,\rangle_{\rm disk, BCFT_*}. \end{aligned} \tag{2.4.22}$$

Here, $\mathcal{A}_0(\mathcal{V})$ is defined in the boundary conformal field theory (BCFT) corresponding to the perturbative vacuum, while $\mathcal{A}_*(\mathcal{V})$ is defined in the different BCFT corresponding to the solution Ψ. Indeed, this is true when Ψ is the tachyon vacuum solution Ψ_0. The tachyon vacuum solution is

$$\Psi_0 = -Q\Big(cB\frac{1}{1-K}\Big) - c\frac{1}{1-K},$$

then, the GIO for Ψ_0 is

$$\begin{aligned} W(\Psi_0, \mathcal{V}) = -W\Big(c\frac{1}{1-K}, \mathcal{V}\Big) &= -\int_0^\infty dx\, e^{-x}\langle\, \mathcal{V}(i\infty)c(0)\,\rangle_{\rm C_x} \\ &= -\int_0^\infty dx\, e^{-x}\langle\, \mathcal{V}(0)h_x \circ c(0)\,\rangle_{\rm disk} \\ &= -\frac{1}{2\pi i}\int_0^\infty dx\, xe^{-x}\langle\, \mathcal{V}(0)c(1)\,\rangle_{\rm disk} \\ &= 0 - \mathcal{A}_0(\mathcal{V}). \end{aligned} \tag{2.4.23}$$

Here, $h_L(\xi)$ is a conformal transformation:

$$h_L : \text{sliver } \mathrm{C}_L \to \text{disk},$$

$$h_L(\xi) := f_{\rm d} \circ f_{\rm s}{}^{-1} \circ f_{L\to 2}(\xi) = e^{2\pi i\xi/L}, \tag{2.4.24}$$

$$\partial_\xi h_L(\xi) = \frac{2\pi i}{L}e^{2\pi i\xi/L}. \tag{2.4.25}$$

Here, $f_{\rm d}(\xi)$ is a conformal transformation:

$$\begin{aligned} f_{\rm d}:\ &\mathrm{UHP}\to\mathrm{disk}, \\ f_{\rm d}(\xi) &= \frac{1+i\xi}{1-i\xi}, \qquad (2.4.26)\\ {f_{\rm d}}^{-1}(\xi) &= i\frac{1-\xi}{1+\xi}, \qquad (2.4.27)\\ \partial_\xi {f_{\rm d}}^{-1}(\xi) &= i\frac{-1}{1+\xi} - i\frac{1-\xi}{(1+\xi)^2}. \qquad (2.4.28)\end{aligned}$$

Moreover, if we choose a vertex operator $\mathcal{V}$ as

$$\mathcal{V}_{\rm G} := \frac{2i}{\pi}\cdot c\tilde{c}\partial X^0\bar{\partial}X^0, \qquad (2.4.29)$$

the value of the closed string one-point function $\mathcal{A}_0(\mathcal{V}_{\rm G})$ is

$$\begin{aligned}\mathcal{A}_0(\mathcal{V}_{\rm G}) &= \frac{1}{2\pi i}\frac{2i}{\pi}\langle\, c(i)c(-i)\Big(-i\frac{1}{2}\Big)^{-1}c(0)\,\rangle^{bc}_{S^2}\times\langle\,\partial X^0(i)\partial X^0(-i)\,\rangle^{\rm ma}_{S^2}\\ &= \frac{1}{2\pi i}\frac{2i}{\pi}(2i)(i+i)(i-0)(-i-0)\times\frac{-\frac{1}{2}\eta^{00}}{(i+i)^2}\\ &= \frac{1}{2\pi^2}, \qquad (2.4.30)\end{aligned}$$

i.e., the tension of the D25-barne T_{25}. Therefore, the GIO of Ψ_0 equals the energy of Ψ_0:

$$W(\Psi_0,\mathcal{V}_{\rm G}) = E(\Psi_0). \qquad (2.4.31)$$

More general discussion can be found in [43].

2.5 Erler–Maccaferri Solution

There exist different type of the solutions by Erler and Maccaferri [25] using the tachyon vacuum solution. The solutions use other BCFT which is different from the BCFT in which the original operators and states are defined. This has been done by using boundary condition changing operators (BCCOs) $\sigma_{\rm L,R}(z)$ which are used in the KOS solution [26]. The form of the solution is

$$\Psi^{\rm a}_{\rm EM} := \Psi_0 + \Phi^{\rm a}, \qquad (2.5.1)$$

then we have

$$Q\Psi^{\rm a}_{\rm EM} + (\Psi^{\rm a}_{\rm EM})^2 = 0 \quad\Leftrightarrow\quad Q_0\Phi^{\rm a} + (\Phi^{\rm a})^2 = 0, \qquad (2.5.2)$$

where we used the fact that Ψ_0 is the solution of the EOM. The equation is satisfied if the string field Φ^{a} is defined as

$$\Phi^{\mathrm{a}} := \Sigma_{\mathrm{L}}^{\mathrm{a}}(-\Psi_0)\Sigma_{\mathrm{R}}^{\mathrm{a}}, \tag{2.5.3}$$

$$\Sigma_{\mathrm{L}}^{\mathrm{a}} := Q_0\big(A_0 V_1(K)\sigma_{\mathrm{L}}^{\mathrm{a}} V_2(K)\big),$$
$$\Sigma_{\mathrm{R}}^{\mathrm{a}} := Q_0(A_0 V_2(K)^{-1}\sigma_{\mathrm{R}}^{\mathrm{a}} V_1(K)^{-1}), \tag{2.5.4}$$

where $V_{1,2}(K)$ are functions of the string field K and the string fields $\sigma_{\mathrm{L,R}}$ are defined by inserting the BCCO $\sigma_{\mathrm{L,R}}(z)$ on the boundary of CFT as the string field c is made by inserting $c(z)$. The EOM follows from the following relations:

$$Q_0\Sigma_{\mathrm{L,R}}^{\mathrm{a}} = {Q_0}^2(\cdots) = 0, \tag{2.5.5}$$
$$\Sigma_{\mathrm{R}}^{\mathrm{a}}\Sigma_{\mathrm{L}}^{\mathrm{a}} = 1. \tag{2.5.6}$$

In the case of the KOS solution, the conformal weights of BCCOs should be 1, however in [25], the BCCOs were modified as

$$\sigma_{\mathrm{L}}^{\mathrm{a}}(z) := \sigma_{*\mathrm{L}}^{\mathrm{a}} e^{i\sqrt{h}X^0}(z), \quad \sigma_{\mathrm{R}}^{\mathrm{a}}(z) := \sigma_{*\mathrm{R}}^{\mathrm{a}} e^{-i\sqrt{h}X^0}(z). \tag{2.5.7}$$

Here, the conformal weight of $\sigma_{\mathrm{L,R}}^{\mathrm{a}}$ is 1, but because of the existence of the operator $e^{\pm i\sqrt{h}X^0}$ which does not have any physical effect, the conformal weight of $\sigma_{*\mathrm{L,R}}^{\mathrm{a}}$ can be some different value h.

The energy of the EM solution is

$$E(\Psi_{\mathrm{EM}}^{\mathrm{a}}) = E(\Psi_0) + \frac{g_{\mathrm{a}}}{2\pi^2}, \tag{2.5.8}$$

where $g_{\mathrm{a}} = \sigma_{\mathrm{L}}^{\mathrm{a}}\sigma_{\mathrm{R}}^{\mathrm{a}}$ is the disk partition function in $\mathrm{BCFT}_{\mathrm{a}}$. The GIO is

$$W(\Psi_{\mathrm{EM}}^{\mathrm{a}}, \mathcal{V}) = \mathcal{A}_{\mathrm{a}}(\mathcal{V}) - \mathcal{A}_0(\mathcal{V}). \tag{2.5.9}$$

By using orthogonal BCCOs, $\sigma_{\mathrm{R}}^{i}\sigma_{\mathrm{L}}^{j} = \delta_{ij}$, the following string fields are also the solutions of the EOM:

$$\Psi_{\mathrm{EM}}^{\mathrm{a+b+\cdots}} := \Psi_0 + \Phi^{\mathrm{a}} + \Phi^{\mathrm{b}} + \cdots. \tag{2.5.10}$$

The energy and the GIO are

$$E(\Psi_{\mathrm{EM}}^{\mathrm{a+b+\cdots}}) = E(\Psi_0) + \frac{1}{2\pi^2}(g_{\mathrm{a}} + g_{\mathrm{b}} + \cdots), \tag{2.5.11}$$
$$W(\Psi_{\mathrm{EM}}^{\mathrm{a+b+\cdots}}, \mathcal{V}) = (\mathcal{A}_{\mathrm{a}}(\mathcal{V}) + \mathcal{A}_{\mathrm{b}}(\mathcal{V}) + \cdots) - \mathcal{A}_0(\mathcal{V}). \tag{2.5.12}$$

When we choose the BCCOs as [44],

$$\sigma_{\mathrm{L}}^{\mathrm{a}_i} := e^{ik_i\cdot X}, \quad \sigma_{\mathrm{R}}^{\mathrm{a}_i} := e^{-ik_i\cdot X},$$
$$k_i^{\mu}k_{i\,\mu} = 0, \quad k_i^{\mu}k_{j\,\mu} < 0, \tag{2.5.13}$$

where k_i^{μ} is e.g., $k_a^{\mu} = (a, 1, \sqrt{a^2-1}, 0, \cdots, 0)$, we obtain the multiple-brane solution whose energy is

$$E(\Psi_{\rm EM}^{\rm a_1+a_2+\cdots+a_\mathit{n}}) = E(\Psi_0) + nT_{25}. \tag{2.5.14}$$

As another concrete example of the EM solution, we review a lump solution. We consider the BCCOs as $\sigma_*^{\rm a} = \sigma_*^{\rm ND}$, i.e., Neumann–Dirichlet (ND) twist operator [45], which changes the Neumann boundary condition of X^1 to the Dirichlet boundary condition. Then, the EM solution describes the D24-brane. The conformal weight of $\sigma_{*\rm L,R}^{\rm ND}$ is equal to $\frac{1}{16}$ [46]. By using the X-X Green function with ND twist operators [47], a three-point function among $\sigma_{*\rm L,R}^{\rm ND}$ and $e^{inX^1/R}$ [48] is

$$\frac{1}{2\pi R}\langle\, e^{-inX^1/R}(z_1)\sigma_{*L}^{\rm ND}(z_2)\sigma_{*R}^{\rm ND}(z_3)\,\rangle_{\rm UHP}^{X^1} = \frac{2^{-2(n/R)^2}}{R}\frac{1}{{z_{12}}^{(n/R)^2}{z_{13}}^{(n/R)^2}{z_{23}}^{1/8-(n/R)^2}}. \tag{2.5.15}$$

Then, the correlator including $\sigma_{\rm L,R}^{\rm ND}$ in the sliver frame [26] is given by[4]

$$\frac{1}{2\pi R}\langle\, f_{\rm s}\circ e^{-inX^1/R}(0)\sigma_{\rm L}^{\rm ND}(t_1)\sigma_{\rm R}^{\rm ND}(t_1+t_2)\,\rangle_{\rm C_\mathit{L}}^{\rm ma} = \frac{2^{-2(n/R)^2}}{R}\left(\frac{2\sin\theta_{t_2}}{L\sin\theta_{t_1}\sin\theta_{t_1+t_2}}\right)^{(n/R)^2}. \tag{2.5.16}$$

Here, we assume that the direction X^1 is compactified as $X^1 \simeq X^1 + 2\pi R$, and end points of the string are at $X^1 = 0$. From the (2.5.16), the OPE between ND twist operators is

$$\sigma_{\rm L}^{\rm ND}(z)\sigma_{\rm R}^{\rm ND}(0) \sim \frac{1}{R}. \tag{2.5.17}$$

Let us compute the tachyon profile of the solution to check that the lump solution describes the lower dimensional D-brane. The tachyon field $T(X^1)$ is expanded as

$$T(X^1) = \sum_{n\in\mathbb{Z}} t_n e^{inX^1/R}. \tag{2.5.18}$$

The coefficient t_n can be computed by using the state $|\tilde{T}_n\rangle$, which is dual to the tachyon state $|T_n\rangle$ satisfying $\mathrm{Tr}[\tilde{T}_n T_m] = \delta_{n,m}$:

$$\begin{aligned} |T_n\rangle &= ce^{inX^1/R}(0)|0\rangle, \\ |\tilde{T}_n\rangle &= -\frac{1}{2\pi R}c\partial ce^{-inX^1/R}(0)|0\rangle. \end{aligned} \tag{2.5.19}$$

For simplicity, by choosing $V_{1,2}(K)$ as $V_1(K) = V_2(K)^{-1} = 1-K$, we have the explicit expression:

$$\Phi^{\rm ND} = cB(1-K)\sigma_{\rm L}^{\rm ND}\frac{1}{1-K}\sigma_{\rm R}^{\rm ND}(1-K)c\frac{1}{1-K}. \tag{2.5.20}$$

[4] We normalize the correlator by dividing by the volume of spacetime.

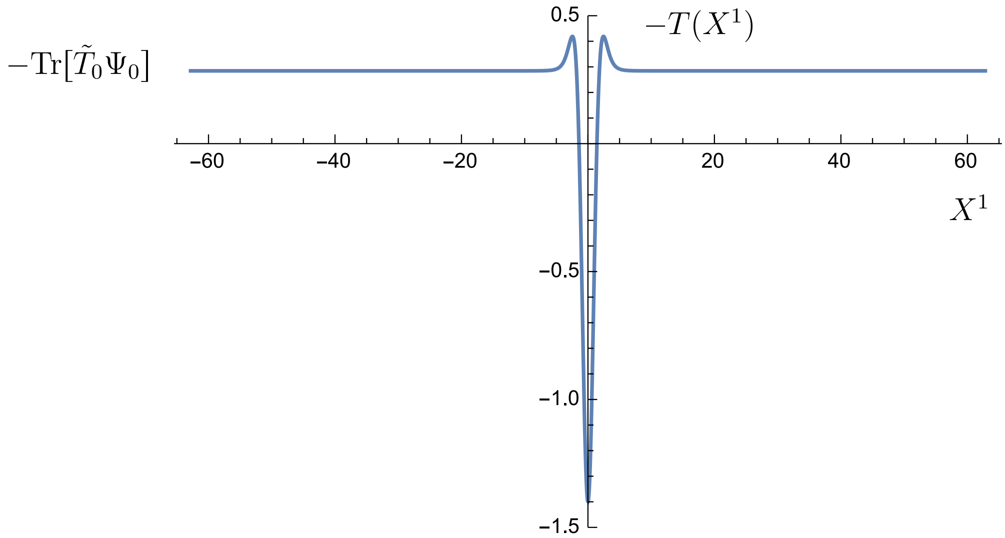


Figure 2.1: The profile of the tachyon field $T(X^1)$ is shown. The compactification radius is taken to be R=20. We numerically computed by setting $|n| \leqslant 100$. The minus sign of $-T(X^1)$ comes form the difference of the notation.

The coefficient t_n [25] is given by

$$\begin{aligned}
t_n &= \mathrm{Tr}\left[\tilde{T}_n\Psi^{\mathrm{ND}}_{\mathrm{EM}}\right] \\
&= \mathrm{Tr}\left[\tilde{T}_n\Psi_0\right] + \mathrm{Tr}\left[\tilde{T}_n\Phi^{\mathrm{a}}\right] \\
&= \mathrm{Tr}\left[\tilde{T}_0\Psi_0\right]\delta_{n,0} \\
&\quad + \left(\frac{\pi}{2}\right)^{-1}\iint_0^\infty dx_1dx_2 \lim_{u_{1,2}\to 0}(-\partial_{u_1})(-\partial_{u_2})\Big\{e^{-(x_1+x_2+u_1+u_2)} \\
&\qquad \times \mathrm{Bccdc}[x_1+\tfrac{1}{2},0,\tfrac{1}{2},u_1+x_2+u_2] \\
&\qquad \times \Big\langle \frac{1}{2\pi i} f_{\mathrm{s}}\circ e^{-ikX^1}(0)\sigma^{\mathrm{ND}}_{\mathrm{L}}(\tfrac{1}{2}+u_1)\sigma^{\mathrm{ND}}_{\mathrm{R}}(\tfrac{1}{2}+u_1+x_1)\Big\rangle^{\mathrm{ma}}_{C_{1+x_1+x_2+u_1+u_2}}\Big\},
\end{aligned} \tag{2.5.21}$$

where $-\mathrm{Tr}[\tilde{T}_0\Psi_0] = 0.284394\ldots$ [29], and the definition of $\mathrm{Bccdc}[t_1,t_2,t_3,t_4]$ is given in appendix A (A.19). Figure 2.1 shows the numerical result for the profile of the tachyon field. Far away from $X^1 = 0$, the value of the tachyon field asymptotically approaches that for the tachyon vacuum solution $-\mathrm{Tr}[\tilde{T}_0\Psi_0]$.

2.6 Multiple-brane Solution

2.6.1 Murata–Schnabl Solution

The multiple-brane solution [16, 17] in the pure-gauge form can be written by using the gauge transformation for the tachyon vacuum solution:

$$\begin{aligned}\Psi_0 \xrightarrow{(U_1{}^{-1})^n} \Psi_n &:= U_1{}^{n-1} Q U_1{}^{-(n-1)} \\ &= cBKG_1{}^{n-1} c(1-G_1{}^{-(n-1)}), \quad n \in \mathbb{Z}_{\geqslant 0}. \end{aligned} \tag{2.6.1}$$

At first, the energy of this solution is expected to reproduce the value which is equal to n times the tension of the D25-brane. However by taking care of the singular string field $1/K$, we obtain the correct energies only for the cases with $n = 0, 1$ and 2.

2.6.2 K_ϵ-Regularization

Except for the tachyon vacuum solution Ψ_0 (and the trivial solution $\Psi_1 = 0$), the string field Ψ_n has the singular string fields $(1/K)$'s. We explain that the string field $1/K$ needs a regularization. We consider $1/K$ is singular because the eigenvalue may be zero, and the (inverse) Laplace transformation is not well-defined. First, we see that $\frac{1}{1-K}$ is a well-defined string field, i.e., it has the inverse and the Laplace transformation:

$$\begin{aligned}\frac{1}{1-K}(1-K) &= \int_0^\infty dx\, e^{-x}\Omega^x \lim_{u\to 0}(-\partial_u)\{e^{-u}\Omega^u\} \\ &= -\int_0^\infty dx \lim_{u\to 0}\, \partial_x\{e^{-(x+u)}\Omega^{x+u}\} \\ &= -[e^{-x}\Omega^x]_0^\infty \\ &= -(\lim_{x\to\infty} e^{-x}\Omega^x - \Omega^0) \\ &= 1, \end{aligned} \tag{2.6.2}$$

where the string field Ω^∞ is finite [39]. Next, if we assume that the Laplace transformation of the string field $1/K$ is

$$\frac{1}{-K} \stackrel{?}{=} \int_0^\infty dz\, \Omega^z, \tag{2.6.3}$$

then, the string field K is not the inverse

$$\begin{aligned}-\frac{1}{-K}K &\stackrel{?}{=} -\int_0^\infty dz\, \Omega^z \lim_{y\to 0}\{\partial_y \Omega^y\} \\ &= -[\Omega^z]_0^\infty \\ &= -\Omega^\infty + 1 \neq 1. \end{aligned} \tag{2.6.4}$$

Let us introduce the so-called "K_ϵ-regularization" [17, 21, 22], in which we replace each string field K in the solution Ψ_n with K_ϵ defined by

$$K_\epsilon := K - \epsilon, \quad 0 < \epsilon \ll 1. \tag{2.6.5}$$

We can check the string field $1/K_\epsilon$ is invertible:

$$\begin{aligned} -\frac{1}{-K_\epsilon} K_\epsilon &= -\int_0^\infty dz\, e^{-\epsilon z} \Omega^z \lim_{y_1 \to 0} \partial_{y_1} \{ e^{-\epsilon y_1} \Omega^{y_1} \} \\ &= -\left[e^{-\epsilon z} \Omega^z \right]_0^\infty \\ &= 1, \end{aligned} \tag{2.6.6}$$

before we take the limit $\epsilon \to 0$. The algebra among K_ϵ, B and c is given by

$$\begin{aligned} [K_\epsilon, B] = 0, \quad B^2 = c^2 = 0, \quad \{B, c\} = 0, \\ QK_\epsilon = 0, \quad Qc = c\partial c, \quad QB = K_\epsilon + \epsilon, \end{aligned} \tag{2.6.7}$$

where $\partial c = [K, c] = [K_\epsilon, c]$. In the following, we use the notation $[\![\bullet]\!]_\epsilon$, in which all K's inside the square bracket with subscript ϵ are replaced with K_ϵ:

$$[\![f(K, B, c)]\!]_\epsilon = f(K_\epsilon, B, c). \tag{2.6.8}$$

From the algebra (2.6.7), we have $Q[\![B]\!]_\epsilon - [\![QB]\!]_\epsilon = \epsilon$ and $Q[\![\varphi]\!]_\epsilon - [\![Q\varphi]\!]_\epsilon = 0$, where φ does not include B. Then, we obtain the relation:

$$Q[\![\bullet]\!]_\epsilon - [\![Q\bullet]\!]_\epsilon = \epsilon \frac{\partial}{\partial B} [\![\bullet]\!]_\epsilon. \tag{2.6.9}$$

Note that the statistics of $\partial/\partial B$ is same as B.

2.6.3 Equation of Motion in the Strong Sense

We introduce an EOM in the strong sense (EOMS):

$$\mathrm{EOMS}(\varphi) := \mathrm{Tr}[\varphi(Q\varphi + \varphi^2)]. \tag{2.6.10}$$

At first sight, if the string field φ is the solution of the EOM algebraically, it seems that $\mathrm{EOMS}(\varphi)$ is automatically zero. However, if the string field φ includes $1/K$ as Ψ_n, we should regularize it and check whether $\mathrm{EOMS}(\varphi)$ is zero or not, even if Ψ_n is in the pure-gauge form, i.e., the solution of the EOM algebraically. By using (2.6.9), the EOMS in the strong sense for $[\![\Psi_n]\!]_\epsilon$ can be written as

$$\mathrm{EOMS}([\![\Psi_n]\!]_\epsilon) = \epsilon \mathrm{Tr}\left[[\![\Psi_n]\!]_\epsilon \frac{\partial}{\partial B} [\![\Psi_n]\!]_\epsilon \right], \tag{2.6.11}$$

where we used the EOM:

$$[\![Q\Psi_n]\!]_\epsilon + [\![\Psi_n]\!]_\epsilon{}^2 = 0. \tag{2.6.12}$$

This quantity is known [16, 17, 21],

$$\lim_{\epsilon\to 0}\mathrm{EOMS}([\![\Psi_n]\!]_\epsilon) = -\frac{n(n-1)}{\pi}\mathrm{Im}[{}_1F_1(2-n,2,2\pi i)], \tag{2.6.13}$$

by using the s-z trick [16, 17]. The EOMS is zero in the case of $n = 0, 1$ and 2, i.e., the tachyon vacuum, the trivial vacuum and the double-brane solution.

Let us show that the double-brane solution,

$$\lim_{\epsilon\to 0}[\![\Psi_2]\!]_\epsilon = \lim_{\epsilon\to 0} cB\frac{K_\epsilon^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}, \tag{2.6.14}$$

satisfies the EOMS without using the s-z trick:

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathrm{EOMS}([\![\Psi_2]\!]_\epsilon) &= \lim_{\epsilon\to 0}\epsilon\mathrm{Tr}\left[[\![\Psi_2]\!]_\epsilon\frac{\partial}{\partial B}[\![\Psi_2]\!]_\epsilon\right] \\
&= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}\left[cB\frac{K_\epsilon^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}c\frac{K_\epsilon^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}\right] \\
&= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}\Bigg[(-\partial c+K_\epsilon c)B\frac{1}{1-K_\epsilon}(\partial c+cK_\epsilon)\frac{1}{-K_\epsilon} \\
&\qquad\qquad \times(-\partial c+K_\epsilon c)\frac{1}{1-K_\epsilon}(\partial c+cK_\epsilon)\frac{1}{-K_\epsilon}\Bigg] \\
&= 2\lim_{\epsilon\to 0}\epsilon\iiint_0^\infty dx_1dx_2dz_1\, e^{-(1+\epsilon)(x_1+x_2)}e^{-\epsilon z_1}\,\mathrm{Bcddd}[x_1,z_1,x_2] \\
&= -\frac{2}{\pi}\lim_{\epsilon\to 0}\epsilon\int_0^\infty a^2da\int_0^1 dc\int_0^c db\, e^{-a(1-b+\epsilon)} \\
&\quad\times\big(\sin(2b\pi)-\sin(2(b-c)\pi)-\sin(2c\pi)\big) \\
&= -2\lim_{\epsilon\to 0}\epsilon(2+\mathrm{O}(\epsilon)) \\
&= 0.
\end{aligned} \tag{2.6.15}$$

Here, we change the variables:

$$z_1 = ab,\quad x_1 = ac-ab,\quad x_2 = a-ac, \tag{2.6.16}$$

$$\left(a = x_1+x_2+z_1,\quad b = \frac{z_1}{x_1+x_2+z_1},\quad c = \frac{z_1+x_1}{x_1+x_2+z_1}\right). \tag{2.6.17}$$

In the case of the triple-brane, the EOMS is checked in the paper [21], and it was found that $\lim_{\epsilon\to 0}\mathrm{EOMS}([\![\Psi_3]\!]_\epsilon) = 6 \neq 0$.

2.6.4 Energy and GIO

Let us compute the energy of the double-brane solution Ψ_2:

$$
\begin{aligned}
\lim_{\epsilon\to 0} E([\![\Psi_2]\!]_\epsilon) = -\lim_{\epsilon\to 0} S([\![\Psi_2]\!]_\epsilon) &= \lim_{\epsilon\to 0}\left(-\frac{1}{6}\mathrm{Tr}[[\![\Psi_2]\!]_\epsilon{}^3] + \frac{1}{2}\mathrm{EOMS}([\![\Psi_2]\!]_\epsilon)\right) \\
&= -\lim_{\epsilon\to 0}\frac{1}{6}\mathrm{Tr}\Bigg[cB\frac{K_\epsilon{}^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon} \\
&\qquad \times cB\frac{K_\epsilon{}^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}cB\frac{K_\epsilon{}^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}\Bigg] \\
&= \frac{1}{6}\mathrm{Tr}\left[Bc\partial c\frac{1}{1-K}\partial c\frac{1}{1-K}\partial c\frac{1}{1-K}\right] \\
&= -E(\Psi_0) \\
&= E(\Psi_0) + 2T_{25}. \qquad (2.6.18)
\end{aligned}
$$

The energy is twice the tension of the D25-brane. It is known that the GIO for the double-brane solution is

$$
\lim_{\epsilon\to 0} W([\![\Psi_2]\!]_\epsilon, \mathcal{V}) = 2A_0(\mathcal{V}) - A_0(\mathcal{V}). \qquad (2.6.19)
$$

We summarize the results for the calculation of the energies and the GIOs we obtained so far:

$$
\begin{aligned}
E(\Psi_0) &= 0 - T_{25}, \\
E(\Psi_1) &= T_{25} + E(\Psi_0), \\
\lim_{\epsilon\to 0} E([\![\Psi_2]\!]_\epsilon) &= 2T_{25} + E(\Psi_0), \\
W(\Psi_0, \mathcal{V}) &= 0 - A_0(\mathcal{V}), \\
W(\Psi_1, \mathcal{V}) &= A_0(\mathcal{V}) - A_0(\mathcal{V}), \\
\lim_{\epsilon\to 0} W([\![\Psi_2]\!]_\epsilon, \mathcal{V}) &= 2A_0(\mathcal{V}) - A_0(\mathcal{V}). \qquad (2.6.20)
\end{aligned}
$$

We can understand that the singular gauge transformation $U_1{}^{-1}$ increases the energy of the solution by one unit of the tension of the D25-brane T_{25}. Furthermore, we can expect that the singular gauge transformation $U_1{}^{-1}$ increases the D25-brane. The results for the GIOs would be regarded as supports for the observation.

Chapter 3

Singular Gauge Transformation and the Erler–Maccaferri Solution

3.1 Construction of the Solution

We will give a support that the singular gauge transformation ${U_1}^{-1}$ increases the D25-brane. Let us discuss the string field constructed by performing the singular gauge transformation ${U_1}^{-1}$ n times for the EM solution $\Psi^{\rm a}_{\rm EM}$ [49]. By performing the gauge transformation once, we obtain

$$\begin{aligned}\Psi^{\rm a}_{\rm EM} \xrightarrow{{U_1}^{-1}} \Psi^{\rm a}_{\rm EM+1} &:= U_1(Q+\Psi^{\rm a}_{\rm EM}){U_1}^{-1} \\ &= U_1\Sigma^{\rm a}_{\rm L}(-\Psi_0)\Sigma^{\rm a}_{\rm R}{U_1}^{-1} \\ &= cBK\sigma^{\rm a}_{\rm L}\frac{1}{1-K}\sigma^{\rm a}_{\rm R}Kc\frac{1}{-K},\end{aligned} \tag{3.1.1}$$

while performing it n times,

$$\begin{aligned}\Psi^{\rm a}_{\rm EM} \xrightarrow{(U_1^{-1})^n} \Psi^{\rm a}_{\rm EM+\mathit{n}} &:= U_1^n(Q+\Psi^{\rm a}_{\rm EM})U_1^{-n} \\ &= \Psi_n + cBK{G_1}^{n-1}\sigma^{\rm a}_{\rm L}\frac{1}{1-K}\sigma^{\rm a}_{\rm R}(-c+\partial c\frac{1}{-K}{G_1}^{-(n-1)}) \\ &= \Psi_n + \Phi^{\rm a}_n.\end{aligned} \tag{3.1.2}$$

Here, we define

$$\Phi^{\rm a}_n := cBK{G_1}^{n-1}\sigma^{\rm a}_{\rm L}\frac{1}{1-K}\sigma^{\rm a}_{\rm R}(-c+\partial c\frac{1}{-K}{G_1}^{-(n-1)}), \tag{3.1.3}$$

as the EM solution

$$\Psi^{\rm a}_{\rm EM} = \Psi_0 + \Phi^{\rm a}.$$

Since the string field $\Psi^{\rm a}_{\mathrm{EM}+n}$ is constructed by using the gauge transformation for the EM solution, $\Psi^{\rm a}_{\mathrm{EM}+n}$ satisfies the EOM formally. However, $\Psi^{\rm a}_{\mathrm{EM}+n}$ has the singular string field $1/K$. Therefore, we should implement the K_ϵ-regularization as the double-brane solution

$\Psi_2 \to [\![\Psi_2]\!]_\epsilon$:

$$\Psi^{\rm a}_{{\rm EM}+n} \to [\![\Psi^{\rm a}_{{\rm EM}+n}]\!]_\epsilon = [\![\Psi_n]\!]_\epsilon + [\![\Phi^{\rm a}_n]\!]_\epsilon. \tag{3.1.4}$$

We expect that the string field $\Psi^{\rm a}_{{\rm EM}+n}$ describes the EM solution with n D25-branes.

3.2 Equation of Motion in the Strong Sense

Since we use the K_ϵ-regularization, we should check the EOMS($[\![\Psi^{\rm a}_{{\rm EM}+n}]\!]_\epsilon$). By using (2.6.9) and (2.6.12), we consider the following difference:

$$\begin{aligned}
&\mathrm{EOMS}([\![\Psi^{\rm a}_{{\rm EM}+n}]\!]_\epsilon) - \mathrm{EOMS}([\![\Psi_n]\!]_\epsilon) \\
&= \epsilon \mathrm{Tr}\left[([\![\Psi_n]\!]_\epsilon + [\![\Phi^{\rm a}_n]\!]_\epsilon)\frac{\partial}{\partial B}([\![\Psi_n]\!]_\epsilon + [\![\Phi^{\rm a}_n]\!]_\epsilon)\right] - \epsilon \mathrm{Tr}\left[[\![\Psi_n]\!]_\epsilon \frac{\partial}{\partial B}[\![\Psi_n]\!]_\epsilon\right].
\end{aligned} \tag{3.2.1}$$

Since each $\Phi^{\rm a}_n$ contains two BCCOs, the right-hand side of the equation seems to be composed of terms with two BCCOs and also four BCCOs. The explicit form of the term with four BCCOs is

$$\begin{aligned}
\mathrm{Tr}\left[[\![\Phi^{\rm a}_n]\!]_\epsilon \left(\epsilon \frac{\partial}{\partial B}[\![\Phi^{\rm a}_n]\!]_\epsilon\right)\right] = -\epsilon \mathrm{Tr}\Big[&cBK_\epsilon {G_{1\epsilon}}^{n-1}\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\frac{1}{-K_\epsilon}{G_{1\epsilon}}^{-(n-1)} \\
&\times cK_\epsilon {G_{1\epsilon}}^{n-1}\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\frac{1}{-K_\epsilon}{G_{1\epsilon}}^{-(n-1)}\Big],
\end{aligned} \tag{3.2.2}$$

where $G_{1\epsilon} := [\![G_1]\!]_\epsilon$. This will be rewritten into the summation of the terms with two BCCOs:

$$\begin{aligned}
(3.2.2) = -\epsilon \mathrm{Tr}\Big[&[c, K_\epsilon {G_{1\epsilon}}^{n-1}]B\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\frac{1}{-K_\epsilon}{G_{1\epsilon}}^{-(n-1)} \\
&\times [c, K_\epsilon {G_{1\epsilon}}^{n-1}]\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\frac{1}{-K_\epsilon}{G_{1\epsilon}}^{-(n-1)}\Big] \\
&+\epsilon \mathrm{Tr}\Big[[c, K_\epsilon {G_{1\epsilon}}^{n-1}]B\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial cc\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\frac{1}{-K_\epsilon}{G_{1\epsilon}}^{-(n-1)}\Big] \\
&+\epsilon \mathrm{Tr}\Big[cB\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\frac{1}{-K_\epsilon}{G_{1\epsilon}}^{-(n-1)}[c, K_\epsilon {G_{1\epsilon}}^{n-1}]\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\Big] \\
&-\epsilon \mathrm{Tr}\Big[cB\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial cc\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\Big].
\end{aligned} \tag{3.2.3}$$

The first term including four BCCOs vanishes, as the result of the following very useful relation:

$$\mathrm{Tr}[B\varphi] = \mathrm{Tr}[BcB\varphi] = \mathrm{Tr}[B^2c\varphi] = 0, \tag{3.2.4}$$

where the string field φ commutes with B, and we use $B = BcB$ and the cyclicity of Tr. The remaining terms reduce to the terms with two BCCOs, by using that BCCOs commute with ghosts, the cyclicity of the Tr and $\sigma^{\rm a}_{\rm R}\sigma^{\rm a}_{\rm L} = 1$. The contribution of Tr with two BCCOs always reduces to the following correlator in the matter sector of CFT :

$$\langle\, \sigma^{\rm a}_{\rm L}(0)\sigma^{\rm a}_{\rm R}(z_1)\,\rangle^{\rm ma}_{{\rm C}_L} = g_{\rm a}. \tag{3.2.5}$$

Since the correlator is independent of the positions fo the BCCOs, we have

$$\langle\, \partial\sigma_{\rm L}^{\rm a}(0)\sigma_{\rm R}^{\rm a}(z_1)\,\rangle_{{\rm C}_L}^{\rm ma} = 0. \tag{3.2.6}$$

When we move the position of $\sigma_{\rm L}^{\rm a}$ to the immediate left of $\sigma_{\rm R}^{\rm a}$, the remnants are commutators among $\sigma_{\rm L}^{\rm a}$ and the function of K which become the derivatives such as $\partial\sigma_{\rm L}^{\rm a} = [K, \sigma_{\rm L}^{\rm a}]$. In Tr, they can be set to zero due to (3.2.6). Then, because the term with four BCCOs vanishes, we can extract the factor $g_{\rm a} = \sigma_{\rm L}^{\rm a}\sigma_{\rm R}^{\rm a}$ on the right-hand side, replacing $\sigma_{\rm L,R}^{\rm a}$ by 1:

$$\begin{aligned}
&{\rm EOMS}([\![\Psi_{{\rm EM}+n}^{\rm a}]\!]_\epsilon) - {\rm EOMS}([\![\Psi_n]\!]_\epsilon) \\
&= g_{\rm a}\left(\epsilon{\rm Tr}\left[([\![\Psi_n]\!]_\epsilon + [\![\Phi_n^{\rm a}]\!]_\epsilon)\frac{\partial}{\partial B}([\![\Psi_n]\!]_\epsilon + [\![\Phi_n^{\rm a}]\!]_\epsilon)\right] - \epsilon{\rm Tr}\left[[\![\Psi_n]\!]_\epsilon\frac{\partial}{\partial B}[\![\Psi_n]\!]_\epsilon\right]\right)\Big|_{\sigma_{\rm L,R}^{\rm a}=1} \\
&= g_{\rm a}\big({\rm EOMS}([\![\Psi_{n+1}]\!]_\epsilon) - {\rm EOMS}([\![\Psi_n]\!]_\epsilon)\big),
\end{aligned} \tag{3.2.7}$$

where we used the relation:

$$\Psi_{{\rm EM}+n}^{\rm a}|_{\sigma_{\rm L,R}^{\rm a}=1} = \Psi_{n+1}. \tag{3.2.8}$$

Finally, the EOMS for $\Psi_{{\rm EM}+n}^{\rm a}$ becomes

$$\lim_{\epsilon\to 0}{\rm EOMS}([\![\Psi_{{\rm EM}+n}^{\rm a}]\!]_\epsilon) = \lim_{\epsilon\to 0}\big((1-g_{\rm a}){\rm EOMS}([\![\Psi_{n+1}]\!]_\epsilon) + g_{\rm a}{\rm EOMS}([\![\Psi_n]\!]_\epsilon)\big). \tag{3.2.9}$$

Here, we recall (2.6.13):

$$\lim_{\epsilon\to 0}{\rm EOMS}([\![\Psi_n]\!]_\epsilon) = -\frac{n(n-1)}{\pi}{\rm Im}[{}_1F_1(2-n,2,2\pi i)],$$

then, we find that $\Psi_{{\rm EM}+n}^{\rm a}$ with $n=1$ satisfies the EOMS, while for $n>1$ it does not for generic $g_{\rm a}$. We also find that there is a special value of $g_{\rm a}$ for each n for which $\Psi_{{\rm EM}+n}^{\rm a}$ accidentally satisfies the EOMS.

3.3 Energy and Gauge Invariant Observable

Let us check the energy of the solution $\Psi^{\rm a}_{\rm EM+1}$ (3.1.1). Since the EOMS vanishes, it can be easily evaluated by using the cubic term

$$\begin{aligned}
\lim_{\epsilon\to 0} E(\llbracket\Psi^{\rm a}_{\rm EM+1}\rrbracket_\epsilon) &= \lim_{\epsilon\to 0}\left(-\frac{1}{6}{\rm Tr}[\llbracket\Psi^{\rm a}_{\rm EM+1}\rrbracket_\epsilon{}^3] + \frac{1}{2}{\rm EOMS}(\llbracket\Psi^{\rm a}_{\rm EM+1}\rrbracket_\epsilon)\right)\\
&= -\frac{1}{6}\lim_{\epsilon\to 0}{\rm Tr}\left[\left(cBK_\epsilon\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}K_\epsilon c\frac{1}{-K_\epsilon}\right)^3\right]\\
&= -\frac{1}{6}\lim_{\epsilon\to 0}{\rm Tr}\left[Bc\partial c\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\partial c\sigma^{\rm a}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm a}_{\rm R}\right]\\
&= -g_{\rm a}\frac{1}{6}{\rm Tr}\left[Bc\partial c\frac{1}{1-K}\partial c\frac{1}{1-K}\partial c\frac{1}{1-K}\right]\\
&= g_{\rm a}\big(-E(\Psi_0)\big)\\
&= E(\Psi^{\rm a}_{\rm EM}) + T_{25}.
\end{aligned}\tag{3.3.1}$$

We realize that the energy of the solution is increased by the tension of the D25-brane T_{25} through the singular gauge transformation ${U_1}^{-1}$. In addition, we calculate the GIO:

$$\begin{aligned}
\lim_{\epsilon\to 0} W(\llbracket\Psi^{\rm a}_{\rm EM+1}\rrbracket_\epsilon,\mathcal{V}) &= \lim_{\epsilon\to 0} W\big(\llbracket U_1\Sigma^{\rm a}_{\rm L}(-\Psi_0)\Sigma^{\rm a}_{\rm R}{U_1}^{-1}\rrbracket_\epsilon,\mathcal{V}\big)\\
&= -W(\Sigma^{\rm a}_{\rm L}\Psi_0\Sigma^{\rm a}_{\rm R},\mathcal{V})\\
&= \mathcal{A}_{\rm a}(\mathcal{V})\\
&= \big(\mathcal{A}_{\rm a}(\mathcal{V}) + \mathcal{A}_0(\mathcal{V})\big) - \mathcal{A}_0(\mathcal{V}).
\end{aligned}\tag{3.3.2}$$

It can be regarded as the increase of a D25-brane from the EM solution.

3.4 An Example: D24+D25-brane

As in the case of the EM solution, let us plot the tachyon profile $T(X^1) = \sum_n t_n e^{inX^1/R}$ for the solution:

$$\llbracket\Psi^{\rm ND}_{\rm EM+1}\rrbracket_\epsilon = cBK_\epsilon\sigma^{\rm ND}_{\rm L}\frac{1}{1-K_\epsilon}\sigma^{\rm ND}_{\rm R}K_\epsilon c\frac{1}{-K_\epsilon}.\tag{3.4.1}$$

The coefficient t_n for $\Psi^{\rm ND}_{\rm EM+1}$ is given by

$$\begin{aligned}
t_n &= \lim_{\epsilon\to 0}{\rm Tr}\left[\tilde{T}_n\llbracket\Psi^{\rm ND}_{\rm EM+1}\rrbracket_\epsilon\right]\\
&= -\lim_{\epsilon\to 0}\frac{\pi}{2}\iint_0^\infty dx_1dz_1\lim_{y_{1,2}\to 0}\partial_{y_1}\partial_{y_2}\\
&\quad\Big\{e^{-\epsilon(z_1+y_1+y_2)}e^{-(1+\epsilon)x_1}{\rm Bccdc}[z_1+\tfrac{1}{2},0,\tfrac{1}{2},x_1+y_1+y_2]\\
&\quad\times\Big\langle\frac{1}{2\pi R}f_{\rm s}\circ e^{-inR/X^1}(0)\sigma^{\rm ND}_{\rm L}(\tfrac{1}{2}+y_1)\sigma^{\rm ND}_{\rm R}(\tfrac{1}{2}+y_1+x_1)\Big\rangle^{\rm ma}_{C_{1+x_1+y_1+y_2+z_1}}\Big\}.
\end{aligned}\tag{3.4.2}$$

Figure 3.1 shows the numerical result for the profile of the tachyon field. Recall that

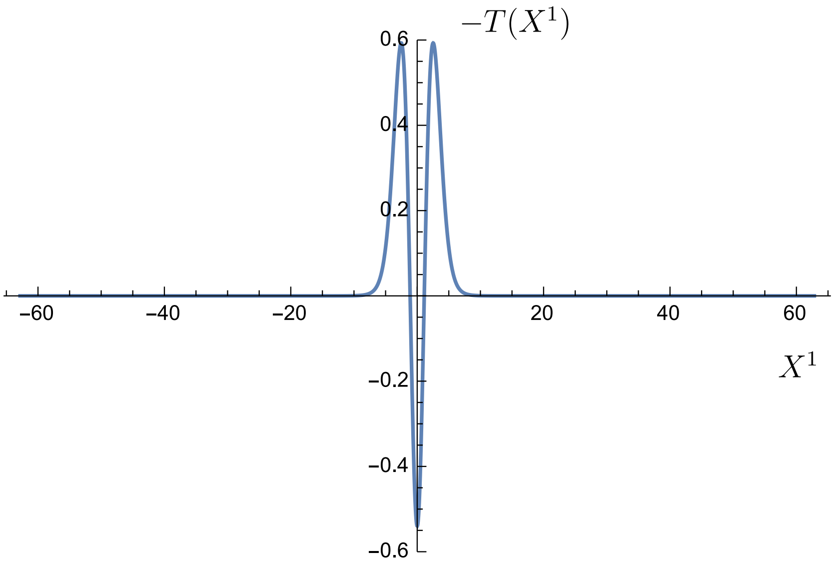


Figure 3.1: The profile of the tachyon field $T(X^1)$ is shown. The compactification radius is taken to be R=20. We numerically computed by setting $|n| \leqslant 100$ and $\epsilon = 0.001$.

in the case of the EM solution $\Psi_{\rm EM}^{\rm ND}$, far away from $X^1 = 0$, the value of the tachyon field asymptotically approaches that for the tachyon vacuum solution $-{\rm Tr}[\tilde{T}_0 \Psi_0]$. In our case of $\Psi_{\rm EM+1}^{\rm ND}$, the tachyon field asymptotically approaches zero, i.e., the value of the perturbative vacuum $\Psi_1 = 0$ representing the D25-brane. Therefore, we interpret the solution $\Psi_{\rm EM+1}^{\rm ND}$ to describe a multiple-brane solution in which the D24-brane is placed near $X^1 = 0$ on the D25-brane.

From the construction of the solution:

$$\Psi_{\rm EM}^{\rm a} \xrightarrow{{U_1}^{-1}} \Psi_{\rm EM+1}^{\rm a},$$

and the result of the energy, the GIO and the tachyon profile:

$$\begin{aligned}
\lim_{\epsilon\to 0} E([\![\Psi_{\rm EM+1}^{\rm a}]\!]_\epsilon) &= E(\Psi_{\rm EM}^{\rm a}) + T_{25}, \\
\lim_{\epsilon\to 0} W([\![\Psi_{\rm EM+1}^{\rm a}]\!]_\epsilon, \mathcal{V}) &= \big(\mathcal{A}_{\rm a}(\mathcal{V}) + \mathcal{A}_0(\mathcal{V})\big) - \mathcal{A}_0(\mathcal{V}), \\
\lim_{\epsilon\to 0} T(X^1)|_{[\![\Psi_{\rm EM+1}^{\rm ND}]\!]_\epsilon} &= T(X^1)|_{\Psi_1} \quad 1 \ll |X^1|, \\
\lim_{\epsilon\to 0} T(X^1)|_{[\![\Psi_{\rm EM+1}^{\rm ND}]\!]_\epsilon} &\neq T(X^1)|_{\Psi_1} \quad |X^1| \sim 1,
\end{aligned}$$

we obtain the supports that the singular gauge transformation ${U_1}^{-1}$ increases the D25-brane.

Chapter 4

Review of the Modified Cubic Superstring Field Theory

4.1 Non-GSO-Projected Action

In the case of the supersymmetric theory, first Witten extended his bosonic cubic action [3] so that it includes the supersymmetry. However, this action suffers from the contact term problem of the picture changing operators (PCOs). Next, the action was modified to avoid the contact problem [4–6]. And then to describe unstable D-branes, the non-GSO-projected action [50] for NS sector was constructed:

$$S(\Psi_+, \Psi_-) = \mathrm{Tr}_{Y_{-2}}\left[\frac{1}{2}\Psi_+ Q\Psi_+ + \frac{1}{3}{\Psi_+}^3 + \frac{1}{2}\Psi_- Q\Psi_- - \Psi_+{\Psi_-}^2\right], \tag{4.1.1}$$

where the string fields $\Psi_\pm$ is in the GSO($\pm$) sector, and Y_{-2} is the double-step inverse PCO:

$$Y_{-2}(i) := Y(i)\tilde{Y}(i) \quad (i \text{ is the string midpoint in the UHP}), \tag{4.1.2}$$

$$Y(z) := c\partial\xi e^{-2\phi}(z), \tag{4.1.3}$$

and the trace (the BPZ inner product) $\mathrm{Tr}_{Y_{-2}}$ is defined as

$$\mathrm{Tr}_{Y_{-2}}[\varphi_{\mathrm{test}}] := \langle\, Y_{-2}(i\infty) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0)\,\rangle_{\mathrm{C}_1}. \tag{4.1.4}$$

The EOMs for the GSO(+) sector and the GSO(−) sector are derived from this action:

$$Y_{-2}\left(Q\Psi_+ + {\Psi_+}^2 - {\Psi_-}^2\right) = 0, \tag{4.1.5}$$

$$Y_{-2}(Q\Psi_- + \Psi_+\Psi_- - \Psi_-\Psi_+) = 0. \tag{4.1.6}$$

We can rewrite the action and the EOMs in a more simple form by using a matrix-valued string field [51–53]:

$$\hat{\Psi} := \Psi_+ \otimes a + \Psi_- \otimes b, \tag{4.1.7}$$

where a and b are $N \times N$ matrices. Q and Y_{-2} are also the matrix-valued ones:

$$\hat{Q} := Q \otimes q, \tag{4.1.8}$$
$$\hat{Y}_{-2} := Y_{-2} \otimes y. \tag{4.1.9}$$

Let us determine the matrices a, b, q, y. When we demand that

$$(4.1.1) = \hat{\mathrm{Tr}}_{\hat{Y}_{-2}} \left[\frac{1}{2} \hat{\Psi} \hat{Q} \hat{\Psi} + \frac{1}{3} \hat{\Psi}^3 \right], \tag{4.1.10}$$

where

$$\hat{\mathrm{Tr}}_{\hat{Y}_{-2}}[\hat{\varphi}_{\mathrm{test}}] := \mathrm{Tr}_{Y_{-2}}[\varphi_{\mathrm{test}}] \times \frac{1}{N} \mathrm{Tr}[y\sigma_\mu], \quad \hat{\varphi}_{\mathrm{test}} := \varphi_{\mathrm{test}} \otimes \sigma_\mu, \tag{4.1.11}$$

we have the follwing conditions:

$$\mathrm{Tr}[yaqa] = \mathrm{Tr}[ybab] = N, \tag{4.1.12}$$
$$\mathrm{Tr}[yaqb] = \mathrm{Tr}[ybqa] = 0, \tag{4.1.13}$$
$$\mathrm{Tr}[ya^3] = N, \tag{4.1.14}$$
$$\mathrm{Tr}[yab^2] = -\mathrm{Tr}[ybab] = \mathrm{Tr}[yb^2a] = -N, \tag{4.1.15}$$
$$\mathrm{Tr}[ya^2b] = \mathrm{Tr}[yaba] = \mathrm{Tr}[ybaa] = 0, \tag{4.1.16}$$

where we used the cyclicity propertiy (a state in the GSO(−) sector has a half-integer conformal weight)

$$\mathrm{Tr}_{Y_{-2}}[\Psi_+ {\Psi_-}^2] = -\mathrm{Tr}_{Y_{-2}}[\Psi_- \Psi_+ \Psi_-] = \mathrm{Tr}_{Y_{-2}}[{\Psi_-}^2 \Psi_+]. \tag{4.1.17}$$

The conditions (4.1.12)-(4.1.16) are satisfies, if we demand the following equations:

$$y = a, \quad q = a, \quad a^2 = I_N, \quad b^2 = -I_N, \quad \{a, b\} = 0. \tag{4.1.18}$$

For $N = 2$, we find a solution for (4.1.18):

$$a = \sigma_3, \quad b = i\sigma_2, \tag{4.1.19}$$

then we have

$$\hat{\Psi} = \Psi_+ \otimes \sigma_3 + \Psi_- \otimes i\sigma_2 = \begin{pmatrix} \Psi_+ & \Psi_- \\ -\Psi_- & -\Psi_+ \end{pmatrix}, \tag{4.1.20}$$

where σ_i are the Pauli matrices. Therefore, the action and the EOMs become[1]

$$S = \hat{\mathrm{Tr}}_{\hat{Y}_{-2}} \left[\frac{1}{2} \hat{\Psi} \hat{Q} \hat{\Psi} + \frac{1}{3} \hat{\Psi}^3 \right], \tag{4.1.21}$$
$$\hat{Y}_{-2}(\hat{Q}\hat{\Psi} + \hat{\Psi}^2) = 0. \tag{4.1.22}$$

[1]If we only consider solutions which are constructed by using the string field in the extended KBc algebra, we may think the EOM as $Q\Psi + \Psi^2 = 0$ since there are no c and γ at the midpoint other than the PCO.

Unless the operator Y_{-2} exists, these are the same forms as the bosonic ones. Therefore, the gauge transformation is same:

$$\hat{\Psi} \xrightarrow{\hat{U}} \hat{U}^{-1}(\hat{Q} + \hat{\Psi})\hat{U}, \tag{4.1.23}$$

where the string field $\hat{U}$ is the matrix-valued gauge parameter.

4.2 $KBcG\gamma$ Algebra

To construct analytic solutions, as in the case of the bosonic theory, we give the extension of the KBc algebra in the superstring theory. We define the string fields K, B, c, G, γ [27] which are closed under Q and $*$:

$$\hat{\mathrm{Tr}}_{\hat{Y}_{-2}}[\hat{\varphi}_{\mathrm{test}} \cdot \hat{K}] := \langle\, Y_{-2}(i\infty) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \cdot \int_{\downarrow \frac{1}{2}} \frac{dz}{2\pi i} T(z) \,\rangle_{\mathrm{C}_1} \times \frac{1}{2}\mathrm{Tr}[\sigma_3 \sigma_\mu \cdot I_2], \tag{4.2.1}$$

$$\hat{\mathrm{Tr}}_{\hat{Y}_{-2}}[\hat{\varphi}_{\mathrm{test}} \cdot \hat{B}] := \langle\, Y_{-2}(i\infty) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \cdot \int_{\downarrow \frac{1}{2}} \frac{dz}{2\pi i} b(z) \,\rangle_{\mathrm{C}_1} \times \frac{1}{2}\mathrm{Tr}[\sigma_3 \sigma_\mu \cdot \sigma_3], \tag{4.2.2}$$

$$\hat{\mathrm{Tr}}_{\hat{Y}_{-2}}[\hat{\varphi}_{\mathrm{test}} \cdot \hat{c}] := \langle\, Y_{-2}(i\infty) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \cdot c(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \times \frac{1}{2}\mathrm{Tr}[\sigma_3 \sigma_\mu \cdot \sigma_3], \tag{4.2.3}$$

$$\hat{\mathrm{Tr}}_{\hat{Y}_{-2}}[\hat{\varphi}_{\mathrm{test}} \cdot \hat{G}] := \langle\, Y_{-2}(i\infty) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \cdot \int_{\downarrow \frac{1}{2}} \frac{dz}{2\pi i} G(z) \,\rangle_{\mathrm{C}_1} \times \frac{1}{2}\mathrm{Tr}[\sigma_3 \sigma_\mu \cdot \sigma_1], \tag{4.2.4}$$

$$\hat{\mathrm{Tr}}_{\hat{Y}_{-2}}[\hat{\varphi}_{\mathrm{test}} \cdot \hat{\gamma}] := \langle\, Y_{-2}(i\infty) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0) \cdot \gamma(\tfrac{1}{2}) \,\rangle_{\mathrm{C}_1} \times \frac{1}{2}\mathrm{Tr}[\sigma_3 \sigma_\mu \cdot i\sigma_2]. \tag{4.2.5}$$

Here,

$$T(z) = \left(-\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu - \partial b \cdot c - 2b\partial c - \frac{1}{2}\partial\beta \cdot \gamma - \frac{3}{2}\beta\partial\gamma \right)(z), \tag{4.2.6}$$

$$G(z) = \left(i\sqrt{2}\psi^\mu \partial X \mu + \partial\beta \cdot c + \frac{3}{2}\beta\partial c - 2b\gamma \right)(z). \tag{4.2.7}$$

$T(z)$ is the energy momentum tensor and $G(z)$ is the super current in the superconformal field theory.

Next, we give the algebra obtained by using the following OPEs:

$$b(z)c(0) \sim \frac{1}{z}, \quad \beta(z)\gamma(0) \sim -\frac{1}{z}, \tag{4.2.8}$$

$$G(z)G(0) \sim \frac{2T(0)}{z} + \ldots, \quad G(z)c(0) \sim \frac{-2\gamma(0)}{z}, \quad G(z)\gamma(0) \sim \frac{-\partial c(0)}{2z}, \tag{4.2.9}$$

$$j_{\mathrm{B}}(z)j_{\mathrm{B}}(0) \sim 0, \quad j_{\mathrm{B}}(z)b(0) \sim \frac{T(0)}{z},$$

$$j_{\mathrm{B}}(z)c(0) \sim \frac{(c\partial c - \gamma^2)(0)}{z}, \quad j_{\mathrm{B}}(z)\gamma(0) \sim \frac{(c\partial\gamma - \frac{1}{2}\partial c\gamma)(0)}{z}, \tag{4.2.10}$$

where,

$$j_{\mathrm{B}}(z) = \left(cT^{\mathrm{m}} + \gamma G^{\mathrm{m}} + \frac{1}{2} cT^{\mathrm{g}} + \frac{1}{2} \gamma G^{\mathrm{g}} \right)(z),$$
$$\hat{Q} = Q \otimes \sigma_3, \quad Q = \oint \frac{dz}{2\pi i} j_{\mathrm{B}}(z), \tag{4.2.11}$$

$$[\hat{K}, \hat{B}] = 0, \quad [\hat{K}, \hat{c}] := \hat{\partial}\hat{c}, \quad [\hat{K}, \hat{\gamma}] := \hat{\partial}\hat{\gamma},$$
$$\{\hat{B}, \hat{B}\} = \{\hat{c}, \hat{c}\} = 0, \quad \{\hat{B}, \hat{c}\} = 1, \quad \{\hat{\gamma}, \hat{c}\} = \{\hat{\gamma}, \hat{B}\} = 0, \tag{4.2.12}$$
$$\hat{\delta}\hat{G} = 2\hat{K}, \quad \hat{\delta}\hat{c} = 2\hat{\gamma}, \quad \hat{\delta}\hat{\gamma} = \frac{1}{2}\hat{\partial}\hat{c}, \quad \hat{\delta}\hat{\gamma}^2 = \hat{\partial}\hat{c}\hat{\gamma}, \quad \hat{\delta}\hat{K} = \hat{\delta}\hat{B} = 0, \tag{4.2.13}$$
$$\hat{Q}\hat{K} = 0, \quad \hat{Q}\hat{B} = \hat{K}, \quad \hat{Q}\hat{c} = \hat{c}\hat{\partial}\hat{c} + \hat{\gamma}^2, \quad \hat{Q}\hat{\gamma} = \hat{c}\hat{\partial}\hat{\gamma} - \frac{1}{2}\hat{\partial}\hat{c}\hat{\gamma}. \tag{4.2.14}$$

Here, the equations (4.2.13) correspond to the superconformal transformation. We give the derivations of the algebra in appendix B.

4.3 Known Solutions

4.3.1 Tachyon Vacuum Solution

We discuss tachyon vacuum solutions in the modified cubic superstring field theory. First, the Schnabl-like tachyon vacuum solution [54] was constructed. Next, the "simple" tachyon vacuum solution [55] was obtained from the following gauge parameter, which is appeared in the bosonic cubic theory:

$$U_1 = Bc + cBG_1, \quad G_1 = \frac{-K}{1-K}. \tag{4.3.1}$$

Then the tachyon vacuum solution is expressed as

$$\begin{aligned} \Psi_1 \xleftarrow{U_1{}^{-1}} \Psi_0 &= U_1{}^{-1} Q U_1 \\ &= (-B\gamma^2 - cB(1-K)c)\frac{1}{1-K} \\ &= -(Q(cB) + c)\frac{1}{1-K}. \end{aligned} \tag{4.3.2}$$

The symbol hat " ˆ " which represents that the string field has the Pauli matrices is usually omitted hereafter.

Let us check that the energy of the solution,

$$E(\Psi_0) = -S(\Psi_0) = -\mathrm{Tr}_{Y_{-2}}\left[\frac{1}{2}\Psi_0 Q \Psi_0 + \frac{1}{3}\Psi_0{}^3\right], \tag{4.3.3}$$

is lower than the perturbative vaccum $\Psi_1 = 0$ by the unit of the tension of the D9-brane

$T_9 := 1/2\pi^2$. We calculate the kinetic term of the action:

$$
\begin{aligned}
\mathrm{Tr}_{Y_{-2}}[\Psi_0 Q\Psi_0] &= \mathrm{Tr}_{Y_{-2}}\left[(Q(cB)+c)\frac{1}{1-K}Q\left((Q(cB)+c)\frac{1}{1-K}\right)\right] \\
&= \mathrm{Tr}_{Y_{-2}}\left[c\frac{1}{1-K}Qc\frac{1}{1-K}\right] \\
&= \mathrm{Tr}_{Y_{-2}}\left[c\frac{1}{1-K}cKc\frac{1}{1-K}+c\frac{1}{1-K}\gamma^2\frac{1}{1-K}\right]. \qquad (4.3.4)
\end{aligned}
$$

The first term in the last line vanishes because of the ϕ momentum conservation. The ϕ momentum conservation means that for non-vanishing $\mathrm{Tr}_{Y_{-2}}[\varphi]$, including the effect of Y_{-2}, the ghost number of φ is the bc-ghost number one and the ghost number three, i.e., φ needs a c and two γ's in the algebra. Then, the kinetic term becomes

$$
\begin{aligned}
\mathrm{Tr}_{Y_{-2}}[\Psi_0 Q\Psi_0] &= \mathrm{Tr}_{Y_{-2}}\left[c\frac{1}{1-K}\gamma^2\frac{1}{1-K}\right] \\
&= \iint_0^\infty dx_1 dx_2\, e^{-(x_1+x_2)}\mathrm{Tr}_{Y_{-2}}\left[c\Omega^{x_1}\gamma^2\Omega^{x_2}\right]. \qquad (4.3.5)
\end{aligned}
$$

By using the following correlator [54], which we will derive in the appendix C:

$$
\mathrm{Tr}_{Y_{-2}}[c\Omega^{t_1}\gamma\Omega^0\gamma\Omega^{t_2}] = \frac{L^2}{2\pi^2}, \qquad (4.3.6)
$$

we have the energy of the tachyon vacuum solution:

$$
\begin{aligned}
E(\Psi_0) &= -\frac{1}{6}\iint_0^\infty dx_1 dx_2\, e^{-(x_1+x_2)}\frac{(x_1+x_2)^2}{2\pi^2} \\
&= -\frac{1}{6}\int_0^\infty a\, da\int_0^1 db\, e^{-a}\frac{a^2}{2\pi^2} \\
&= -\frac{1}{2\pi^2} = -T_9. \qquad (4.3.7)
\end{aligned}
$$

Here, we used the EOMS since there is no singularity in the tachyon vacuum solution.

4.3.2 Half-brane Solution

Next, we discuss the half-brane solution [27][2]. By using the gauge parameter,

$$
U_{1/2} := Bc + cB\frac{-G}{1-G}, \qquad (4.3.8)
$$

[2]For related work, see [56].

the half-brane solution in the pure-gauge form is written formally as

$$\Psi_1 \xleftarrow{{U_{1/2}}^{-1}} \Psi_{1/2} := {U_{1/2}}^{-1} Q U_{1/2}, \tag{4.3.9}$$

$$= -(B\gamma^2 + cB(1-G)Gc)\frac{1}{1-G}$$

$$= -(Q(cB) + cBGc)\frac{1}{1-G}. \tag{4.3.10}$$

Note that the gauge parameter $U_{1/2}$ can be obtained from U_1 by replacing K with G.

Let us give a brief summary of the calculation of the energy; detailed calculations are given in the appendix D. Since we do not have to introduce the regularization, we can evaluate the energy from the cubic term of the action:

$$E(\Psi_{1/2}) = -S(\Psi_{1/2}) = \frac{1}{6}\mathrm{Tr}_{Y_{-2}}[{\Psi_{1/2}}^3]. \tag{4.3.11}$$

The cubic term is

$$\mathrm{Tr}_{Y_{-2}}\left[{\Psi_{1/2}}^3\right] = -3\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2 Gc\frac{1}{1-G}cG\right] \tag{4.3.12}$$

$$+ \mathrm{Tr}_{Y_{-2}}\left[\left(cB(1-G)Gc\frac{-1}{1-G}\right)^3\right]. \tag{4.3.13}$$

By using the following relation for the string field φ whose "internal CP factor" is σ_2[3] :

$$\mathrm{Tr}_{Y_{-2}}[G\varphi] = \frac{1}{2}\mathrm{Tr}_{Y_{-2}}[G\varphi + \varphi G] = \frac{1}{2}\mathrm{Tr}_{Y_{-2}}[\delta\varphi], \tag{4.3.14}$$

where the string field $\delta\varphi$ is the superconformal transformation of the string field φ, the two terms of the cubic term are

$$(4.3.12) = \frac{3}{2\pi^2},$$

$$(4.3.13) = -\frac{6(\pi^2-6)}{\pi^4} - 3\times\frac{12-\pi^2}{\pi^4} = -\frac{3}{\pi^2}. \tag{4.3.15}$$

Therefore, we have

$$E(\Psi_{1/2}) = \frac{1}{6}\left(\frac{3}{2\pi^2} - \frac{3}{\pi^2}\right) = -\frac{1}{4\pi^2} = E(\Psi_0) + \frac{1}{2}T_9. \tag{4.3.16}$$

The energy of the half-brane solution is one half the tension of the D9-brane T_9.

[3] If φ has another Pauli matrix, the $\mathrm{Tr}_{Y_{-2}}[G\varphi]$ vanishes since $\mathrm{Tr}[\sigma_3\sigma_1\sigma_{i\neq 2}] = 0$.

Chapter 5

Multiple-half-brane Solution

5.1 Gauge Equivalence between ${U_{1/2}}^2$ and U_1

We would like to construct a new solution by using the $KBcG\gamma$ algebra which reproduces the energy as n-half times the tension of D9-brane, i.e., a multiple-half-brane solution. The first step to construct it is to construct the tachyon vacuum solution by using the algebra. The gauge transformation with the gauge parameter $U_{1/2}$ decreases the energy by one half the tension of the D9-brane, $T_9/2$. So, we guess that the gauge parameter ${U_{1/2}}^2$ is equivalent to U_1 up to a regular gauge transformation. We see that this is fact.

5.1.1 From the Form of the Gauge Parameter

In [27], the pure-gauge-form solutions are classified from the form of the function of K and G in the gauge parameter. In our notation, the gauge parameter is as follows:

$$U = Bc + cBg(K,G). \tag{5.1.1}$$

By using the properties $G^2 = K$ and $[G,K] = 0$, we can rewrite the function $g(K,G)$ as follows:[1]

$$g(K,G) = g_+(K) + Gg_-(K). \tag{5.1.2}$$

The solutions are classified by $g_\pm(K)$:

$$\begin{aligned} \text{Pure Gauge}: &\quad g_+(0) \neq 0, && (5.1.3)\\ \text{Half Brane}: &\quad g_+(0) = 0,\ g_-(0) \neq 0, && (5.1.4)\\ \text{Tachyon Vacuum}: &\quad g_+(0) = 0,\ g_-(0) = 0,\ \partial_K g_+(0) \neq 0. && (5.1.5) \end{aligned}$$

Let us assume that the gauge parameter U_{half} belongs to the class (5.1.4), i.e.,

$$U_{\text{half}} = Bc + cB\big(h_+(K) + Gh_-(K)\big), \quad h_+(0) = 0, \quad h_-(0) \neq 0. \tag{5.1.6}$$

[1]We assume $f(G)$ is the polynomyal of the G.

Then, we can show that ${U_{\rm half}}^2 =: Bc + cB(\tilde{g}_+(K) + G\tilde{g}_-(K))$ satisfies the properties of the tachyon vacuum (5.1.5). First, by using $G^2 = K$, $[G, K] = 0$ and (2.3.8), ${U_{\rm half}}^2$ becomes

$$ {U_{\rm half}}^2 = Bc + cB\big(h_+(K)^2 + h_-(K)^2 K + G \cdot 2h_+(K)h_-(K)\big), \tag{5.1.7} $$

and then we find $\tilde{g}_\pm(K)$ and $\partial_K \tilde{g}_+(K)$ as

$$ \tilde{g}_+(K) = h_+(K)^2 + h_-(K)^2 K, \tag{5.1.8} $$
$$ \tilde{g}_-(K) = 2h_+(K)h_-(K), \tag{5.1.9} $$
$$ \partial_K \tilde{g}_+(K) = 2\partial_K h_+(K) \cdot h_+(K) + 2\partial_K h_-(K) \cdot h_-(K)K + h_-(K)^2. \tag{5.1.10} $$

Therefore, we obtain that $\tilde{g}(K)$ satisfies the tachyon vacuum properties (5.1.5):

$$ \tilde{g}_+(0) = h_-(0)^2 \times 0 = 0, \tag{5.1.11} $$
$$ \tilde{g}_-(0) = h_-(0) \times 0 = 0, \tag{5.1.12} $$
$$ \partial_K \tilde{g}_+(0) = 2\partial_K h_+(0) \times 0 + 2\partial_K h_-(0) \times 0 + h_-(0)^2 = h_-(0)^2 \neq 0. \tag{5.1.13} $$

Here, we assume that $|h_-(0)|$, $|\partial_K h_\pm(0)| < \infty$.

5.1.2 From the Energy of the Pure-gauge Solution $\tilde{U}^{-1}Q\tilde{U}$

We show that the gauge parameter ${U_{1/2}}^2$ is gauge equivalent to U_1 more directly. We think that the gauge transformation $\tilde{U}$ connecting ${U_{1/2}}^2$ and U_1, and we show that the gauge transformation $\tilde{U}$ is regular. The regularity can be read by calculating the energy of the solution $\tilde{\Psi} := \tilde{U}^{-1}Q\tilde{U}$, since the change of the action for the finite gauge transformation is (2.1.20). The gauge parameter $\tilde{U}$ s.t.

$$ {U_{1/2}}^{-2} Q {U_{1/2}}^2 =: \Psi_{0/2} \xrightarrow{\tilde{U}} \Psi_0 = {U_1}^{-1} Q U_1, \tag{5.1.14} $$

is given by

$$ \begin{aligned} \tilde{U} &= {U_{1/2}}^{-2} U_1 \\ &= Bc + cB\left(\frac{1-G}{-G}\right)^2 \left(\frac{-K}{1-K}\right) \\ &= Bc + cB\left(-\frac{1-G}{1+G}\right). \end{aligned} \tag{5.1.15} $$

Here the explicit form of $\hat{\Psi}_{0/2}$ is given by

$$\begin{aligned}\hat{\Psi}_{0/2} &= (\hat{B}\hat{\gamma}^2 - \hat{c}\hat{B}(1-\hat{G})^2\hat{c})\frac{2\hat{G}-1}{(1-\hat{G})^2} \\ &= \left(Q\left(cB\frac{3K-1}{(1-K)^2}\right) - 4cBGc\frac{KG}{(1-K)^2}\right)\otimes\sigma_3 \\ &\quad + \left(-2Q\left(cB\frac{KG}{(1-K)^2}\right) - 2cBGc\frac{3K-1}{(1-K)^2}\right)\otimes i\sigma_2. \end{aligned} \tag{5.1.16}$$

Note that the solution $\hat{\Psi}_{0/2}$ has the string fields in the GSO(−) sector.

Let us consider the pure-gauge-form solution whose gauge parameter is $\tilde{U}$ is

$$\begin{aligned}\Psi_1 \xrightarrow{\tilde{U}} \tilde{\Psi} &:= \tilde{U}^{-1}Q\tilde{U} \\ &= \left(B\gamma^2 + cBK\frac{1+G}{1-G}c\right)\frac{-2}{1+G} \\ &= \left(Q(cB) + 2cB\frac{K}{1-G}c\right)\frac{-2}{1+G}. \end{aligned} \tag{5.1.17}$$

We check wether the energy of this solution $\tilde{\Psi}$ is zero or not. This solution $\tilde{\Psi}$ does not contain the singular string field, therefore we evaluate the energy only from the cubic term:

$$\begin{aligned}\mathrm{Tr}_{Y_{-2}}\big[\tilde{\Psi}^3\big] &= \mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{-2}{1+G}B\gamma^2\frac{-2}{1+G}B\gamma^2\frac{-2}{1+G}\right] \\ &\quad + 3\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{-2}{1+G}B\gamma^2\frac{-2}{1+G}cBK\frac{1+G}{1-G}c\frac{-2}{1+G}\right] \\ &\quad + 3\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{-2}{1+G}cBK\frac{1+G}{1-G}c\frac{-2}{1+G}cBK\frac{1+G}{1-G}c\frac{-2}{1+G}\right] \\ &\quad + \mathrm{Tr}_{Y_{-2}}\left[\left(cBK\frac{1+G}{1-G}c\frac{-2}{1+G}\right)^3\right] \\ &= -3\cdot 2^3\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{K}{1-G}c\frac{1}{1+G}c\frac{K}{1-G}\right] \qquad (5.1.18) \\ &\quad - 2^3\mathrm{Tr}_{Y_{-2}}\left[\left(cBK\frac{1+G}{1-G}c\frac{1}{1+G}\right)^3\right]. \qquad (5.1.19)\end{aligned}$$

The first term (5.1.18) in the last form is computed as

$$
\begin{aligned}
(5.1.18) &= -3\cdot 2^3 \mathrm{Tr}_{Y_{-2}}\left[B\gamma^2 \frac{K}{1-K}(1+G)c\frac{1}{1+G}c\frac{K}{1-G}\right] \\
&= -3\cdot 2^3 \mathrm{Tr}_{Y_{-2}}\left[B\gamma^2 \frac{K}{1-K}\big([1+G,c]+c(1+G)\big)\frac{1}{1+G}c\frac{K}{1-G}\right] \\
&= -3\cdot 2^3 \mathrm{Tr}_{Y_{-2}}\left[B\gamma^2 \frac{K}{1-K}\delta c\frac{1}{1+G}c\frac{K}{1-G}\right] \\
&= -3\cdot 2^3 \mathrm{Tr}_{Y_{-2}}\left[B\gamma^2 \frac{K}{1-K}\delta c\frac{1}{1+G}\left(\left[c,\frac{1}{1-G}\right]+\frac{1}{1-G}c\right)K\right] \\
&= 0, \qquad (5.1.20)
\end{aligned}
$$

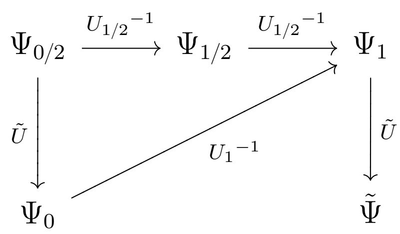
where we used $\mathrm{Tr}_{Y_{-2}}[B\varphi]=0$ for φ s.t. $[B,\varphi]=0$, and the ϕ momentum conservation. The second term (5.1.19) is computed as

$$
\begin{aligned}
(5.1.19) &= -2^3 \mathrm{Tr}_{Y_{-2}}\left[cBK\frac{1+G}{1-G}c\frac{1}{1+G}cBK\frac{1+G}{1-G}c\frac{1}{1+G}cBK\frac{1+G}{1-G}c\frac{1}{1+G}\right] \\
&= -2^3 \mathrm{Tr}_{Y_{-2}}\left[cB\frac{K}{1-G}\delta c\frac{K}{1+G}\frac{1}{1-G}\delta c\frac{K}{1-G}\frac{1}{1+G}\delta c\frac{1}{1+G}\right] \\
&= -2^3 \mathrm{Tr}_{Y_{-2}}\left[cB\frac{K}{1-K}\delta c\frac{K}{1-K}\delta c\frac{K}{1-K}\delta c\right] \\
&= 0, \qquad (5.1.21)
\end{aligned}
$$

where we used the ϕ momentum conservation. Therefore, the energy of the solution $\tilde{\Psi}$ is zero:

$$E(\tilde{\Psi}) = (5.1.20) + (5.1.21) = 0. \qquad (5.1.22)$$

Namely, the gauge transformation $\tilde{U}$ is regular. Since the regular gauge transformation does not change the physics, the solution $\Psi_{0/2} = {U_{1/2}}^{-2} Q {U_{1/2}}^{2}$ is gauge equivalent to the tachyon vacuum $\Psi_0 = {U_1}^{-1} Q U_1$. We show this result in the following diagram.



The arrows represent the gauge transformations whose gauge parameters are denoted with the arrows.

We list the energy of the solutions in the above diagram:

$$
\begin{aligned}
E(\Psi_{0/2}) &= E(\Psi_0), \\
E(\Psi_{1/2}) &= E(\Psi_0) + \frac{1}{2}T_9, \\
E(\Psi_1) = E\big(\tilde{\Psi}\big) &= E(\Psi_0) + T_9.
\end{aligned}
\tag{5.1.23}
$$

5.2 Multiple-half-brane Solution $\Psi_{3/2}$

5.2.1 Solution

Next, we construct a new solution by performing the singular gauge transformation ${U_{1/2}}^{-1}$ for the tachyon vacuum solution $\Psi_{0/2}$ three times:

$$
\Psi_{0/2} \xrightarrow{{U_{1/2}}^{-1}} \Psi_{1/2} \xrightarrow{{U_{1/2}}^{-1}} \Psi_1 \xrightarrow{{U_{1/2}}^{-1}} \Psi_{3/2}.
$$

The explicit form of the solution $\Psi_{3/2}$ is given by

$$
\begin{aligned}
\Psi_{3/2} &= {U_{1/2}}^3 (Q + \Psi_{0/2}) {U_{1/2}}^{-3} \\
&= U_{1/2} Q {U_{1/2}}^{-1} \\
&= \left(B\gamma^2 + cB \frac{GK}{1-G} c \right) \frac{1}{-G}.
\end{aligned}
\tag{5.2.1}
$$

We conjecture that the energy of this solution $\Psi_{3/2}$ is $3/2$ times the tension of the D9-brane T_9, $E(\Psi_{3/2}) \stackrel{?}{=} E(\Psi_0) + \frac{3}{2}T_9$, because the number of times of the gauge transformation from the tachyon vacuum is 3 and the gauge transformation increases the energy by one half of the tension of the D9-brane T_9.

5.2.2 G_ϵ-Regularization

Since the string field $1/\hat{G}$ [2] has the singular string field $1/\hat{K}$ as follows:

$$
\frac{1}{\hat{G}} = \hat{G} \frac{1}{\hat{K}},
\tag{5.2.2}
$$

we need a regularization for $\hat{G}$, as $\hat{K}$. Since the K_ϵ-regularization works well in the bosonic cubic SFT, we would like to keep the regularization for $\hat{K}$, so we demand

$$
[\![\hat{G}]\!]_\epsilon^{\,2} = [\![\hat{K}]\!]_\epsilon .
\tag{5.2.3}
$$

[2] We again denote the hat in this subsection.

This is satisfied by the following regularization[3]:

$$\hat{G}_\epsilon \coloneqq \hat{G} - \sqrt{-\epsilon} \otimes \sigma_3 = \begin{pmatrix} -\sqrt{-\epsilon} & G \\ G & \sqrt{-\epsilon} \end{pmatrix}. \tag{5.2.4}$$

Here we denote $[\![f(\hat{G}, \hat{K}, \hat{B}, \hat{c})]\!]_\epsilon = f(\hat{G}_\epsilon, \hat{K}_\epsilon, \hat{B}, \hat{c})$. We can check the G_ϵ-regularization satisfies the equation (5.2.3):

$$\begin{aligned}\{\hat{G}_\epsilon, \hat{G}_\epsilon\} &= 2\hat{G}_\epsilon \cdot \hat{G}_\epsilon \\ &= 2\big(\hat{G}_\epsilon{}^2 - \sqrt{-\epsilon} G \otimes \sigma_1\sigma_3 - \sqrt{-\epsilon} G \otimes \sigma_3\sigma_1 + \sqrt{-\epsilon}^2 \otimes \sigma_3{}^2\big) \\ &= 2(\hat{K} - \hat{\epsilon}) \\ &= 2\hat{K}_\epsilon, \end{aligned} \tag{5.2.5}$$

where $\{\sigma_i, \sigma_j\} = 0,\ i \neq j$. Then, we have the inverse of $\hat{G}_\epsilon$:

$$\frac{1}{\hat{G}_\epsilon} = \hat{G}_\epsilon \frac{1}{\hat{K}_\epsilon} = G\frac{1}{K_\epsilon} \otimes \sigma_1 - \sqrt{-\epsilon}\frac{1}{K_\epsilon} \otimes \sigma_3 = \begin{pmatrix} -\sqrt{-\epsilon}\frac{1}{K_\epsilon} & G\frac{1}{K_\epsilon} \\ G\frac{1}{K_\epsilon} & \sqrt{-\epsilon}\frac{1}{K_\epsilon} \end{pmatrix}. \tag{5.2.6}$$

Similarly, we can check other relations in the algebra including the string field G:

$$[\hat{G}_\epsilon, \hat{c}] = 2\hat{\gamma}, \quad [\hat{G}_\epsilon, \hat{B}] = 0, \quad [\hat{G}_\epsilon, \hat{K}] = 0, \quad \{\hat{G}_\epsilon, \hat{\gamma}\} = \frac{1}{2}\hat{\partial}\hat{c}, \tag{5.2.7}$$

and

$$\hat{Q}\hat{G}_\epsilon = \hat{Q}\hat{G} - Q\sqrt{-\epsilon} \otimes I_2 = 0. \tag{5.2.8}$$

We regularize the solution $\Psi_{3/2}$ by using the G_ϵ-regularization as

$$[\![\hat{\Psi}_{3/2}]\!]_\epsilon = \left(\hat{B}\hat{\gamma}^2 + \hat{c}\hat{B}\frac{\hat{G}_\epsilon \hat{K}_\epsilon}{1 - \hat{G}_\epsilon}\hat{c}\right)\frac{1}{-\hat{G}_\epsilon}. \tag{5.2.9}$$

Once it is regularized, $[\![\hat{\Psi}_{3/2}]\!]_\epsilon$ has the definition in the sliver frame.

[3]We also have the same result by taking the regularization as $\hat{G}_\epsilon \coloneqq \hat{G} + \sqrt{-\epsilon} \otimes \sigma_3$.

5.2.3 Equation of Motion in the Strong Sense

Since we have regularized the solution, we have to check the EOMS as before. The EOMS for the solution $\Psi_{3/2}$ is given by

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathrm{EOMS}([\![\Psi_{3/2}]\!]_\epsilon) &= \lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[[\![\Psi_{3/2}]\!]_\epsilon\frac{\partial}{\partial B}[\![\Psi_{3/2}]\!]_\epsilon\right] \\
&= \lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[\left(B\gamma^2+cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\right)\frac{1}{-G_\epsilon}\left(\gamma^2-c\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\right)\frac{1}{-G_\epsilon}\right] \\
&= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}c\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right] \qquad (5.2.10)\\
&\quad +\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\gamma^2\frac{1}{-G_\epsilon}\right] \qquad (5.2.11)\\
&\quad -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}c\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right]. \qquad (5.2.12)
\end{aligned}$$

The EOMS decomposes into three terms. The first term (5.2.10) becomes

$$\begin{aligned}
(5.2.10) &= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}c\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right] \\
&= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}(-\delta c+G_\epsilon c)\frac{1}{1-G_\epsilon}(\partial c+cK_\epsilon)\frac{1}{-G_\epsilon}\right] \\
&= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}(-\delta c)\frac{1}{1-G_\epsilon}(cK_\epsilon)\frac{1}{-G_\epsilon}\right] \qquad (5.2.13)\\
&\quad -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}(G_\epsilon c)\frac{1}{1-G_\epsilon}(\partial c)\frac{1}{-G_\epsilon}\right], \qquad (5.2.14)
\end{aligned}$$

where we used the following equations:

$$\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}(-\delta c)\frac{1}{1-G_\epsilon}(\partial c)\frac{1}{-G_\epsilon}\right]=0, \qquad (5.2.15)$$

$$\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}(G_\epsilon c)\frac{1}{1-G_\epsilon}(cK_\epsilon)\frac{1}{-G_\epsilon}\right]=0. \qquad (5.2.16)$$

Here we use $\mathrm{Tr}_{Y_{-2}}[B\varphi]=0$ for φ s.t. $[B,\varphi]=0$ and take the limit $\epsilon\to 0$ since there is no $1/K_\epsilon$ or $1/G_\epsilon$. The first term (5.2.13) becomes

$$
\begin{aligned}
&(5.2.13)\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}\delta c\frac{1}{1-G_\epsilon}cG_\epsilon\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2G_\epsilon\frac{1}{-K_\epsilon}\delta c(1+G_\epsilon)\frac{1}{1-K_\epsilon}cG_\epsilon\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2G_\epsilon\frac{1}{-K_\epsilon}\delta cG_\epsilon\frac{1}{1-K_\epsilon}cG_\epsilon\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2G_\epsilon\frac{1}{-K_\epsilon}\delta c\frac{1}{1-K_\epsilon}cK_\epsilon\right]\\
&=-\lim_{\epsilon\to 0}\frac{\epsilon}{2}\mathrm{Tr}_{Y_{-2}}\left[\delta\left(\delta c\frac{1}{1-K_\epsilon}cK_\epsilon B\gamma^2\frac{1}{-K_\epsilon}\right)\right]\\
&=-\lim_{\epsilon\to 0}\left(\frac{\epsilon}{2}\mathrm{Tr}_{Y_{-2}}\left[(\partial c)\frac{1}{1-K_\epsilon}cK_\epsilon B\gamma^2\frac{1}{-K_\epsilon}\right]+\frac{\epsilon}{2}\mathrm{Tr}_{Y_{-2}}\left[\delta c\frac{1}{1-K_\epsilon}cK_\epsilon B(\partial c\gamma)\frac{1}{-K_\epsilon}\right]\right)\\
&=\lim_{\epsilon\to 0}\left(\frac{\epsilon}{2}\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\frac{K_\epsilon}{1-K_\epsilon}\gamma^2\frac{1}{-K_\epsilon}\right]+\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\gamma\frac{1}{-K_\epsilon}\partial\gamma\frac{1}{1-K_\epsilon}\right]\right). \qquad (5.2.17)
\end{aligned}
$$

We can show that both terms in the last line vanish. The first term vanishes in the following manner:

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{\epsilon}{2}\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\frac{K_\epsilon}{1-K_\epsilon}\gamma^2\frac{1}{-K_\epsilon}\right]&=-\lim_{\epsilon\to 0}\frac{\epsilon}{2}\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\frac{1}{1-K_\epsilon}\gamma^2\right]\\
&\sim\lim_{\epsilon\to 0}\epsilon\times(\text{finite})\\
&=0. \qquad (5.2.18)
\end{aligned}
$$

Here, in the first equality, we use the following relation:

$$
\begin{aligned}
&\mathrm{Tr}_{Y_{-2}}[Bc\partial^2c\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}]\\
&=\mathrm{Tr}_{Y_{-2}}[BcK\partial c\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}]-\mathrm{Tr}_{Y_{-2}}[Bc\partial cK\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}]\\
&=\mathrm{Tr}_{Y_{-2}}[B([c,K]+Kc)\partial c\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}]-\mathrm{Tr}_{Y_{-2}}[Bc\partial cK\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}]\\
&=\lim_{y\to 0}\partial_y\big\{\mathrm{Bcdgg}[t_2;t_1+t_2+t_3+y]-\mathrm{Bcdgg}[t_2;t_1+t_2+t_3+y]\big\}\\
&=0. \qquad (5.2.19)
\end{aligned}
$$

Here, the correlator $\mathrm{Bcdgg}[t_1;t_2]$ is defined in appendix C. The second term in (5.2.17) also vanishes as follows:

$$\begin{aligned}\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\gamma\frac{1}{-K_\epsilon}\partial\gamma\frac{1}{1-K_\epsilon}\right] &= \lim_{\epsilon\to0}\iint_0^\infty dx_1dz_1\lim_{y\to0}\partial_y e^{-(1+\epsilon)x_1}e^{-\epsilon z_1}\\ &\qquad\times\epsilon\big\{\mathrm{Bcdgg}[z_1+y;x_1+y+z_1]-\mathrm{d}[z_1;x_1+y+z_1]\big\}\\ &=\lim_{\epsilon\to0}\frac{\epsilon}{2}(\cos(\pi\epsilon)\mathrm{Ci}(\pi\epsilon)+\cdots)\\ &=0, \end{aligned} \tag{5.2.20}$$

where $\mathrm{Ci}(\pi\epsilon)=-\int_{\pi\epsilon}^\infty dt\,\frac{\cos t}{t}=\log\epsilon+\gamma_\mathrm{E}+\log\pi+\mathrm{O}(\epsilon^2)$, $\gamma_\mathrm{E}=0.577\cdots$.

We can check that the remaining terms (5.2.11), (5.2.12) and (5.2.14) are also zero in the limit $\epsilon\to0$, and hence the solution $\lim_{\epsilon\to0}[\![\Psi_{3/2}]\!]_\epsilon$ satisfies the EOMS. We will give the calculations of the remaining terms in appendix E.

5.2.4 Energy

We conjectured that the energy of the solution $\Psi_{3/2}$ is $3/2$ times the tension of the D9-brane. So we check the conjecture. Since the EOMS is zero:

$$\lim_{\epsilon\to0}\mathrm{EOMS}([\![\Psi_{3/2}]\!]_\epsilon)=0,$$

the energy can be calculated only by using the cubic term in the action:

$$\lim_{\epsilon\to0}E([\![\Psi_{3/2}]\!]_\epsilon)=-\lim_{\epsilon\to0}S([\![\Psi_{3/2}]\!]_\epsilon)=\lim_{\epsilon\to0}\frac{1}{6}\mathrm{Tr}_{Y_{-2}}[[\![\Psi_{3/2}]\!]_\epsilon{}^3]. \tag{5.2.21}$$

The cubic term decomposes into two terms:

$$\begin{aligned}\lim_{\epsilon\to0}\mathrm{Tr}_{Y_{-2}}[[\![\Psi_{3/2}]\!]_\epsilon{}^3] &= \lim_{\epsilon\to0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}B\gamma^2\frac{1}{-G_\epsilon}B\gamma^2\frac{1}{-G_\epsilon}\right]\\ &\quad+3\lim_{\epsilon\to0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}B\gamma^2\frac{1}{-G_\epsilon}cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right]\\ &\quad+3\lim_{\epsilon\to0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right]\\ &\quad+\lim_{\epsilon\to0}\mathrm{Tr}_{Y_{-2}}\left[\left(cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right)^3\right]\\ &=3\lim_{\epsilon\to0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}c\frac{K_\epsilon}{1-G_\epsilon}\right] \qquad (5.2.22)\\ &\quad+\lim_{\epsilon\to0}\mathrm{Tr}_{Y_{-2}}\left[\left(cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right)^3\right]. \qquad (5.2.23)\end{aligned}$$

The first term (5.2.22) becomes

$$
\begin{aligned}
(5.2.22) &= 3\lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2 K_\epsilon(1+G_\epsilon)\frac{1}{1-K_\epsilon}c\frac{G_\epsilon}{-K_\epsilon}cK_\epsilon(1+G_\epsilon)\frac{1}{1-K_\epsilon}\right]\\
&= 3\lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{K_\epsilon}{1-K_\epsilon}c\frac{G_\epsilon}{-K_\epsilon}c\frac{K_\epsilon G_\epsilon}{1-K_\epsilon}\right]\\
&\quad + 3\lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{K_\epsilon G_\epsilon}{1-K_\epsilon}c\frac{G_\epsilon}{-K_\epsilon}c\frac{K_\epsilon}{1-K_\epsilon}\right]\\
&= 3\lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{K_\epsilon}{1-K_\epsilon}(\delta c)\frac{G_\epsilon}{-K_\epsilon}c\frac{K_\epsilon}{1-K_\epsilon}\right]\\
&\quad - 3\lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{K_\epsilon}{1-K_\epsilon}c\frac{G_\epsilon}{-K_\epsilon}(\delta c)\frac{K_\epsilon}{1-K_\epsilon}\right]\\
&= -\frac{3}{2}\mathrm{Tr}_{Y_{-2}}\left[\delta\left(c\frac{1}{1-K}B\gamma^2\frac{K}{1-K}\delta c\right)\right]+\frac{3}{2}\mathrm{Tr}_{Y_{-2}}\left[\delta\left(\delta c\frac{K}{1-K}B\gamma^2\frac{1}{1-K}c\right)\right]\\
&= -\frac{3}{2}\mathrm{Tr}_{Y_{-2}}\left[c\frac{1}{1-K}B(2\delta\gamma\cdot\gamma)\frac{K}{1-K}\delta c\right]-\frac{3}{2}\mathrm{Tr}_{Y_{-2}}\left[c\frac{1}{1-K}B\gamma^2\frac{K}{1-K}(\partial c)\right]\\
&\quad + \frac{3}{2}\mathrm{Tr}_{Y_{-2}}\left[(\partial c)\frac{K}{1-K}B\gamma^2\frac{1}{1-K}c\right]-\frac{3}{2}\mathrm{Tr}_{Y_{-2}}\left[\delta c\frac{K}{1-K}B(2\delta\gamma\cdot\gamma)\frac{1}{1-K}c\right]\\
&= 6\iint_0^\infty dx_1dx_2\lim_{y\to 0}\partial_y e^{-(x_1+x_2)}\\
&\qquad\qquad \times\Big\{\mathrm{Bcdgg}[y+x_1;x_1+x_2+y]+3\cdot\mathrm{Bcdgg}[0;x_1+x_2+y]\Big\}\\
&= -\frac{3}{2\pi^2}.
\end{aligned}
\tag{5.2.24}
$$

The second term (5.2.23) reduces to the same term (D.5), which we find in appendix D where the energy of the half-brane solution is calculated:

$$
\begin{aligned}
(5.2.23) &= \lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}\left[cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right]\\
&= -\mathrm{Tr}_{Y_{-2}}\left[cB\frac{G_\epsilon}{1-G_\epsilon}\delta c\frac{G_\epsilon}{1-G_\epsilon}\delta c\frac{G_\epsilon}{1-G_\epsilon}\delta c\right]\\
&= -(\mathrm{D}.5)\\
&= \frac{3}{\pi^2}.
\end{aligned}
\tag{5.2.25}
$$

Therefore, we have checked our conjecture:

$$
\lim_{\epsilon\to 0}E([\![\Psi_{3/2}]\!]_\epsilon)=\frac{1}{6}\big((5.2.24)+(5.2.25)\big)=+\frac{1}{4\pi^2}=E(\Psi_0)+\frac{3}{2}T_9. \tag{5.2.26}
$$

5.3 Double-brane Solution

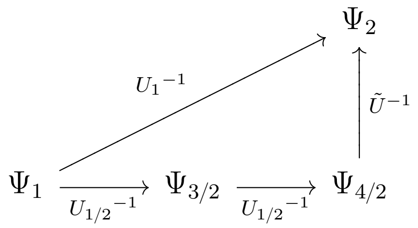
We do not consider $\Psi_{4/2}$:

$$
\Psi_{3/2}\xrightarrow{U_{1/2}^{-1}}\Psi_{4/2}, \tag{5.3.1}
$$

since the gauge equivalent string field Ψ_2 does not satisfy the EOMS. The latter string field Ψ_2 can be written in the pure-gauge form by using the gauge parameter ${U_1}^{-1}$:

$$\Psi_1 \xrightarrow{{U_1}^{-1}} \Psi_2 = U_1 Q {U_1}^{-1} = \left(B\gamma^2 + cB\frac{K^2}{1-K}c \right) \frac{1}{-K}, \tag{5.3.2}$$

and its relation to other string fields are summarized in the following diagram.



The EOMS for Ψ_2 decomposes into four terms:

$$\begin{aligned}
\lim_{\epsilon\to 0} \mathrm{EOMS}(\llbracket \Psi_2 \rrbracket_\epsilon) &= \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}}[\llbracket \Psi_2 \rrbracket_\epsilon \partial_B \llbracket \Psi_2 \rrbracket_\epsilon] \\
&= \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[\left(B\gamma^2 + cB\frac{K_\epsilon^2}{1-K_\epsilon}c \right) \frac{1}{-K_\epsilon} \left(\gamma^2 - c\frac{K_\epsilon^2}{1-K_\epsilon}c \right) \frac{1}{-K_\epsilon} \right] \\
&= \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[B\gamma^2 \frac{1}{-K_\epsilon} \gamma^2 \frac{1}{-K_\epsilon} \right] \\
&\quad - \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[B\gamma^2 \frac{1}{-K_\epsilon} c \frac{K_\epsilon^2}{1-K_\epsilon} c \frac{1}{-K_\epsilon} \right] \\
&\quad + \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[cB \frac{K_\epsilon^2}{1-K_\epsilon} c \frac{1}{-K_\epsilon} \gamma^2 \frac{1}{-K_\epsilon} \right] \\
&\quad - \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[cB \frac{K_\epsilon^2}{1-K_\epsilon} c \frac{1}{-K_\epsilon} c \frac{K_\epsilon^2}{1-K_\epsilon} c \frac{1}{-K_\epsilon} \right].
\end{aligned} \tag{5.3.3}$$

The first and the forth terms vanish:

$$\mathrm{Tr}_{Y_{-2}} \left[B\gamma^2 \frac{1}{-K_\epsilon} \gamma^2 \frac{1}{-K_\epsilon} \right] = 0, \tag{5.3.4}$$

$$\mathrm{Tr}_{Y_{-2}} \left[cB \frac{K_\epsilon^2}{1-K_\epsilon} c \frac{1}{-K_\epsilon} c \frac{K_\epsilon^2}{1-K_\epsilon} c \frac{1}{-K_\epsilon} \right] = 0, \tag{5.3.5}$$

because of the ϕ momentum conservation. The second term is

$$\begin{aligned}
&- \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[B\gamma^2 \frac{1}{-K_\epsilon} c \frac{K_\epsilon^2}{1-K_\epsilon} c \frac{1}{-K_\epsilon} \right] \\
&= 2 \lim_{\epsilon\to 0} \epsilon \iint_0^\infty dx_1 dz_1\, e^{-(1+\epsilon)x_1} e^{-\epsilon z_1} \mathrm{Bcdgg}[0; x_1 + z_1].
\end{aligned} \tag{5.3.6}$$

The third term reduces to the second term:

$$\begin{aligned}\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[cB\frac{K_\epsilon^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}\gamma^2\frac{1}{-K_\epsilon}\right] &= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[\gamma^2\frac{1}{-K_\epsilon}(-\partial c+K_\epsilon c)\frac{1}{1-K_\epsilon}\right]\\ &\quad -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-K_\epsilon}c\frac{K_\epsilon^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}\right]\\ &= -\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-K_\epsilon}c\frac{K_\epsilon^2}{1-K_\epsilon}c\frac{1}{-K_\epsilon}\right]. \quad (5.3.7)\end{aligned}$$

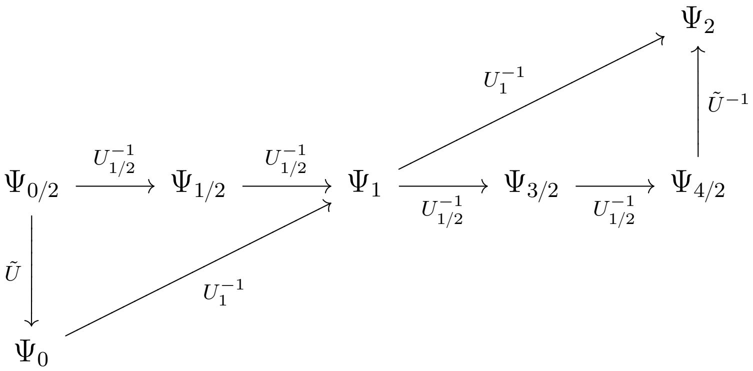
Therefore,

$$\begin{aligned}\lim_{\epsilon\to 0}\mathrm{EOMS}([\Psi_2]_\epsilon) &= 4\lim_{\epsilon\to 0}\epsilon\iint_0^\infty dx_1dz_1\, e^{-(1+\epsilon)x_1}e^{-\epsilon z_1}\mathrm{Bcdgg}[0;x_1+z_1]\\ &= \lim_{\epsilon\to 0}\left(\frac{2}{\pi^2\epsilon}+\mathrm{O}(\epsilon)\right)\neq 0. \quad (5.3.8)\end{aligned}$$

However, the value of the cubic term coincides with the expected value for the double-brane solution [57]:

$$\frac{\pi^2}{3}\lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}[[\![\Psi_2]\!]_\epsilon{}^3] = 1. \quad (5.3.9)$$

We show the solutions (at least algebraically) studied in this section in the following diagram.



Ψ_0 and $\Psi_{0/2}$ are the tachyon vacuum, $\Psi_{1/2}$ is the half-brane solution [27], $\Psi_{3/2}$ is our new multiple-half-brane solution, while the string fields Ψ_2 and $\Psi_{4/2}$ do not satisfy the EOMS. In all the cases of these string fields the following relation holds:

$$\frac{\pi^2}{3}\lim_{\epsilon\to 0}\mathrm{Tr}_{Y_{-2}}[[\![\Psi_n]\!]_\epsilon{}^3]+1 = n, \quad \{n=0,\ 0/2,\ 1/2,\ 1,\ 3/2,\ 2\}. \quad (5.3.10)$$

Chapter 6

Review of the Berkovits' Open Superstring Field Theory

6.1 Action

The action of the Berkovits' open superstring field theory [8, 9] is given by

$$S(g) = -\frac{1}{2}\int_0^1 dt\, \mathrm{Tr}\big[\partial_t(\Psi_\eta \Psi_Q) + \Psi_t\{\Psi_\eta, \Psi_Q\}\big], \tag{6.1.1}$$

where each Ψ_D is a "connection":

$$\Psi_Q := g(t)^{-1}Qg(t), \quad \Psi_\eta := g(t)^{-1}\eta_0 g(t), \quad \Psi_t := g(t)^{-1}\partial_t g(t), \tag{6.1.2}$$

and $g(t),\ t \in [0,1]$ is defined as

$$g(0) = 1, \quad g(1) = g. \tag{6.1.3}$$

The string field g is in the NS sector, GSO($\pm$)[1] sector, and in the large Hilbert space $\mathcal{H}^{\mathrm{large}}$, i.e., the Hilbert space includes the zero mode of the ξ ghost, which does not exist in the bosonization of the $\beta\gamma$ ghost:

$$\beta(z) = \partial\xi e^{-\phi}(z), \quad \gamma(z) = \eta e^{\phi}(z). \tag{6.1.4}$$

Since, this action is in the same form as the WZW action by replacing

$$\eta_0 \to \partial, \quad Q \to \bar{\partial}, \tag{6.1.5}$$

the EOM is given by

$$\eta_0(g^{-1}Qg) = 0, \tag{6.1.6}$$

[1] We attach the "internal CP factor" as in the case of the modified cubic superstring field theory.

and the action is invariant under the gauge transformation:

$$g \xrightarrow{(\Lambda,\Omega)} g^{\mathrm{g}} = \Lambda g \Omega, \qquad Q\Lambda = 0, \quad \eta_0 \Omega = 0, \tag{6.1.7}$$

where we denote the pair of the gauge parameter Λ and Ω as (Λ, Ω) with the arrow. Indeed, if we demand "axioms":

$$Q^2 = \eta_0^2 = \{Q, \eta_0\} = 0, \tag{6.1.8}$$

$$Q, \eta_0 \text{ are derivatives under } * \text{ product}, \tag{6.1.9}$$

$$\mathrm{Tr}[Q\varphi] = \mathrm{Tr}[\eta_0 \varphi] = 0, \tag{6.1.10}$$

$$(\varphi_1 * \varphi_2) * \varphi_3 = \varphi_1 * (\varphi_2 * \varphi_3) = \varphi_1 * \varphi_2 * \varphi_3, \tag{6.1.11}$$

$$\mathrm{Tr}[\varphi_1 \varphi_2] = (-)^{\epsilon(\varphi_1)\epsilon(\varphi_2)} \mathrm{Tr}[\varphi_2 \varphi_1], \tag{6.1.12}$$

we can derive the EOM and show the invariance of the action. The action can be rewritten as the following equation [58]:

$$S(g) = -\int_0^1 dt\, \mathrm{Tr}[(\eta_0 \Psi_t)\Psi_Q]. \tag{6.1.13}$$

Note that the action does not use the PCO, then the action is free from the contact term problem.

6.2 Tachyon Vacuum Solution

The tachyon vacuum solution in Berkovits' SFT found by Erler [28][2] can be written formally as

$$g_0 = Q\left((1 + q \cdot \zeta)\frac{B}{K}\right) U_1, \tag{6.2.1}$$

where $\zeta := c\gamma^{-1}$, $U_1 = Bc + cB\frac{-K}{1-K}$, $q \in \mathbb{C}$. Here, the string field γ^{-1} is constructed by inserting $e^{-\phi}\xi(z)$ on the boundary in the sliver frame with the Pauli matrix $-i\sigma_2$, then γ^{-1} has the ξ zero mode. The explicit form of the solution g_0 is

$$\begin{aligned} g_0 &= \left(q \cdot Q\zeta \cdot \frac{B}{K} + (1 + q \cdot \zeta)\right)\left(Bc + cB\frac{-K}{1-K}\right) \\ &= 1 - cB\frac{1}{1-K} - q \cdot cVB\frac{1}{1-K} - q \cdot \gamma B \frac{1}{1-K} + q \cdot \zeta, \end{aligned} \tag{6.2.2}$$

where $V := \frac{1}{2}\partial c \gamma^{-1}$, $Q\zeta = cV + \gamma$.

We introduce a "matrix" notation [28]:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} := \gamma(M_{11}B\zeta + M_{12}B) + c(M_{21}B\zeta + M_{22}B), \tag{6.2.3}$$

[2] A related work can be found in [59].

where $[M_{ij}, B\} = 0$. We can show that a product among the above string fields is like a matrix:

$$
\begin{aligned}
&\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} * \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \\
&= \Big((\gamma M_{11} + cM_{21})B\zeta + (\gamma M_{12} + cM_{22})B \Big) \\
&\quad \times \Big(\gamma(N_{11}B\zeta + N_{12}B) + c(N_{21}B\zeta + N_{22}B) \Big) \\
&= \gamma M_{11}B\zeta(\gamma N_{11}B\zeta + \gamma N_{12}B) + \gamma M_{12}B(cN_{21}B\zeta + cN_{22}B) \\
&\quad + cM_{21}B\zeta(\gamma N_{11}B\zeta + \gamma N_{12}B) + cM_{22}B(cN_{21}B\zeta + cN_{22}B) \\
&= \gamma(M_{11}N_{11}B\zeta + M_{11}N_{12}B) + \gamma(M_{12}N_{21}B\zeta + M_{12}N_{22}B) \\
&\quad + c(M_{21}N_{11}B\zeta + M_{21}N_{12}B) + c(M_{22}N_{21}B\zeta + M_{22}N_{22}B) \\
&= \Big(\gamma(M_{11}N_{11} + M_{12}N_{21}) + c(M_{21}N_{11} + M_{22}N_{21}) \Big) B\zeta \\
&\quad + \Big(\gamma(M_{12}N_{22} + M_{11}N_{12}) + c(M_{22}N_{22} + M_{21}N_{12}) \Big) B \\
&= \begin{bmatrix} M_{11}N_{11} + M_{12}N_{21} & M_{11}N_{12} + M_{12}N_{22} \\ M_{21}N_{11} + M_{22}N_{21} & M_{21}N_{12} + M_{22}N_{22} \end{bmatrix}.
\end{aligned}
\tag{6.2.4}
$$

This notation is especially useful when we search for the inverse of the string fields under the $*$ product.

We rewrite the tachyon vacuum solution in the "matrix" notation. The each factor on the right-hand side of (6.2.1) can be written as

$$
\begin{aligned}
Q\left((1 + q\cdot\zeta)\frac{B}{K} \right) &= q\cdot cV\frac{B}{K} + q\cdot\gamma\frac{B}{K} + 1 + q\cdot\zeta \\
&= \begin{bmatrix} 1 & q\cdot\frac{1}{K} \\ q & 1 + q\cdot V\frac{1}{K} \end{bmatrix},
\end{aligned}
\tag{6.2.5}
$$

and

$$
U_1 = Bc + cB\frac{-K}{1-K} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-K}{1-K} \end{bmatrix}.
\tag{6.2.6}
$$

Then, g_0 in the "matrix" notation is given by

$$
\begin{aligned}
g_0 &= \begin{bmatrix} 1 & q\cdot\frac{1}{K} \\ q & 1 + q\cdot V\frac{1}{K} \end{bmatrix} \begin{bmatrix} 1 & \\ & \frac{-K}{1-K} \end{bmatrix} \\
&= \begin{bmatrix} 1 & q\cdot\frac{1}{K} \\ q & 1 + q\cdot V\frac{1}{K} \end{bmatrix} \begin{bmatrix} 1 & \\ & -K \end{bmatrix} \begin{bmatrix} 1 & \\ & \frac{1}{1-K} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -q \\ q & -K - q\cdot V \end{bmatrix} \begin{bmatrix} 1 & \\ & \frac{1}{1-K} \end{bmatrix}.
\end{aligned}
\tag{6.2.7}
$$

We define $g_0(t)$ s.t. $g_0(1) = g_0$ and $g_0(0) = 1$. Here we take $g_0(t)$ as follows

$$g_0(t) = \bar{t} + t g_0, \tag{6.2.8}$$

where $\bar{t} := 1 - t$, then we have

$$\begin{aligned} g_0(t) &= \begin{bmatrix} \bar{t} & \\ & \bar{t} \end{bmatrix} + \begin{bmatrix} t & -\alpha \\ \alpha & -t \cdot K - \alpha \cdot V \end{bmatrix} \begin{bmatrix} 1 & \\ & \frac{1}{1-K} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\alpha \\ \alpha & \bar{t} - K - \alpha \cdot V \end{bmatrix} \begin{bmatrix} 1 & \\ & \frac{1}{1-K} \end{bmatrix} \\ &= v_0 u_0. \end{aligned} \tag{6.2.9}$$

Here, $\alpha := qt$, and string fields v_0 and u_0 are difined as

$$v_0 := \begin{bmatrix} 1 & -\alpha \\ \alpha & \bar{t} \cdot I - K - \alpha \cdot V \end{bmatrix}, \tag{6.2.10}$$

$$u_0 := \begin{bmatrix} 1 & \\ & \frac{1}{1-K} \end{bmatrix}, \tag{6.2.11}$$

where $I = 1$ is identity string field, and for latter calculations we leave it in the following calculations.

Next, we derive the inverse $g_0(t)^{-1}$ of the string field $g_0(t)$. We define the string field $\det_0$ as

$$\det{}_0 := \bar{t} \cdot I - K - \alpha \cdot V + \alpha^2, \tag{6.2.12}$$

then the inverse of the string field v_0 in its "matrix" notation can be found:

$$v_0{}^{-1} = \begin{bmatrix} 1 - \alpha^2 \cdot \frac{1}{\det_0} & \alpha \cdot \frac{1}{\det_0} \\ -\alpha \cdot \frac{1}{\det_0} & \frac{1}{\det_0} \end{bmatrix}. \tag{6.2.13}$$

We will give the definition of the string field $\frac{1}{\det_0} := (\bar{t} \cdot I - K - \alpha \cdot V + \alpha^2)^{-1}$ in appendix F. Indeed, this is the inverse:

$$\begin{aligned} v_0{}^{-1} v_0 &= \begin{bmatrix} 1 - \alpha^2 \cdot \frac{1}{\det_0} & \alpha \cdot \frac{1}{\det_0} \\ -\alpha \cdot \frac{1}{\det_0} & \frac{1}{\det_0} \end{bmatrix} \begin{bmatrix} 1 & -\alpha \\ \alpha & \det_0 - \alpha^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = Bc + cB \\ &= 1. \end{aligned} \tag{6.2.14}$$

As for $v_0 v_0{}^{-1} = 1$, we can show it in a same way. For the later convenience we introduce

$\tilde{v}_0 := {v_0}^{-1} - Bc$, the "matrix" notation of which is given by

$$\begin{aligned}
\tilde{v}_0 &= \begin{bmatrix} -\alpha^2 \cdot \frac{1}{\det_0} & \alpha \cdot \frac{1}{\det_0} \\ -\alpha \cdot \frac{1}{\det_0} & \frac{1}{\det_0} \end{bmatrix} \\
&= \begin{bmatrix} -\alpha^2 & \alpha \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\det_0} & \\ & \frac{1}{\det_0} \end{bmatrix} \\
&= w D_0. \qquad (6.2.15)
\end{aligned}$$

Here the string fields w and D_0 are defined as

$$w := \begin{bmatrix} -\alpha^2 & \alpha \\ -\alpha & 1 \end{bmatrix}, \qquad (6.2.16)$$

$$D_0 := \begin{bmatrix} \frac{1}{\det_0} & \\ & \frac{1}{\det_0} \end{bmatrix}. \qquad (6.2.17)$$

Note that w and D_0 commute $[w, D_0] = 0$ since $\alpha \in \mathbb{C}$. Therefore, we have

$$g_0(t) = v_0 u_0, \qquad g_0(t)^{-1} = {u_0}^{-1}(wD_0 + Bc). \qquad (6.2.18)$$

Then, we compute the energy of the solution:

$$E(g_0) = -S(g_0) = \int_0^1 dt \, \mathrm{Tr} \left[\eta_0 \big(g_0(t)^{-1} \partial_t g_0(t) \big) g_0(t)^{-1} Q g_0(t) \right] \qquad (6.2.19)$$

by using these results. The computation becomes very long, though it is straightforward. So, we will give it in appendix F. The result reproduces the correct value of the tachyon vacuum:

$$E(g_0) = \int_0^1 dt \left[\bar{t}(2q\alpha - 1) \frac{\alpha^2}{(\bar{t} + \alpha^2)^3 2^2} \frac{4}{\pi^2} - \bar{t} q \frac{\alpha}{(\bar{t} + \alpha^2)^2 2^2} \frac{8}{\pi^2} \right] = -\frac{1}{2\pi^2}. \qquad (6.2.20)$$

For $|\mathrm{Re}[q]| > |\mathrm{Im}[q]|$, this is independent on q.

Chapter 7

Double-brane Solution in Berkovits' Open SFT

7.1 Perturbative Vacuum

Since we find the tachyon vacuum g_0 can be written in the pure-gauge form formally, let us consider the solutions constructed by using singular gauge transformations. We try to construct the double-brane solution as another non-trivial solution. We suggest that if we replace U_1 in (6.2.1) to ${U_1}^{-1}$ by using the gauge transformation, we obtain the double-brane solution.

First, we check that the string field g_1 which is made by performing the gauge transformation (Λ, Ω) once,

$$g_0 \xrightarrow{(1,{U_1}^{-1})} g_1 := Q\left((1+q\zeta)\frac{B}{K}\right), \tag{7.1.1}$$

is gauge equivalent to the trivial solution $g = 1$. Though g_1 includes the ill-defined string field $1/K$, if we assume that $1/K$ is Q-closed (since $QK = 0$) and also that $1/K$ is the inverse of K algebraically, the EOM is satisfied because

$${g_1}^{-1}Qg_1 = 0. \tag{7.1.2}$$

The above two assumptions ($Q\frac{1}{K} = 0$ and $K\frac{1}{K} = 1$) are satisfied if we implement the K_ϵ-regularization for the fundamental variable g as $g \to [\![g]\!]_\epsilon$ for finite ϵ. However, for the reason discussed later, we do not take this regularization. Hence, we may refer to them as assumptions. Then, the energy of the solution g_1 is $E(g_1) = E(g_0) + T_9$ trivially, since $Qg_1 = 0$.

7.2 Double-brane Solution

Let us consider the string field g_2 which is constructed by performing the singular gauge transformation $(1, U_1^{-1})$ twice for the tachyon vacuum g_0:

$$g_0 \xrightarrow{(1,U_1{}^{-1})} g_1 \xrightarrow{(1,U_1{}^{-1})} g_2 := Q\left((1+q\zeta)\frac{B}{K}\right)U_1{}^{-1}. \tag{7.2.1}$$

Then, g_2 is the solution of the EOM $\eta_0(g^{-1}Qg) = 0$ since

$$g_2{}^{-1}Qg_2 = U_1QU_1{}^{-1} = \Psi_2 \in \mathcal{H}^{\text{small}}. \tag{7.2.2}$$

Here, we assume $1/K \in \mathcal{H}^{\text{small}}$, i.e., it does not have ξ_0. The solution g_0, s.t. $g_0^{-1}Qg_0 = \Psi_0$, reproduces the energy of the tachyon vacuum solution, therefore, we expect the solution g_2, s.t. $g_2^{-1}Qg_2 = \Psi_2$, reproduces the energy of the double brane solution. To check this expectation, we try to compute the energy. First, we write g_2 in the "matrix" notation:

$$\begin{aligned}
g_2 &= Q\left((1+q\zeta)\frac{B}{K}\right)U_1{}^{-1}\\
&= \begin{bmatrix}1 & -q\\ q & -K-qV\end{bmatrix}\begin{bmatrix}1 & \\ & \frac{1}{1-K}\end{bmatrix}\begin{bmatrix}1 & \\ & (\frac{1-K}{-K})^2\end{bmatrix}\\
&= \begin{bmatrix}1 & -q\\ q & -K-qV\end{bmatrix}\begin{bmatrix}1 & \\ & \frac{1-K}{K^2}\end{bmatrix},
\end{aligned} \tag{7.2.3}$$

and then we give $g_2(t)$ as in the case of the tachyon vacuum solution:

$$\begin{aligned}
g_2(t) = \bar{t} + tg_2 &= \begin{bmatrix}\bar{t} & \\ & \bar{t}\end{bmatrix}\begin{bmatrix}1 & \\ & \frac{K^2}{1-K}\end{bmatrix}\begin{bmatrix}1 & \\ & \frac{1-K}{K^2}\end{bmatrix} + \begin{bmatrix}t & -\alpha\\ \alpha & -tK-\alpha V\end{bmatrix}\begin{bmatrix}1 & \\ & \frac{1-K}{K^2}\end{bmatrix}\\
&= \begin{bmatrix}1 & -\alpha\\ \alpha & \bar{t}\frac{K^2}{1-K} - tK - \alpha V\end{bmatrix}\begin{bmatrix}1 & \\ & \frac{1-K}{K^2}\end{bmatrix}\\
&= \begin{bmatrix}1 & -\alpha\\ \alpha & \bar{t}\Omega' - K - \alpha V\end{bmatrix}\begin{bmatrix}1 & \\ & \frac{1-K}{K^2}\end{bmatrix},
\end{aligned} \tag{7.2.4}$$

where

$$\Omega' := \frac{K}{1-K}. \tag{7.2.5}$$

Then, by defining

$$v_2 := \begin{bmatrix} 1 & -\alpha \\ \alpha & \bar{t}\Omega' - K - \alpha V \end{bmatrix}, \tag{7.2.6}$$

$$u_2 := \begin{bmatrix} 1 & \\ & \frac{1-K}{K^2} \end{bmatrix}, \tag{7.2.7}$$

$$\det{}_2 := \bar{t}\Omega' - K - \alpha V + \alpha^2, \tag{7.2.8}$$

$$D_2 := \begin{bmatrix} \frac{1}{\det_2} & \\ & \frac{1}{\det_2} \end{bmatrix}, \tag{7.2.9}$$

$g_2(t)^{-1}$ is given by

$$g_2(t)^{-1} = {u_2}^{-1}(wD_2 + Bc), \tag{7.2.10}$$

as in the case of the tachyon vacuum

$$g_0(t)^{-1} = {u_0}^{-1}(wD_0 + Bc). \tag{7.2.11}$$

Then precise definition of the string field $1/\det_2$ is given in the next subsection.

7.2.1 Energy

To evaluate the energy of the solution, we need some regularization since the string field $1/\det_2$ is singular at $t = 0$ as we will see below. First, we might try to regularize the solution as $g_2 \to [\![g_2]\!]_\epsilon$. However, this is not a desirable regularization. If we regularize the solution g_2 itself, this means that the solution is constructed by using a regular gauge transformation $(1, [\![{U_1}^{-1}]\!]_\epsilon)$ from the perturbative vacuum. So the result is gauge equivalent to the perturbative vacuum. To avoid this, we introduce the K_ϵ-regularization for the "connection" Ψ_D (6.1). This seems to be good because in the bosonic cubic theory the regularization $[\![U_1 Q {U_1}^{-1}]\!]_\epsilon$ works well, and the regularization for Ψ_D includes $[\![{g_2}^{-1} Q g_2]\!]_\epsilon = [\![U_1 Q {U_1}^{-1}]\!]_\epsilon$ as a case with $D = Q$. In the bosonic cubic SFT and the modified cubic SFT, we checked the EOMS as a condition for an acceptable solution. However, in the present case, a suitable condition is not clear. Since extra ξ zero modes do not seem to appear in the K_ϵ-regularization, the inner product between $\eta_0[\![g^{-1}Qg]\!]_\epsilon$ with any other test string fields seems to be zero.

The computation of the energy can be obtained from the one of the tachyon vacuum solution (F.43) in appendix G by replacing as $I \to \Omega'$ and $\det_0 \to \det_2$. The result is given by

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\mathrm{Tr}\left[\eta_0\big([\![g_2(t)^{-1}\partial_t g_2(t)]\!]_\epsilon\big)[\![g_2(t)^{-1}Qg_2(t)]\!]_\epsilon\right]\\
&=\lim_{\epsilon\to 0}\bar{t}\Bigg(2q\alpha^2\mathrm{Tr}\left[\left[\!\!\left[\left(\eta_0\frac{1}{\det_2}\right)\frac{1}{\det_2}Bc\frac{1}{1-K}\partial\gamma\frac{1}{1-K}\right]\!\!\right]_\epsilon\right] &(7.2.12)\\
&\qquad -2q\alpha\mathrm{Tr}\left[\left[\!\!\left[\left(\eta_0\frac{1}{\det_2}\right)\frac{1}{\det_2}B\Omega' c\partial c\Omega'\right]\!\!\right]_\epsilon\right] &(7.2.13)\\
&\qquad +q\alpha\mathrm{Tr}\left[\left[\!\!\left[\frac{1}{\det_2}V\left(\eta_0\frac{1}{\det_2}\right)\frac{1}{1-K}\partial\gamma\frac{1}{1-K}Bc\right]\!\!\right]_\epsilon\right] &(7.2.14)\\
&\qquad -q\mathrm{Tr}\left[\left[\!\!\left[\frac{1}{\det_2}V\left(\eta_0\frac{1}{\det_2}\right)B\Omega' c\partial c\Omega'\right]\!\!\right]_\epsilon\right] &(7.2.15)\\
&\qquad -\alpha\mathrm{Tr}\left[\left[\!\!\left[\left(\eta_0\frac{1}{\det_2}\right)\Omega'\frac{1}{\det_2}\frac{1}{1-K}\partial\gamma\frac{1}{1-K}Bc\right]\!\!\right]_\epsilon\right] &(7.2.16)\\
&\qquad +\mathrm{Tr}\left[\left[\!\!\left[\left(\eta_0\frac{1}{\det_2}\right)\Omega'\frac{1}{\det_2}B\Omega' c\partial c\Omega'\right]\!\!\right]_\epsilon\right]\Bigg), &(7.2.17)
\end{aligned}
$$

where we used

$$
\begin{aligned}
[\Omega',\gamma] &= \frac{1}{1-K}\partial\gamma\frac{1}{1-K},\\
\Omega' c\partial c - Kc[\Omega',c] &= -\Omega' c\partial c\Omega' - K\left[c,\frac{1}{1-K}\right]\partial c\frac{1}{1-K}. &(7.2.18)
\end{aligned}
$$

We give the definition of the string field $1/\det_2$ as $1/\det_0$ (F.47):

$$
\begin{aligned}
\frac{1}{\det_2} &:= (\bar{t}\Omega' - K - \alpha V + \alpha^2)^{-1}\\
&= \left(\big(\bar{t}K - K(1-K) - \alpha V(1-K) + \alpha^2(1-K)\big)\frac{1}{1-K}\right)^{-1}\\
&= (1-K)\big(K^2 - (\alpha^2+t)K + \alpha^2 - \alpha V(1-K)\big)^{-1}\\
&= (1-K)\left(\frac{1}{F_2} - \alpha V(1-K)\right)^{-1}\\
&= (1-K)\left((1-\alpha V(1-K)F_2)\frac{1}{F_2}\right)^{-1}\\
&= (1-K)F_2(1-\alpha V(1-K)F_2)^{-1}\\
&= (1-K)F_2\big(1+\alpha V(1-K)F_2 + \alpha V(1-K)F_2\alpha V(1-K)F_2 + \cdots\big)\\
&= (1-K)\big(F_2 + F_2\alpha V(1-K)F_2 + F_2\alpha V(1-K)F_2\alpha V(1-K)F_2 + \cdots\big). &(7.2.19)
\end{aligned}
$$

Here, we defined

$$
F_2 := \frac{1}{K^2 - (\alpha^2+t)K + \alpha^2}, \qquad (7.2.20)
$$

which becomes singular string field at $t = 0$:

$$F_2|_{t=0} = \frac{1}{K^2} \tag{7.2.21}$$

and hence the regularization is needed. We adopt the K_ϵ-regularization and define the string field F_2 in terms of the Laplace transformation:

$$\begin{aligned}
[\![F_2]\!]_\epsilon &= \frac{1}{{K_\epsilon}^2 - (\alpha^2 + t)K_\epsilon + \alpha^2} \\
&= \frac{1}{k_+ - K}\frac{1}{k_- - K} \\
&= \int_0^\infty dl \int_0^\infty dm\, e^{-k_+ l} e^{-k_- m} \Omega^{l+m} \\
&= \frac{1}{k_+ - k_-}\int_0^\infty dn\, (e^{-k_- n} - e^{-k_+ n})\Omega^n,
\end{aligned} \tag{7.2.22}$$

where

$$k_\pm := \frac{(\alpha^2 + t) \pm \sqrt{(\alpha^2 + t)^2 - 4\alpha^2}}{2} + \epsilon. \tag{7.2.23}$$

If we choose q as $q \in (0, \frac{1}{2}]$, then $k_\pm$ satisfy

$$\mathrm{Re}[k_\pm] > 0, \quad \mathrm{Im}[k_\pm] = 0. \tag{7.2.24}$$

Then the Laplace transformation is well-defined at least for $q \in (0, \frac{1}{2}]$.

We do not write all the calculation in this section since it will be long. We give the calculation of the remaining terms in appendix G. So we only write the first term of

(7.2.12)-(7.2.17):

$$
\begin{aligned}
(7.2.12) &= 2\lim_{\epsilon\to 0}\bar{t}q\alpha^2\mathrm{Tr}\left[\left[\!\left[\left(\eta_0\frac{1}{\det_2}\right)\frac{1}{\det_2}Bc\frac{1}{1-K}\partial\gamma\frac{1}{1-K}\right]\!\right]_\epsilon\right]\\
&= 2\lim_{\epsilon\to 0}\bar{t}q\alpha^2\mathrm{Tr}\Big[\Big[\!\!\Big[(\eta_0\big((1-K)F_2\alpha V(1-K)F_2\alpha V(1-K)F_2)\big)\\
&\qquad\qquad \times(1-K)(F_2\alpha V(1-K)F_2)Bc\frac{1}{1-K}\partial\gamma\frac{1}{1-K}\Big]\!\!\Big]_\epsilon\Big]\\
&\quad + 2\lim_{\epsilon\to 0}\bar{t}q\alpha^2\mathrm{Tr}\Big[\Big[\!\!\Big[(\eta_0\big((1-K)F_2\alpha V(1-K)F_2)\big)\\
&\qquad\qquad \times(1-K)(F_2\alpha V(1-K)F_2\alpha V(1-K)F_2)Bc\frac{1}{1-K}\partial\gamma\frac{1}{1-K}\Big]\!\!\Big]_\epsilon\Big]\\
&= -2\lim_{\epsilon\to 0}\bar{t}q\alpha^5\mathrm{Tr}\Big[[\![F_2]\!]_\epsilon V(1-K_\epsilon)[\![F_2]\!]_\epsilon V(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2(\eta_0 V)[\![F_2]\!]_\epsilon Bc\partial\gamma\Big]\\
&\quad + 2\lim_{\epsilon\to 0}\bar{t}q\alpha^5\mathrm{Tr}\Big[[\![F_2]\!]_\epsilon(\eta_0 V)(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2 V(1-K_\epsilon)[\![F_2]\!]_\epsilon V[\![F_2]\!]_\epsilon Bc\partial\gamma\Big]\\
&= -\lim_{\epsilon\to 0}\frac{\bar{t}q\alpha^5}{2^2}\mathrm{Tr}\Big[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon\partial c\gamma^{-1}\\
&\qquad\qquad \times(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2\partial c\Big] \qquad (7.2.25)\\
&\quad + \lim_{\epsilon\to 0}\frac{\bar{t}q\alpha^5}{2^2}\mathrm{Tr}\Big[(\eta_0\gamma^{-1})(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon\\
&\qquad\qquad \times\partial c\gamma^{-1}[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c\Big]. \qquad (7.2.26)
\end{aligned}
$$

The first term (7.2.25) is

$$
\begin{aligned}
&\int_0^1 dt\,(7.2.25)|_{q=\frac{1}{2}}\\
&= -\frac{1}{2^8}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^5\mathrm{Tr}\big[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c\gamma^{-1}\\
&\qquad\qquad \times(1-K_\epsilon)[\![F_2]\!]_\epsilon\partial c\gamma^{-1}(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2\partial c\big]\\
&= -\frac{1}{2^8}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^5\int_0^\infty\prod_{i=1}^5 dn_i\prod_{j=1}^3\lim_{u_j\to 0}(-\partial_{u_j})\Big\{\left(\frac{1}{k_+-k_-}\right)^5(e^{-k_-n_i}-e^{-k_+n_i})e^{-(1+\epsilon)u_j}\\
&\quad\cdot\mathrm{eidii}[n_1,n_2,u_1+n_3,u_2+u_3+n_4+n_5]\\
&\quad\cdot\mathrm{Bcddd}[u_1+n_3,u_2+u_3+n_4+n_5,n_1+n_2]\Big\}\\
&\quad\times\frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)i\sigma_2\sigma_3\sigma_3\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3]. \qquad (7.2.27)
\end{aligned}
$$

Here we define

$$
\begin{aligned}
&\mathrm{eigii}[t_1,t_2,t_3,t_4]\\
&:=\Big\langle\oint_0\frac{dz}{2\pi i}\eta(z)\xi(0)e^{-\phi}(0)\eta e^{\phi}(t_1)\xi e^{-\phi}(t_1+t_2)\xi e^{-\phi}(t_1+t_2+t_3)\Big\rangle^{\xi\eta\phi}_{\mathrm{C}_{t_1+t_2+t_3+t_4}}\\
&= -\frac{\pi}{L}\frac{\sin\theta_{t_1}}{\sin\theta_{t_1+t_2}\sin\theta_{t_1+t_2+t_3}}, \qquad (7.2.28)
\end{aligned}
$$

$$\begin{aligned}\mathrm{eidii}[t_1,t_2,t_3,t_4] &:= \lim_{y\to 0}\partial_y\{\mathrm{eigii}[t_1+y,t_2,t_3,t_4]-\mathrm{eigii}[t_1,t_2+y,t_3,t_4]\}\\ &= -\frac{\pi^2}{L^2}\cos\theta_{t_1}\csc\theta_{t_1+t_2}\csc\theta_{t_1+t_2+t_3}. \end{aligned} \tag{7.2.29}$$

However, the computations are difficult, because the number of the integrals and that of the terms in the integrands are large. Therefore, we try to evaluate the solution another way.

7.2.2 Gauge Invariant Observable

The GIO [30] in Berkovits' SFT is defined as

$$W(g, \mathcal{V}) := \mathrm{Tr}_{\mathcal{V}}[g^{-1}Qg], \tag{7.2.30}$$

where

$$\mathrm{Tr}_{\mathcal{V}}[\varphi_{\mathrm{test}}] := \langle\, \mathcal{V}(i\infty) f_{\mathrm{s}} \circ \varphi_{\mathrm{test}}(0)\, \rangle_{\mathrm{C}_1}. \tag{7.2.31}$$

Here, $\mathcal{V}$ is an NS-NS on-shell vertex operator:

$$\mathcal{V} = (\xi + \tilde{\xi}) c\tilde{c} e^{-\phi} e^{-\tilde{\phi}} V^{(\frac{1}{2},\frac{1}{2})}. \tag{7.2.32}$$

Let us show the gauge invariance. The gauge transformation (6.1.7) of the GIO is given by

$$\begin{aligned} W(g^{\mathrm{g}}, \mathcal{V}) &= \mathrm{Tr}_{\mathcal{V}}[\Lambda^{-1} g^{-1} \Omega^{-1} Q(\Omega g \Lambda)] \\ &= \mathrm{Tr}_{\mathcal{V}}[g^{-1}Qg + \Lambda^{-1}Q\Lambda]. \end{aligned} \tag{7.2.33}$$

We define

$$\Sigma_\tau := {\Lambda_\tau}^{-1} Q \Lambda_\tau, \quad \Lambda_\tau := e^{\tau\lambda}, \tag{7.2.34}$$

where λ is a string field in the small Hilbert space, i.e., $\eta_0 \lambda = 0$. Then, we can show that $\mathrm{Tr}_{\mathcal{V}}[\Sigma_\tau]$ does not depend on the parameter τ:

$$\begin{aligned} \partial_\tau \mathrm{Tr}_{\mathcal{V}}[\Sigma_\tau] &= \mathrm{Tr}_{\mathcal{V}}[-\lambda\Sigma_\tau] + \mathrm{Tr}_{\mathcal{V}}[\Lambda_\tau^{-1} Q(\lambda\Lambda_\tau)] \\ &= -\mathrm{Tr}_{\mathcal{V}}[\lambda\Sigma_\tau] + \mathrm{Tr}_{\mathcal{V}}[Q\lambda] + \mathrm{Tr}_{\mathcal{V}}[\lambda\Sigma_\tau] \\ &= \mathrm{Tr}_{\mathcal{V}}[Q\lambda] \\ &= 0. \end{aligned} \tag{7.2.35}$$

Since $\Sigma_0 = 0$, this means

$$\mathrm{Tr}_{\mathcal{V}}[\Sigma_\tau] = 0. \tag{7.2.36}$$

Then, we find the gauge invariance:

$$\begin{aligned} W(g^{\mathrm{g}}, \mathcal{V}) &= W(g, \mathcal{V}) + \mathrm{Tr}_{\mathcal{V}}[\Sigma_1] \\ &= W(g, \mathcal{V}). \end{aligned} \tag{7.2.37}$$

For the tachyon vacuum solution g_0, the GIO becomes

$$\begin{aligned}
W(g_0, \mathcal{V}) &= \mathrm{Tr}_{\mathcal{V}}[\Psi_0] \\
&= \mathrm{Tr}_{\mathcal{V}}\left[(Q(Bc) - c)\frac{1}{1-K}\right] \\
&= -\mathrm{Tr}_{\mathcal{V}}\left[c\frac{1}{1-K}\right] \\
&= 0 - \mathcal{A}_0(\mathcal{V}).
\end{aligned} \tag{7.2.38}$$

For the trivial solution g_1, it becomes

$$W(g_1, \mathcal{V}) = 0 = \mathcal{A}_0(\mathcal{V}) - \mathcal{A}_0(\mathcal{V}). \tag{7.2.39}$$

For our solution g_2, it is calculated as

$$\begin{aligned}
\lim_{\epsilon \to 0} [\![W(g_2, \mathcal{V})]\!]_\epsilon = \lim_{\epsilon \to 0} \mathrm{Tr}_{\mathcal{V}}[[\![\Psi_2]\!]_\epsilon] &= \lim_{\epsilon \to 0} \mathrm{Tr}_{\mathcal{V}}\left[\left(B\gamma^2 + cB\frac{{K_\epsilon}^2}{1-K_\epsilon}c\right)\frac{1}{-K_\epsilon}\right] \\
&= \lim_{\epsilon \to 0} \mathrm{Tr}_{\mathcal{V}}\left[cB\frac{K_\epsilon}{1-K_\epsilon}\partial c\frac{1}{-K_\epsilon}\right] \\
&= -\mathrm{Tr}_{\mathcal{V}}\left[cB\frac{1}{1-K}\partial c\right] \\
&= \mathrm{Tr}_{\mathcal{V}}\left[BQc\frac{1}{1-K}\right] \\
&= \mathrm{Tr}_{\mathcal{V}}\left[c\frac{K}{1-K}\right] \\
&= \mathrm{Tr}_{\mathcal{V}}\left[c\frac{1}{1-K}\right] \\
&= 2\mathcal{A}_0(\mathcal{V}) - \mathcal{A}_0(\mathcal{V}).
\end{aligned} \tag{7.2.40}$$

Here, we used

$$\mathrm{Tr}_{\mathcal{V}}[B\gamma^2 f(K)] = 0, \tag{7.2.41}$$

which follows from the ϕ-momentum conservation, and also

$$\mathrm{Tr}_{\mathcal{V}}[\varphi_1 Q \varphi_2] = -(-)^{\epsilon(\varphi_1)}\mathrm{Tr}_{\mathcal{V}}[Q\varphi_1 \cdot \varphi_2], \tag{7.2.42}$$

since $\mathcal{V}$ is on-shell. Our solution satisfies a needed property of the double brane solution. Namely, the value of the GIO of our solution (7.2.40) is larger than that of the tachyon vacuum solution by the value which seems to be consistent with the existence of two D9-branes. When we choose the vertex operator $\mathcal{V}$ to be the time-like component of the graviton

$$\mathcal{V}_{\mathrm{G}} := \frac{i}{2\pi}(\xi + \tilde{\xi})c\tilde{c}e^{-\phi}e^{-\tilde{\phi}}\psi^0\tilde{\psi}^0, \tag{7.2.43}$$

then the disk amplitude corresponds to the energy of the solution:

$$\begin{aligned}\mathcal{A}_0(\mathcal{V}_{\rm G}) &= \frac{1}{2\pi i}\frac{i}{2\pi}\langle\, \xi+\tilde{\xi}\,\rangle^{\xi}_{\rm UHP}\langle\, e^{-\phi}(i)e^{-\phi}(-i)\,\rangle^{\phi}_{S^2} \\ &\quad \times\langle\, c(i)c(-i)(-\frac{1}{2}i)^{-1}c(0)\,\rangle^{bc}_{S^2}\,\langle\, \psi^0(i)\psi^0(-i)\,\rangle^{\rm ma}_{S^2} \\ &= \frac{1}{2\pi i}\frac{i}{2\pi}(-2)\frac{1}{i+i}\times(2i)(i+i)(i-0)(-i-0)\times\frac{\eta^{00}}{i+i} \\ &= \frac{1}{2\pi^2} \\ &= T_9. \end{aligned} \tag{7.2.44}$$

Then, we obtain

$$\begin{aligned} W(g_0,\mathcal{V}_{\rm g}) &= E(g_0), \\ W(g_1,\mathcal{V}_{\rm g}) &= E(g_0)+T_9, \\ \lim_{\epsilon\to 0}[\![W(g_2,\mathcal{V}_{\rm g})]\!]_\epsilon &= E(g_0)+2T_9. \end{aligned} \tag{7.2.45}$$

Chapter 8

Conclusion

We constructed three types of the new multiple-brane solutions by using singular gauge transformations in different theories.

First, we constructed candidates for the solution of the EOM in the bosonic cubic SFT. They were obtained by performing the singular gauge transformation whose gauge parameter is ${U_1}^{-1}$ for the EM soltuion; the number of times of the singular gauge transformation is equal to n. Since, in general, these candidates include the singular string field $1/K$, we adopted the K_ϵ-regularization and checked the EOM in the strong sense (EOMS). After this checking, we realize that the only candidate which satisfies the EOMS is the one for $n = 1$. We evaluated the energy of our solution, and then we found that the singular gauge transformation increases the energy by the value of the tension of the D25-brane. We also calculated the tachyon profile, by using the Neumann–Dirichlet twist operators as the boundary condition changing operators. The plotted figure shows that our solution describes the D24-brane on the D25-brane; these D-branes are originated from the EM solution and the gauge transformation, respectively. This result gives a support for that the singular gauge transformation ${U_1}^{-1}$ creates the D25-brane in this case.

Second, in the modified cubic superstring field theory, we constructed a candidate for the solution of the EOM by performing the singular gauge transformation for the tachyon vacuum solution three times. Here we took the singular gauge parameter ${U_{1/2}}^{-1}$. As our first solution, this solution includes the singular string field $1/G$, then we introduced the G_ϵ-regularization, and we checked that the solution satisfies the EOMS. We also evaluated the energy, and the result is expected one, i.e., the energy of our solution is increased from the energy of the tachyon vacuum solution by $3/2$ times the tension of the D9-brane. Since Ψ_2 does not satisfy the EOMS, a pure-gauge-form string field ${U_{1/2}}^2 Q {U_{1/2}}^{-2}$, which is gauge equivalent to Ψ_2, does not satisfy the EOMS. Therefore, we did not consider further gauge transformations with ${U_{1/2}}^{-1}$.

Third, we constructed a candidate for the double-brane solution by performing the singular gauge transformation from the tachyon vacuum in the Berkovits' superstring field theory. We gave the integral form of the energy of the candidate but did not reach the final result because the integral is complicated and lengthy. We also discussed the regularization in this theory. We gave another support that the candidate is the double-brane solution. This was given by evaluating the GIO, and we found the value which is consistent with the double-brane solution.

Let us give some comments regarding future directions. Since we have not yet completed the computation of the energy of the double-brane solution in Berkovits' SFT, to accomplish this task should be the important future work. Since in this thesis, we mainly evaluated the solutions by using their energies and EOMSs, it would be interesting to investigate other properties of the solutions. Regarding such a direction, we would say that the D-brane charge should be an interesting quantity to be studied. However, solutions studied in this thesis does not include any stable BPS D-branes. What is more, since the D-brane construction studied in this thesis is based on the singular gauge transformations connecting the unstable perturbative vacuum and the tachyon vacuum, It is not clear whether this method is, in any sense, useful to construct stable BPS D-branes with charges. These issues including the investigations of further method of constructing D-branes are important and interesting future directions.

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Appendix A

Correlators and Formulae in the Bosonic Cubic String Field Theory

We give formulae of the correlators [15] in the sliver frame. The basic formula is

$$\mathrm{Tr}[c\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}] = -\left(\frac{L}{\pi}\right)^3 \sin\theta_{t_1}\sin\theta_{t_2}\sin\theta_{t_3}, \tag{A.1}$$

where L is the circumference of the sliver, now $L = t_1 + t_2 + t_3$, and $\theta_t := \frac{\pi t}{L}$. This can be derived from the three point function of $c(z)$

$$\langle\, c(z_1)c(z_2)c(z_3)\,\rangle^{bc}_{\mathrm{UHP}} = z_{12}z_{13}z_{23}, \qquad (z_{ij} := z_i - z_j) \tag{A.2}$$

as follows:

$$\begin{aligned}
\mathrm{Tr}[c\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}] &= \langle\, c(0)c(t_1)c(t_1+t_2)\,\rangle^{bc}_{\mathrm{C}_L} \\
&= \langle\, f_{L\to 2}\circ c(0) f_{L\to 2}\circ c(t_1) f_{L\to 2}\circ c(t_1+t_2)\,\rangle^{bc}_{\mathrm{C}_2} \\
&= \left(\frac{2}{L}\right)^{-3} \langle\, c(0)c\left(\frac{2t_1}{L}\right)c\left(\frac{2(t_1+t_2)}{L}\right)\rangle^{bc}_{\mathrm{C}_2} \\
&= \left(\frac{2}{L}\right)^{-3} \langle\, f_{\mathrm{s}}{}^{-1}\circ c(0) f_{\mathrm{s}}{}^{-1}\circ c\left(\frac{2t_1}{L}\right) f_{\mathrm{s}}{}^{-1}\circ c\left(\frac{2(t_1+t_2)}{L}\right)\rangle^{bc}_{\mathrm{UHP}} \\
&= \left(\frac{2}{L}\right)^{-3}\left(\frac{\pi}{2}\right)^{-3} \cos^2\theta_{t_1}\cos^2\theta_{t_1+t_2}\langle\, c(0)c(\tan\theta_{t_1})c(\tan\theta_{t_1+t_2})\,\rangle^{bc}_{\mathrm{UHP}} \\
&= \left(\frac{L}{\pi}\right)^3 \cos^2\theta_{t_1}\cos^2\theta_{t_1+t_2} \\
&\quad \times (0-\tan\theta_{t_1})(0-\tan\theta_{t_1+t_2})(\tan\theta_{t_1}-\tan\theta_{t_1+t_2}) \\
&= \left(\frac{L}{\pi}\right)^3 \sin\theta_{t_1}\sin\theta_{t_1+t_2}(\sin\theta_{t_1}\cos\theta_{t_1+t_2} - \sin\theta_{t_1+t_2}\cos\theta_{t_1}) \\
&= -\left(\frac{L}{\pi}\right)^3 \sin\theta_{t_1}\sin\theta_{t_1+t_2}\sin\theta_{t_2} \\
&= -\left(\frac{L}{\pi}\right)^3 \sin\theta_{t_1}\sin\theta_{t_2}\sin\theta_{t_3},
\end{aligned} \tag{A.3}$$

where ${f_{\rm s}}^{-1}(\xi)=\tan\frac{\pi\xi}{2}$, $\partial_\xi {f_{\rm s}}^{-1}(\xi)=\frac{\pi}{2}\frac{1}{\cos^2\frac{\pi\xi}{2}}$, $\sin\theta_{L-t}=\sin\theta_t$. Next we give the correlator with a string field B and four c's:

$$
\begin{aligned}
{\rm Tr}[Bc\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}c\Omega^{t_4}] &= -\frac{t_1}{L}\left(\frac{L}{\pi}\right)^3\sin\theta_{t_1+t_2}\sin\theta_{t_3}\sin\theta_{t_4}\\
&\quad+\frac{t_1+t_2}{L}\left(\frac{L}{\pi}\right)^3\sin\theta_{t_1}\sin\theta_{t_2+t_3}\sin\theta_{t_4}\\
&\quad-\frac{t_1+t_2+t_3}{L}\left(\frac{L}{\pi}\right)^3\sin\theta_{t_1}\sin\theta_{t_2}\sin\theta_{t_3+t_4} \qquad (A.4)\\
&= -\frac{L^2}{4\pi^3}(t_3\sin 2\theta_{t_1}-(t_2+t_3)\sin 2\theta_{t_1+t_2}+t_2\sin 2\theta_{t_1+t_2+t_3}\\
&\quad+t_1\sin 2\theta_{t_3}-(t_1+t_2)\sin 2\theta_{t_2+t_3}+(t_1+t_2+t_3)\sin 2\theta_{t_2}). \qquad (A.5)
\end{aligned}
$$

To derive this, we consider the following correlator:

$$
{\rm Tr}[Bc\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}c\Omega^{t_4}]=\Big\langle\int_{\downarrow 0}\frac{dz}{2\pi i}b(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}. \qquad (A.6)
$$

We start with the following relation:

$$
\begin{aligned}
&\Big\langle\int_{\downarrow -0}\frac{dz}{2\pi i}zb(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&=\Big\langle\int_{\downarrow -0}\frac{dz}{2\pi i}zb(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&\quad+\Big\langle\left(\int_{\downarrow +0}+\int_{\uparrow +0}\right)\frac{dz}{2\pi i}zb(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&=\Big\langle\left(\int_{\downarrow -0}+\int_{\uparrow +0}\right)\frac{dz}{2\pi i}zb(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&\quad+(-)^{\epsilon(b)\epsilon(c)}\Big\langle c(0)\int_{\downarrow +0}\frac{dz}{2\pi i}zb(z)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&=\Big\langle\left(\oint_0\frac{dz}{2\pi i}zb(z)c(0)\right)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&\quad-\Big\langle c(0)\int_{\downarrow +0}\frac{dz}{2\pi i}zb(z)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}. \qquad (A.7)
\end{aligned}
$$

After repeated uses of the similar relations around $z=t_1$, $z=t_1+t_2$ and $z=t_1+t_2+t_3$, we obtain

$$
\begin{aligned}
(A.7)&=\Big\langle\left(\oint_0\frac{dz}{2\pi i}zb(z)c(0)\right)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&\quad-\Big\langle c(0)\left(\oint_{t_1}\frac{dz}{2\pi i}zb(z)c(t_1)\right)c(t_1+t_2)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}\\
&\quad+\Big\langle c(0)c(t_1)\left(\oint_{t_1+t_2}\frac{dz}{2\pi i}zb(z)c(t_1+t_2)\right)c(t_1+t_2+t_3)\Big\rangle^{bc}_{{\rm C}_L}
\end{aligned}
$$

$$
\begin{aligned}
&-\big\langle\, c(0)c(t_1)c(t_1+t_2)\left(\oint_{t_1+t_2+t_3}\frac{dz}{2\pi i}zb(z)c(t_1+t_2+t_3)\right)\big\rangle^{bc}_{\mathrm{C}_L}\\
&+\big\langle\, c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\int_{\downarrow L-0}\frac{dz}{2\pi i}zb(z)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
=&\big\langle\,(0)\cdot c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
&-\big\langle\, c(0)\cdot(t_1)\cdot c(t_1+t_2)c(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
&+\big\langle\, c(0)c(t_1)\cdot(t_1+t_2)\cdot c(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
&-\big\langle\, c(0)c(t_1)c(t_1+t_2)\cdot(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
&+\big\langle\int_{\downarrow -0}\frac{dz}{2\pi i}(z+L)b(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L},
\end{aligned}
\tag{A.8}
$$

where we used the periodicity $z \simeq z+L$ of the cylinder. Then, we obtain the following correlator:

$$
\begin{aligned}
&\big\langle\int_{\downarrow -0}\frac{dz}{2\pi i}zb(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
&=-t_1\big\langle\, c(0)c(t_1+t_2)c(t_1+t_2+t_3)\big\rangle^{bc}_{\mathrm{C}_L}\\
&\quad+(t_1+t_2)\big\langle\, c(0)c(t_1)c(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
&\quad-(t_1+t_2+t_3)\big\langle\, c(0)c(t_1)c(t_1+t_2)\,\big\rangle^{bc}_{\mathrm{C}_L}\\
&\quad+\big\langle\int_{\downarrow -0}\frac{dz}{2\pi i}(z+L)b(z)c(0)c(t_1)c(t_1+t_2)c(t_1+t_2+t_3)\,\big\rangle^{bc}_{\mathrm{C}_L}.
\end{aligned}
\tag{A.9}
$$

Therefore, (A.4) is obtained:

$$
\begin{aligned}
\mathrm{Tr}[Bc\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}c\Omega^{t_4}] =&\ \frac{t_1}{L}\mathrm{Tr}[c\Omega^{t_1+t_2}c\Omega^{t_3}c\Omega^{t_4}]\\
&-\frac{t_1+t_2}{L}\mathrm{Tr}[c\Omega^{t_1}c\Omega^{t_2+t_3}c\Omega^{t_4}]\\
&+\frac{t_1+t_2+t_3}{L}\mathrm{Tr}[c\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3+t_4}]\\
=&-\frac{t_1}{L}\left(\frac{L}{\pi}\right)^3\sin\theta_{t_1+t_2}\sin\theta_{t_3}\sin\theta_{t_4}\\
&+\frac{t_1+t_2}{L}\left(\frac{L}{\pi}\right)^3\sin\theta_{t_1}\sin\theta_{t_2+t_3}\sin\theta_{t_4}\\
&-\frac{t_1+t_2+t_3}{L}\left(\frac{L}{\pi}\right)^3\sin\theta_{t_1}\sin\theta_{t_2}\sin\theta_{t_3+t_4}.
\end{aligned}
\tag{A.10}
$$

Furthermore, by using

$$
\begin{aligned}
\sin x_1\sin x_2\sin x_3 =&\ \frac{1}{4}\big(-\sin(x_1+x_2+x_3)+\sin(x_1+x_2-x_3)\\
&+\sin(x_1-x_2+x_3)-\sin(x_1-x_2-x_3)\big),
\end{aligned}
\tag{A.11}
$$

which can be derived from

$$\sin(x_1+x_2+x_3) = \sin x_1 \cos x_2 \cos x_3 - \sin x_1 \sin x_2 \sin x_3 \\ + \cos x_1 \sin x_2 \cos x_3 + \cos x_1 \cos x_2 \sin x_3, \tag{A.12}$$

we can rewrite it:

$$\begin{aligned}
(\text{A.10}) = &-\frac{t_1}{L}\left(\frac{L}{\pi}\right)^3 \frac{1}{4}\big(-\sin\theta_{(t_1+t_2)+t_3+t_4} + \sin\theta_{(t_1+t_2)+t_3-t_4} \\
&\qquad + \sin\theta_{(t_1+t_2)-t_3+t_4} - \sin\theta_{(t_1+t_2)-t_3-t_4}\big) \\
&+ \frac{t_1+t_2}{L}\left(\frac{L}{\pi}\right)^3 \frac{1}{4}\big(-\sin\theta_{t_1+(t_2+t_3)+t_4} + \sin\theta_{t_1+(t_2+t_3)-t_4} \\
&\qquad + \sin\theta_{t_1-(t_2+t_3)+t_4} - \sin\theta_{t_1-(t_2+t_3)-t_4}\big) \\
&- \frac{t_1+t_2+t_3}{L}\left(\frac{L}{\pi}\right)^3 \frac{1}{4}\big(-\sin\theta_{t_1+t_2+(t_3+t_4)} + \sin\theta_{t_1+t_2-(t_3+t_4)} \\
&\qquad + \sin\theta_{t_1-t_2+(t_3+t_4)} - \sin\theta_{t_1-t_2-(t_3+t_4)}\big) \\
= &-\frac{L^2}{4\pi^3}(t_3 \sin 2\theta_{t_1} - (t_2+t_3)\sin 2\theta_{t_1+t_2} + t_2 \sin 2\theta_{t_1+t_2+t_3} \\
&\qquad + t_1 \sin 2\theta_{t_3} - (t_1+t_2)\sin 2\theta_{t_2+t_3} + (t_1+t_2+t_3)\sin 2\theta_{t_2}),
\end{aligned} \tag{A.13}$$

where $t_4 = L-(t_1+t_2+t_3)$ and $\sin(\pi-\theta) = \sin\theta$. For simplicity, we define

$$\text{Bcccc}[t_1,t_2,t_3,t_4] := \text{Tr}[Bc\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}c\Omega^{t_4}]. \tag{A.14}$$

By using this notation,

$$\begin{aligned}
\text{Bcccc}[t_1,t_2,t_3,t_4] = &-\frac{L^2}{4\pi^3}(t_3 \sin 2\theta_{t_1} - (t_2+t_3)\sin 2\theta_{t_1+t_2} + t_2 \sin 2\theta_{t_1+t_2+t_3} \\
&+ t_1 \sin 2\theta_{t_3} - (t_1+t_2)\sin 2\theta_{t_2+t_3} + (t_1+t_2+t_3)\sin 2\theta_{t_2}).
\end{aligned} \tag{A.15}$$

We further define

$$\begin{aligned}
\text{Bcddd}[t_1,t_2,t_3] &:= \text{Tr}[Bc\partial c\Omega^{t_1}\partial c\Omega^{t_2}\partial c\Omega^{t_3}], \\
\text{Bccdd}[t_1,t_2,t_3,t_4] &:= \text{Tr}[Bc\Omega^{t_1}c\Omega^{t_2}\partial c\Omega^{t_3}\partial c\Omega^{t_4}], \\
\text{Bcccd}[t_1,t_2,t_3,t_4] &:= \text{Tr}[Bc\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}\partial c\Omega^{t_4}], \\
\text{Bccdc}[t_1,t_2,t_3,t_4] &:= \text{Tr}[Bc\Omega^{t_1}c\Omega^{t_2}\partial c\Omega^{t_3}c\Omega^{t_4}],
\end{aligned} \tag{A.16}$$

and by using

$$K = \partial_y(\Omega^y)|_{y\to 0}, \tag{A.17}$$

we obtain

$$\text{Bcddd}[t_1,t_2,t_3] = \lim_{y\to 0}\partial_{y'}\text{Bccdd}[y,t_1,t_2,t_3]$$

$$
\begin{aligned}
&= \lim_{y,y'\to 0} \partial_y \partial_{y'} \Big\{ \mathrm{Bcccd}[y, t_1 + y', t_2, t_3] - \mathrm{Bcccd}[y, t_1, t_2 + y', t_3] \Big\} \\
&= \lim_{y,y',y''\to 0} \partial_y \partial_{y'} \partial_{y''} \\
&\quad \Big\{ \mathrm{Bcccc}[y, t_1 + y', t_2 + y'', t_3] - \mathrm{Bcccc}[y, t_1 + y', t_2, t_3 + y''] \\
&\qquad - \mathrm{Bcccc}[y, t_1, t_2 + y' + y'', t_3] + \mathrm{Bcccc}[y, t_1, t_2 + y', t_3 + y''] \Big\} \\
&= -\frac{1}{\pi} (\sin 2\theta_{t_2} + \sin 2\theta_{t_3} - \sin 2\theta_{t_2+t_3})
\end{aligned} \tag{A.18}
$$

Similary,

$$
\begin{aligned}
&\mathrm{Bccdc}[t_1, t_2, t_3, t_4] \\
&= \frac{L^2}{4\pi^3} \Bigg(\left(\frac{2\pi}{L} \right) \big(-(t_1 + t_2 + t_3) \cos 2\theta_{t_2} + (t_2 + t_3) \cos 2\theta_{t_1+t_2} + t_1 \cos 2\theta_{t_3} \big) \\
&\qquad + \sin 2\theta_{t_1} + \sin 2\theta_{t_2+t_3} - \sin 2\theta_{t_1+t_2+t_3} \Bigg).
\end{aligned} \tag{A.19}
$$

Appendix B

$KBcG\gamma$ Algebra

We summarize the derivations of the $KBcG\gamma$ algebras.

- $\hat{B}^2 = \hat{c}^2 = 0$

$$B^2 \otimes I = c^2 \otimes I = 0 \tag{B.1}$$

- $\{\hat{B}, \hat{c}\} = 1$

$$\begin{aligned}
\{\hat{B}, \hat{c}\} = (Bc + cB) \otimes I_2 \rightarrow_{\text{CFT}} & \int_{\downarrow -0} \frac{dz}{2\pi i} b(z)c(0) + \int_{\downarrow +0} \frac{dz}{2\pi i} c(0)b(z) \\
&= \left(\int_{\downarrow -0} + \int_{\uparrow +0} \right) \frac{dz}{2\pi i} b(z)c(0) \\
&= \oint_0 \frac{dz}{2\pi i} b(z)c(0) \\
&= 1 \\
\rightarrow_{\text{SFT}} & \ 1 \otimes I_2 = \hat{1} := 1
\end{aligned} \tag{B.2}$$

- $\{\hat{\gamma}, \hat{B}\} = \{\hat{\gamma}, \hat{c}\} = 0$

$$\{\hat{\gamma}, \hat{c}\} = (\gamma c - c\gamma) \otimes i\sigma_2\sigma_3 = 0, \quad \{\hat{\gamma}, \hat{B}\} = (\gamma B - B\gamma) \otimes i\sigma_2\sigma_3 = 0. \tag{B.3}$$

$$\because \ [\gamma, B] = [\gamma, c] = 0.$$

- $\hat{\delta}\hat{G} = 2\hat{K}$

$$\begin{aligned}
\hat{\delta}\hat{G} = \{\hat{G}, \hat{G}\} &= (GG + GG) \otimes I_2 \\
&\rightarrow_{\text{CFT}} \int_{\downarrow 0} \frac{dz}{2\pi i} \int_{\downarrow +0} \frac{dw}{2\pi i} G(z)G(w) + \int_{\downarrow -0} \frac{dw}{2\pi i} \int_{\downarrow 0} \frac{dz}{2\pi i} G(w)G(z) \\
&= \int_{\downarrow 0} \frac{dz}{2\pi i} \left(\int_{\downarrow -0} + \int_{\uparrow +0} \right) \frac{dw}{2\pi i} G(w)G(z) \\
&= \int_{\downarrow 0} \frac{dz}{2\pi i} \oint_z \frac{dw}{2\pi i} \frac{2T(w)}{w - z}
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_{\downarrow 0} \frac{dz}{2\pi i} T(z) \\
&\to_{\rm SFT}\ 2K \otimes I_2 = 2\hat{K}
\end{aligned}
\tag{B.4}
$$

- $\hat{\delta}\hat{c} = 2\hat{\gamma}$

$$
\begin{aligned}
[\hat{G}, \hat{c}] = (Gc + cG) \otimes \sigma_1\sigma_3 \to_{\rm CFT} & \int_{\downarrow -0} \frac{dz}{2\pi i} G(z)c(0) + \int_{\downarrow +0} \frac{dz}{2\pi i} c(0)G(z) \\
&= \left(\int_{\downarrow -0} + \int_{\uparrow +0} \right) \frac{dz}{2\pi i} G(z)c(0) \\
&= \oint_0 \frac{dz}{2\pi i} \frac{-2\gamma(0)}{z} \\
&= -2\gamma(0) \\
&\to_{\rm SFT} \ -2\gamma \otimes -i\sigma_2 = 2\hat{\gamma}
\end{aligned}
\tag{B.5}
$$

- $\hat{\delta}\hat{\gamma} = \hat{\partial}\hat{c}/2$

$$
\begin{aligned}
\{\hat{G}, \hat{\gamma}\} = (G\gamma - \gamma G) \otimes \sigma_1 i\sigma_2 \to_{\rm CFT} & \int_{\downarrow -0} \frac{dz}{2\pi i} G(z)\gamma(0) - \int_{\downarrow +0} \frac{dz}{2\pi i} \gamma(0)G(z) \\
&= \left(\int_{\downarrow -0} + \int_{\uparrow +0} \right) \frac{dz}{2\pi i} G(z)\gamma(0) \\
&= \oint_0 \frac{dz}{2\pi i} \frac{-\partial c(0)}{2z} \\
&= -\frac{1}{2} \left(\int_{\downarrow 0-} \frac{dz}{2\pi i} T(z)c(0) - \int_{\downarrow 0+} c(0)T(z) \right) \\
&\to_{\rm SFT} \ -\frac{1}{2}\partial c \otimes (-\sigma_3) = \frac{1}{2}\hat{\partial}\hat{c}
\end{aligned}
\tag{B.6}
$$

- $\hat{\delta}\hat{\gamma}^2 = 2\hat{\delta}\hat{\gamma} \cdot \hat{\gamma}$

$$
[\hat{G}, \hat{\gamma}^2] = \{\hat{G}, \hat{\gamma}\}\hat{\gamma} - \hat{\gamma}\{\hat{G}, \hat{\gamma}\} = \hat{\partial}\hat{c}\hat{\gamma}
\tag{B.7}
$$

- $\hat{Q}\hat{B} = \hat{K}$

$$
\begin{aligned}
\hat{Q}\hat{B} = QB \otimes I_2 \to_{\rm CFT} & \oint_w \frac{dz}{2\pi i} \int_{\downarrow 0} \frac{dw}{2\pi i} j_{\rm B}(z)b(w) \\
&= \int_{\downarrow 0} \frac{dw}{2\pi i} T(w) \\
&\to_{\rm SFT} \ K \otimes I_2 = \hat{K}
\end{aligned}
\tag{B.8}
$$

- $[\hat{K}, \hat{B}] = 0$

$$
0 = \hat{Q}(\hat{B} \cdot \hat{B})
$$

$$
\begin{aligned}
&= \hat{Q}\hat{B}\cdot\hat{B} - \hat{B}\hat{Q}\hat{B} \\
&= \hat{K}\hat{B} - \hat{B}\hat{K}
\end{aligned} \tag{B.9}
$$

- $\hat{Q}\hat{K} = 0$

$$
\hat{Q}\hat{K} = \hat{Q}^2\hat{B} = 0 \tag{B.10}
$$

- $\hat{Q}\hat{c} = \hat{c}\hat{\partial}\hat{c} + \hat{\gamma}^2$

$$
\begin{aligned}
\hat{Q}\hat{c} = Qc\otimes I_2 \rightarrow_{\rm CFT} &\oint_0 \frac{dz}{2\pi i} j_{\rm B}(z)c(0) \\
&= c\partial c - \gamma^2(0) \\
\rightarrow_{\rm SFT}\ & c\partial c\otimes I_2 + \gamma^2\otimes(i\sigma_2)^2 \\
&= \hat{c}\hat{\partial}\hat{c} + \hat{\gamma}^2
\end{aligned} \tag{B.11}
$$

- $\hat{Q}\hat{G} = 0$

$$
\hat{Q}\hat{G} = QG\otimes i\sigma_2 = 0 \tag{B.12}
$$

- $\hat{Q}\hat{\gamma} = \hat{c}\hat{\partial}\hat{\gamma} - \hat{\partial}\hat{c}\hat{\gamma}/2$

$$
\begin{aligned}
\hat{Q}\hat{\gamma} = Q\gamma\otimes\sigma_3 i\sigma_2 \rightarrow_{\rm CFT} &\oint_0 \frac{dz}{2\pi i} j_{\rm B}(z)\gamma(0) \\
&=(c\partial\gamma - \frac{1}{2}\partial c\gamma)(0) \\
\rightarrow_{\rm SFT}\ & (c\partial\gamma - \frac{1}{2}\partial c\gamma)\otimes\sigma_3 i\sigma_2 \\
&=\hat{c}\hat{\partial}\hat{\gamma} - \frac{1}{2}\hat{\partial}\hat{c}\hat{\gamma}
\end{aligned} \tag{B.13}
$$

Appendix C

Correlators in Modified Cubic String Field Theory

To calculate energy of solutions, we introduce the formulae of the correlators. The non-vanishing correlator in this theory is now normalized as

$$\langle c\partial c\partial^2 c e^{-2\phi}(z)\rangle^{\mathrm{gh}}_{\mathrm{UHP}} = -2. \tag{C.1}$$

In the correlator, the ghost number is 3, the bc-ghost number is 3, the ϕ momentum is -2, and the picture is -2. We derive the basic correlator:

$$\begin{aligned}
&\mathrm{Tr}_{Y_{-2}}[c\gamma\Omega^{t_1}\gamma\Omega^{t_2}] \\
&= \langle\, Y_{-2}(i\infty)c(0)\gamma(0)\gamma(t_1)\,\rangle^{\mathrm{gh}}_{\mathrm{C}_L} \times \frac{1}{2}\mathrm{Tr}[\sigma_3\sigma_3 i\sigma_2 i\sigma_2] \\
&= -\langle\, Y_{-2}(i) f_{\mathrm{s}}^{-1}\circ f_{L\to 2}\circ c(0) f_{\mathrm{s}}^{-1}\circ f_{L\to 2}\circ\gamma(0) f_{\mathrm{s}}^{-1}\circ f_{L\to 2}\circ\gamma(t_1)\,\rangle^{\mathrm{gh}}_{\mathrm{UHP}} \\
&= -\Big(\frac{2}{L}\Big)^{-1-\frac{1}{2}\times 2}\Big(\frac{\pi}{2}\Big)^{-1-\frac{1}{2}\times 2}\Big(\frac{1}{\cos^2\theta_{t_1}}\Big)^{-\frac{1}{2}} \\
&\quad\times\langle\, c\partial\xi e^{-2\phi}(i)c\partial\xi e^{-2\phi}(-i)c(0)\eta e^{\phi}(0)\eta e^{\phi}(\tan\theta_{t_1})\,\rangle^{\mathrm{gh}}_{S^2} \\
&= -\Big(\frac{\pi}{L}\Big)^{-2}\cos\theta_{t_1}\langle c(i)c(-i)c(0)\rangle^{bc}_{S^2} \\
&\quad\times\langle\, e^{-2\phi}(i)e^{-2\phi}(-i)e^{\phi}(0)e^{\phi}(\tan\theta_{t_1})\,\rangle^{\phi}_{S^2} \\
&\quad\times\partial_{s_1}\partial_{s_2}\langle\xi(s_1)\xi(s_2)\eta(0)\eta(\tan\theta_{t_1})\,\rangle^{\xi\eta}_{S^2}|_{s_1=i,\,s_2=-i} \\
&= -(\frac{\pi}{L})^{-2}\cos\theta_{t_1}(i+i)(i-0)(-i-0) \\
&\quad\times(i+i)^{-4}(i-0)^2(i-\tan\theta_{t_1})^2(-i-0)^2(-i-\tan\theta_{t_1})^2(0-\tan\theta_{t_1})^{-1} \\
&\quad\times\partial_{s_1}\partial_{s_2}\{(s_1-s_2)(s_1-0)^{-1}(s_1-\tan\theta_{t_1})^{-1} \\
&\qquad\qquad\cdot(s_2-0)^{-1}(s_2-\tan\theta_{t_1})^{-1}(0-\tan\theta_{t_1})\}|_{s_1=i,\,s_2=-i} \\
&= \frac{L^2\cos\theta_{t_1}}{2\pi^2}.
\end{aligned} \tag{C.2}$$

Here, the width of the sliver is L, and we used the doubling trick, and OPE:

$$\eta \simeq e^{-\chi}, \quad \xi \simeq e^{\chi}, \quad \chi(z)\chi(0) \sim \ln z, \\ \phi(z)\phi(0) \sim -\ln z. \tag{C.3}$$

Next, we derive the correlator $\mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_1}c\gamma\Omega^{t_2}\gamma\Omega^{t_3}]$ by using the technique used in the derivation of $\mathrm{Tr}[Bc\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3}c\Omega^{t_4}]$. To derive this, let us consider the following equations

$$\begin{aligned}
&\langle Y_{-2}(i\infty)\int_{\downarrow -0}\frac{dz}{2\pi i}zb(z)c(0)c\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&=\langle Y_{-2}(i\infty)\left(\int_{\downarrow -0}+\int_{\uparrow +0}\right)\frac{dz}{2\pi i}zb(z)c(0)c\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&\quad+\langle Y_{-2}(i\infty)\int_{\downarrow +0}\frac{dz}{2\pi i}zb(z)c(0)c\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&=\langle Y_{-2}(i\infty)\{\oint_0\frac{dz}{2\pi i}zb(z)c(0)\}c\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&\quad-\langle Y_{-2}(i\infty)\int_{\downarrow +0}\frac{dz}{2\pi i}c(0)zb(z)c(t_1)\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&=-\langle Y_{-2}(i\infty)\left(\int_{\downarrow t_1-0}+\int_{\uparrow t_1+0}\right)\frac{dz}{2\pi i}c(0)zb(z)c(t_1)\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&\quad-\langle Y_{-2}(i\infty)\int_{\downarrow t_1+0}\frac{dz}{2\pi i}c(0)zb(z)c(t_1)\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&=-\langle Y_{-2}(i\infty)c(0)\{\oint_{t_1}\frac{dz}{2\pi i}zb(z)c(t_1)\}\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&\quad+\langle Y_{-2}(i\infty)\int_{\downarrow L-0}\frac{dz}{2\pi i}c(0)c(t_1)\gamma(t_1)\gamma(t_1+t_2)zb(z)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&=-t_1\langle Y_{-2}(i\infty)c(0)\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&\quad+\langle Y_{-2}(i\infty)\int_{\downarrow -0}\frac{dz}{2\pi i}(z+L)b(z)c(0)c(t_1)\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}.
\end{aligned}\tag{C.4}$$

Therefore,

$$\begin{aligned}
\mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_1}c\gamma\Omega^{t_2}\gamma\Omega^{t_3}]&=\langle Y_{-2}(i\infty)\int_{\downarrow -0}\frac{dz}{2\pi i}b(z)c(0)c\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&\quad\times\frac{1}{2}\mathrm{Tr}[\sigma_3\sigma_3\sigma_3\sigma_3 i\sigma_2 i\sigma_2]\\
&=-\frac{t_1}{L}\langle Y_{-2}(i\infty)c(0)\gamma(t_1)\gamma(t_1+t_2)\rangle^{\mathrm{gh}}_{\mathrm{C}_L}\\
&=-\frac{t_1 L\cos\theta_{t_2}}{2\pi^2}.
\end{aligned}\tag{C.5}$$

We define

$$\mathrm{Bccgg}[t_1,t_2;L]:=\mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_1}c\gamma\Omega^{t_2}\gamma\Omega^{t_3}],$$

$$\mathrm{Bcdgg}[t_1; L] := \mathrm{Tr}_{Y_{-2}}[Bc\partial c\gamma\Omega^{t_1}\gamma\Omega^{t_2}]. \tag{C.6}$$

Formulae for the inner product including Bc, ∂c, γ, γ is

$$\begin{aligned}\mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_1}\partial c\Omega^{t_2}\gamma\Omega^{t_3}\gamma\Omega^{t_4}] &= \mathrm{Tr}_{Y_{-2}}[B([c,\Omega^{t_1}] + \Omega^{t_1}c)\partial c\Omega^{t_2}\gamma\Omega^{t_3}\gamma\Omega^{t_4}] \\ &= \mathrm{Tr}_{Y_{-2}}[Bc\partial c\Omega^{t_2}\gamma\Omega^{t_3}\gamma\Omega^{t_4}\Omega^{t_1}] \\ &= \mathrm{Bcdgg}[t_3; L]. \end{aligned}\tag{C.7}$$

Here, the first term in the first line vanishes because $\mathrm{Tr}[B\varphi] = \mathrm{Tr}[B^2 c\varphi] = 0$, for φ s.t. $[B,\varphi] = 0$. Similarly,

$$\begin{aligned}\mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_1}\gamma\Omega^{t_2}\partial c\Omega^{t_3}\gamma\Omega^{t_4}] &= \mathrm{Tr}_{Y_{-2}}[Bc\gamma\Omega^{t_2}\partial c\Omega^{t_3}\gamma\Omega^{t_4}\Omega^{t_1}] \\ &= \mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_2}\partial c\Omega^{t_3}\gamma\Omega^{t_4}\Omega^{t_1}\gamma] \\ &= \mathrm{Tr}_{Y_{-2}}[Bc\partial c\Omega^{t_3}\gamma\Omega^{t_4}\Omega^{t_1}\gamma\Omega^{t_2}] \\ &= \mathrm{Bcdgg}[t_4 + t_1; L], \end{aligned}\tag{C.8}$$

$$\begin{aligned}\mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}\partial c\Omega^{t_4}] &= \mathrm{Tr}_{Y_{-2}}[Bc\Omega^{t_4}\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}\partial c] \\ &= \mathrm{Tr}_{Y_{-2}}[Bc\partial c\Omega^{t_4}\Omega^{t_1}\gamma\Omega^{t_2}\gamma\Omega^{t_3}] \\ &= \mathrm{Bcdgg}[t_2; L]. \end{aligned}\tag{C.9}$$

These only depend on the width between γ's and total width L.

Appendix D

Detailed Calculation of the Energy of the Half-brane Solution

We give detailed calculation of the energy of the half-brane solution. We can compute it from the cubic term in the action:

$$\begin{aligned}
\mathrm{Tr}_{Y_{-2}}[{\Psi_{1/2}}^3] &= \mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{-1}{1-G}B\gamma^2\frac{-1}{1-G}B\gamma^2\frac{-1}{1-G}\right] \\
&\quad + 3\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{-1}{1-G}B\gamma^2\frac{-1}{1-G}cB(1-G)Gc\frac{-1}{1-G}\right] \\
&\quad + 3\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{-1}{1-G}cB(1-G)Gc\frac{-1}{1-G}cB(1-G)Gc\frac{-1}{1-G}\right] \\
&\quad + \mathrm{Tr}_{Y_{-2}}\left[\left(cB(1-G)Gc\frac{-1}{1-G}\right)^3\right] \\
&= -3\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2 Gc\frac{1}{1-G}cG\right] && \text{(D.1)} \\
&\quad + \mathrm{Tr}_{Y_{-2}}\left[\left(cB(1-G)Gc\frac{-1}{1-G}\right)^3\right]. && \text{(D.2)}
\end{aligned}$$

The first term (D.1) becomes

$$\begin{aligned}
\text{(D.1)} &= -3\mathrm{Tr}_{Y_{-2}}[B(\partial c\gamma+\gamma^2 G)Gc(1+G)\Omega_x c] \\
&= -\frac{3}{2}\mathrm{Tr}_{Y_{-2}}[\delta(c\Omega_x cB\partial c\gamma)] - 3\mathrm{Tr}_{Y_{-2}}[B\gamma^2 Kc\Omega_x c] \\
&= 6\mathrm{Tr}_{Y_{-2}}[Bc\partial c\gamma^2\Omega_x] - 3\mathrm{Tr}_{Y_{-2}}[Bc\Omega_x c\gamma^2 K] \\
&= \int_0^\infty dx\, e^{-x_1}\left(6\cdot \mathrm{Bcdgg}[0;x_1] - 3\lim_{y\to 0}\partial_y\cdot \mathrm{Bccgg}[x_1,0;x_1+y]\right) \\
&= \frac{3}{2\pi^2}, && \text{(D.3)}
\end{aligned}$$

where we used the following equation:

$$\frac{1}{1-G} = \frac{1+G}{(1-G)(1+G)} = (1+G)\Omega_x, \tag{D.4}$$

and we define $\Omega_x := \frac{1}{1-K}$. The second term (D.2) becomes

$$\begin{aligned}
\text{(D.2)} &= \text{Tr}_{Y_{-2}}\left[cB(1-G)Gc\frac{-1}{1-G}cB(1-G)Gc\frac{-1}{1-G}cB(1-G)Gc\frac{-1}{1-G}\right] \\
&= \text{Tr}_{Y_{-2}}\Big[cBG\big([1-G,c]+c(1-G)\big)\frac{-1}{1-G} \\
&\quad \times cBG([1-G,c]+c(1-G))\frac{-1}{1-G}cBG\big([1-G,c]+c(1-G)\big)\frac{-1}{1-G}\Big] \\
&= \text{Tr}_{Y_{-2}}\left[cBG\delta c\frac{1}{1-G}cBG\delta c\frac{1}{1-G}cBG\delta c\frac{1}{1-G}\right] \\
&= \text{Tr}_{Y_{-2}}\left[cB\delta c\frac{G}{1-G}\delta c\frac{G}{1-G}\delta c\frac{G}{1-G}\right] && \text{(D.5)} \\
&= \text{Tr}_{Y_{-2}}[cB\delta c(G+K)\Omega_x\delta c(G+K)\Omega_x\delta c(G+K)\Omega_x] \\
&= \text{Tr}_{Y_{-2}}[cB\delta cG\Omega_x\delta cG\Omega_x\delta cG\Omega_x] && \text{(D.6)} \\
&\quad + \text{Tr}_{Y_{-2}}[cB\delta cG\Omega_x\delta cK\Omega_x\delta cK\Omega_x] && \text{(D.7)} \\
&\quad + \text{Tr}_{Y_{-2}}[cB\delta cK\Omega_x\delta cG\Omega_x\delta cK\Omega_x] && \text{(D.8)} \\
&\quad + \text{Tr}_{Y_{-2}}[cB\delta cK\Omega_x\delta cK\Omega_x\delta cG\Omega_x]. && \text{(D.9)}
\end{aligned}$$

The first term (D.6) becomes

$$\begin{aligned}
\text{(D.6)} &= \text{Tr}_{Y_{-2}}[cB\delta c\Omega_x(\partial c-\delta cG)G\Omega_x\delta cG\Omega_x] \\
&= \frac{1}{2}\text{Tr}_{Y_{-2}}[\delta(cB\delta c\Omega_x\partial c\Omega_x\partial c\Omega_x)] - \text{Tr}_{Y_{-2}}[cB\delta c\Omega_x\partial c\Omega_x\delta cK\Omega_x] \\
&\quad - \frac{1}{2}\text{Tr}_{Y_{-2}}[\delta(cB\delta c\Omega_x\delta cK\Omega_x\delta c\Omega_x)] \\
&= -\frac{1}{2}\text{Tr}_{Y_{-2}}[cB\delta c\Omega_x\partial(\delta c)\Omega_x\partial c\Omega_x] - \frac{1}{2}\text{Tr}_{Y_{-2}}[cB\delta c\Omega_x\partial c\Omega_x\partial(\delta c)\Omega_x] \\
&\quad + \text{Tr}_{Y_{-2}}[Bc\delta c\Omega_x\partial c\Omega_x\delta cK\Omega_x] - \frac{1}{2}\text{Tr}_{Y_{-2}}[cB(\partial c)\Omega_x\delta cK\Omega_x\delta c\Omega_x] \\
&\quad + \frac{1}{2}\text{Tr}_{Y_{-2}}[cB\delta c\Omega_x(\partial c)K\Omega_x\delta c\Omega_x] - \frac{1}{2}\text{Tr}_{Y_{-2}}[cB\delta c\Omega_x\delta cK\Omega_x(\partial c)\Omega_x] \\
&= 6\iiint_0^\infty dx_1dx_2dx_3\, e^{-(x_1+x_2+x_3)}\lim_{y\to 0}\partial_y\,\text{Bcdgg}[y+x_1;x_1+x_2+x_3+y] \\
&= -\frac{6\,(\pi^2-6)}{\pi^4}. && \text{(D.10)}
\end{aligned}$$

The second term (D.7) becomes

$$\begin{aligned}
\text{(D.7)} &= \frac{1}{2}\text{Tr}_{Y_{-2}}[\delta(\delta cK\Omega_x\delta cK\Omega_x cB\delta c\Omega_x)] \\
&= \frac{1}{2}\text{Tr}_{Y_{-2}}[(\partial c)K\Omega_x\delta cK\Omega_x cB\delta c\Omega_x]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}_{Y_{-2}}[\delta c K\Omega_x(\partial c)K\Omega_x cB\delta c\Omega_x] \\
&+\frac{1}{2}\mathrm{Tr}_{Y_{-2}}[\delta c K\Omega_x \delta c K\Omega_x cB(\partial c)\Omega_x] \\
&= \iiint_0^\infty dx_1 dx_2 dx_3\, e^{-(x_1+x_2+x_3)} \lim_{y,y'\to 0}\partial_y\partial_{y'} \\
&\quad\times\Big\{-4\cdot \mathrm{Bcdgg}[y+x_1; x_1+x_2+x_3+y+y'] \\
&\qquad + 2\cdot \mathrm{Bcdgg}[x_1; x_1+x_2+x_3+y+y']\Big\} \\
&= -\frac{24-2\pi^2}{\pi^4} - \frac{\pi^2-12}{\pi^4} \\
&= -\frac{12-\pi^2}{\pi^4}.
\end{aligned} \tag{D.11}
$$

We can show that the remaining terms (D.8) and (D.9) are equal to (D.7):

$$
(\mathrm{D}.7) = (\mathrm{D}.8) = (\mathrm{D}.9). \tag{D.12}
$$

Therefore,

$$
(\mathrm{D}.2) = (\mathrm{D}.10) + 3\times(\mathrm{D}.7) = -\frac{3}{\pi^2}. \tag{D.13}
$$

We obtain the energy of the half-brane solution by adding (D.1) and (D.2):

$$
\frac{1}{6}\mathrm{Tr}_{Y_{-2}}[{\Psi_{1/2}}^3] = \frac{1}{6}\left(\frac{3}{2\pi^2} - \frac{3}{\pi^2}\right) = -\frac{1}{4\pi^2} = E(\Psi_0) + \frac{1}{2}T_9. \tag{D.14}
$$

Appendix E

Detailed Calculations of the EOMS for $\Psi_{3/2}$

We give the detailed calculations of the remaining terms of the EOMS for $\Psi_{3/2}$. The term (5.2.14) becomes

$$\begin{aligned}
(5.2.14) &= \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[B\gamma^2 c \frac{1}{1-G_\epsilon} \partial c \frac{1}{-G_\epsilon} \right] \\
&= \lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[B\gamma^2 c G_\epsilon \Omega_\epsilon \partial c G_\epsilon \frac{1}{-K_\epsilon} \right] \\
&= \lim_{\epsilon\to 0} \frac{\epsilon}{2} \mathrm{Tr}_{Y_{-2}} \left[\delta \left(B\gamma^2 c \Omega_\epsilon \partial(\delta c) \frac{1}{-K_\epsilon} \right) \right] \\
&= \lim_{\epsilon\to 0} \left(\frac{\epsilon}{2} \mathrm{Tr}_{Y_{-2}} \left[B(\partial c\gamma) c \Omega_\epsilon \partial(\delta c) \frac{1}{-K_\epsilon} \right] + \frac{\epsilon}{2} \mathrm{Tr}_{Y_{-2}} \left[B\gamma^2 c \Omega_\epsilon \partial^2 c \frac{1}{-K_\epsilon} \right] \right) \\
&= -\lim_{\epsilon\to 0} \epsilon \mathrm{Tr}_{Y_{-2}} \left[B c \partial c \Omega_\epsilon \partial\gamma \frac{1}{-K_\epsilon} \gamma \right] \\
&= -\lim_{\epsilon\to 0} \epsilon \cdot \lim_{y\to 0} \partial_y \big\{ \mathrm{Bcdgg}[z_1; x_1 + y + z_1] - \mathrm{Bcdgg}[z_1 + y; x_1 + y + z_1] \big\} \\
&= (5.2.20) \\
&= 0,
\end{aligned} \tag{E.1}$$

where $\Omega_\epsilon := \frac{1}{1-K_\epsilon}$, and other notations are explained shortly. Therefore, the first term (5.2.10) vanishes:

$$(5.2.10) = (5.2.20) + (\mathrm{E}.1) = 0. \tag{E.2}$$

For simplicity, we use the certain letters x_i and z_i, as Schwinger parameters corresponding to the following Laplace transformations:

$$\begin{aligned}
\frac{1}{1-K_\epsilon} &= \int_0^\infty dx_i \, e^{-(1+\epsilon)x_i} \Omega^{x_i}, \\
\frac{1}{-K_\epsilon} &= \int_0^\infty dz_i \, e^{-\epsilon z_i} \Omega^{z_i}.
\end{aligned} \tag{E.3}$$

In the following, we omit $\int_0^\infty dx_i$ and $\int_0^\infty dz_i$ and also the exponential factors. For example, we abbreviate the term:

$$\begin{aligned}
&\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\Omega_\epsilon\partial\gamma\frac{1}{-K_\epsilon}\gamma\right]\\
&=\lim_{y\to 0}\partial_y\iint_0^\infty dx_1dz_1e^{-(1+\epsilon)x_1}e^{-\epsilon z_1}\\
&\quad\times\{\mathrm{Bcdgg}[z_1;x_1+y+z_1]-\mathrm{Bcdgg}[z_1+y;x_1+y+z_1]\},
\end{aligned}\tag{E.4}$$

as

$$\lim_{y\to 0}\partial_y\{\mathrm{Bcdgg}[z_1;x_1+y+z_1]'-\mathrm{Bcdgg}[z_1+y;x_1+y+z_1]'\}.\tag{E.5}$$

The second term (5.2.11) becomes

$$\begin{aligned}
(5.2.11)&=\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[(1-Bc)\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\gamma^2\frac{1}{-G_\epsilon}\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[B\gamma^2\frac{1}{-G_\epsilon}c\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right]+\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\gamma^2\frac{1}{-G_\epsilon}\right]\\
&=(5.2.10)-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[\frac{K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\gamma^2\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[K_\epsilon G_\epsilon\Omega_\epsilon cG_\epsilon\frac{1}{-K_\epsilon}\gamma^2\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[G_\epsilon\Omega_\epsilon\partial cG_\epsilon\frac{1}{-K_\epsilon}\gamma^2\right]\\
&=0.
\end{aligned}\tag{E.6}$$

Here, we use the following equation which holds for the string field φ anti-commuting with B:

$$\begin{aligned}
\mathrm{Tr}_{Y_{-2}}[\varphi]&=\mathrm{Tr}_{Y_{-2}}[(Bc+cB)\varphi]\\
&=\mathrm{Tr}_{Y_{-2}}[Bc\varphi]+\mathrm{Tr}_{Y_{-2}}[cB\varphi]\\
&=\mathrm{Tt}_{Y_{-2}}[Bc\varphi]-\mathrm{Tr}_{Y_{-2}}[Bc\varphi]\\
&=0.
\end{aligned}\tag{E.7}$$

The third term (5.2.12) becomes

$$\begin{aligned}
(5.2.12)&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[cB\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}c\frac{G_\epsilon K_\epsilon}{1-G_\epsilon}c\frac{1}{-G_\epsilon}\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[cB\frac{K_\epsilon}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}c\frac{K_\epsilon}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}\right]\\
&=-\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[(-\partial c+K_\epsilon)B\frac{1}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}(-\partial c+K_\epsilon c)\frac{1}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}\right]\\
&=\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[\partial cB\frac{1}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}K_\epsilon c\frac{1}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[K_\epsilon cB\frac{1}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}\partial c\frac{1}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}\right]\\
=&\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\frac{1}{1-G_\epsilon}\delta c\frac{1}{-G_\epsilon}\partial c\frac{1}{1-G_\epsilon}\delta cG_\epsilon\right]\\
=&\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc(1+G_\epsilon)\Omega_\epsilon\delta cG_\epsilon\frac{1}{-K_\epsilon}\partial c(1+G_\epsilon)\Omega_\epsilon\delta cG_\epsilon\right]\\
=&\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\Omega_\epsilon\delta cG_\epsilon\frac{1}{-K_\epsilon}\partial c\Omega_\epsilon\delta cG_\epsilon\right] \qquad \text{(E.8)}\\
&+\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[BcG_\epsilon\Omega_\epsilon\delta cG_\epsilon\frac{1}{-K_\epsilon}\partial cG_\epsilon\Omega_\epsilon\delta cG_\epsilon\right]. \qquad \text{(E.9)}
\end{aligned}
$$

The terms (E.8) and (E.9) vanish as follows:

$$
\begin{aligned}
\text{(E.8)} =&\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\Omega_\epsilon G_\epsilon\delta cG_\epsilon\frac{1}{-K_\epsilon}\partial c\Omega_\epsilon\delta c\right]\\
=&\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\Omega_\epsilon(\partial c-\delta cG_\epsilon)G_\epsilon\frac{1}{-K_\epsilon}\partial c\Omega_\epsilon\delta c\right]\\
=&\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[\delta\left(\partial c\Omega_\epsilon\delta cBc\Omega_\epsilon\partial c\frac{1}{-K_\epsilon}\right)\right]\\
=&\lim_{\epsilon\to 0}\left(\epsilon\mathrm{Tr}_{Y_{-2}}\left[\partial(\delta c)\Omega_\epsilon\delta cBc\Omega_\epsilon\partial c\frac{1}{-K_\epsilon}\right]-\epsilon\mathrm{Tr}_{Y_{-2}}\left[\partial c\Omega_\epsilon\delta cBc\Omega_\epsilon\partial(\delta c)\frac{1}{-K_\epsilon}\right]\right)\\
=&\,4\lim_{\epsilon\to 0}\left(\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\frac{1}{-K_\epsilon}\partial\gamma\Omega_\epsilon\gamma\Omega_\epsilon\right]-\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\partial c\Omega_\epsilon\gamma\Omega_\epsilon\partial\gamma\frac{1}{-K_\epsilon}\right]\right)\\
=&\,8\lim_{\epsilon\to 0}\lim_{y\to 0}\partial_y\{\mathrm{Bcdgg}[x_1;z_1+y+x_1+x_2]'-\mathrm{Bcdgg}[x_1+y;z_1+y+x_1+x_2]'\}\\
=&-\lim_{\epsilon\to 0}\epsilon\int_0^\infty da\frac{4a\left(2a^2-\pi^2\left(e^a-1\right)\right)e^{-a(\epsilon+1)}}{\pi^2\left(a^2+\pi^2\right)}\\
=&-\lim_{\epsilon\to 0}4\epsilon(\mathrm{Ci}(\pi\epsilon)\cos(\pi\epsilon)+\cdots)\\
=&\,0, \qquad \text{(E.10)}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(E.9)} =&\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\Omega_\epsilon\delta cG_\epsilon\frac{1}{-K_\epsilon}\partial cG_\epsilon\Omega_\epsilon\delta cK_\epsilon\right]\\
=&-\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\Omega_\epsilon\delta cG_\epsilon\frac{1}{-K_\epsilon}\partial(\delta c)\Omega_\epsilon\delta cK_\epsilon\right]\\
=&\lim_{\epsilon\to 0}2\epsilon\mathrm{Tr}_{Y_{-2}}\left[Bc\Omega_\epsilon\delta cG_\epsilon\frac{1}{-K_\epsilon}\delta cK_\epsilon\Omega_\epsilon\delta cK_\epsilon\right]\\
=&\lim_{\epsilon\to 0}\epsilon\mathrm{Tr}_{Y_{-2}}\left[\delta\left(\delta cK_\epsilon\Omega_\epsilon\delta cK_\epsilon Bc\Omega_\epsilon\delta c\frac{1}{-K_\epsilon}\right)\right]\\
=&\lim_{\epsilon\to 0}\left(\epsilon\mathrm{Tr}_{Y_{-2}}\left[(\partial c)K_\epsilon\Omega_\epsilon\delta cK_\epsilon Bc\Omega_\epsilon\delta c\frac{1}{-K_\epsilon}\right]-\epsilon\mathrm{Tr}_{Y_{-2}}\left[\delta cK_\epsilon\Omega_\epsilon(\partial c)K_\epsilon Bc\Omega_\epsilon\delta c\frac{1}{-K_\epsilon}\right]\right.\\
&\left.+\epsilon\mathrm{Tr}_{Y_{-2}}\left[\delta cK_\epsilon\Omega_\epsilon\delta cK_\epsilon Bc\Omega_\epsilon(\partial c)\frac{1}{-K_\epsilon}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\Bigg(4\epsilon \mathrm{Tr}_{Y_{-2}}\left[Bc\partial c K_\epsilon \Omega_\epsilon \gamma \Omega_\epsilon \partial\gamma \frac{1}{-K_\epsilon}\right] + 4\epsilon \mathrm{Tr}_{Y_{-2}}\left[Bc\partial c \Omega_\epsilon \partial\gamma \frac{1}{-K_\epsilon}\partial\gamma\Omega_\epsilon\right] \\
&\qquad - 4\epsilon \mathrm{Tr}_{Y_{-2}}\left[Bc\partial c \frac{1}{-K_\epsilon}\partial\gamma \Omega_\epsilon \gamma K_\epsilon \Omega_\epsilon\right]\Bigg) \\
&= \lim_{\epsilon\to 0} 4\epsilon \mathrm{Tr}_{Y_{-2}}\left[Bc\partial c \Omega_\epsilon \partial\gamma \frac{1}{-K_\epsilon}\partial\gamma \Omega_\epsilon\right] \\
&= \lim_{\epsilon\to 0} 4\epsilon \lim_{y,y'\to 0} \partial_y \partial_{y'} \Big\{ \mathrm{Bcdgg}[z_1+y'; x_1+y+z_1+y'+x_2]' \\
&\quad - \mathrm{Bcdgg}[z_1; x_1+y+z_1+y'+x_2]' - \mathrm{Bcdgg}[y+z_1+y'; x_1+y+z_1+y'+x_2]' \\
&\quad + \mathrm{Bcdgg}[y+z_1; x_1+y+z_1+y'+x_2]' \Big\} \\
&= \lim_{\epsilon\to 0} \epsilon \int_0^\infty da \frac{2a\left(a^3+(e^a+1)\,a^2+\pi^2 a-\pi^2\,(e^a+1)\right)e^{-a(\epsilon+1)}}{\left(a^2+\pi^2\right)^2} \\
&= -\lim_{\epsilon\to 0} 2\epsilon(\mathrm{Ci}(\pi\epsilon)\cos(\pi\epsilon)+\cdots) \\
&= 0.
\end{aligned}
\tag{E.11}
$$

Then we obtain

$$
\lim_{\epsilon\to 0}\mathrm{EOMS}([\![\Psi_{3/2}]\!]_\epsilon) \sim \lim_{\epsilon\to 0}\epsilon \times \big(\log\epsilon + \mathrm{O}(\epsilon^0)\big) = 0. \tag{E.12}
$$

The solution $\lim_{\epsilon\to 0}[\![\Psi_{3/2}]\!]_\epsilon$ satisfies the EOMS.

Appendix F

Detailed Calculation of the Energy of the Tachyon Vacuum Solution in Berkovits' SFT

We give detailed calculation of the energy of the tachyon vacuum solution in Berkovits' SFT. The energy is given by

$$E(g_0) = -S(g_0) = \int_0^1 dt \mathrm{Tr}[\eta_0\big(g_0(t)^{-1}\partial_t g_0(t)\big) \cdot g_0(t)^{-1}Qg_0(t)]. \tag{F.1}$$

We can rewrite the integrand:

$$\begin{aligned}
&\mathrm{Tr}[\eta_0\big(g_0(t)^{-1}\partial_t g_0(t)\big) \cdot g_0(t)^{-1}Qg_0(t)] \\
&= \mathrm{Tr}[\eta_0\big((v_0u_0)^{-1}\partial_t(v_0u_0)\big)(v_0u_0)^{-1}Q(v_0u_0)] \\
&= \mathrm{Tr}[\eta_0(v_0^{-1}\partial_t v_0)({v_0}^{-1}Qv_0 + Qu_0 \cdot {u_0}^{-1})] \\
&= \mathrm{Tr}[\eta_0(\tilde{v}_0\partial_t v_0)\tilde{v}_0 Qv_0].
\end{aligned} \tag{F.2}$$

Here we used $g_0(t) = v_0u_0$, $\partial_t u_0 = 0$, ${v_0}^{-1} = \tilde{v}_0 + Bc$ and the fact that the string fields u_0, ${u_0}^{-1}$, $Qu_0 \cdot {u_0}^{-1}$, $Bc\partial_t v_0$, and $BcQv_0 \in \mathcal{H}^{\mathrm{small}}$. We write down v_0 expicitly:

$$\begin{aligned}
v_0 = \begin{bmatrix} 1 & -\alpha \\ \alpha & \bar{t}\cdot I - K - \alpha V \end{bmatrix} &= 1 + \alpha\zeta - \alpha\gamma B + c(\bar{t}\cdot I - K - \alpha V - 1)B \\
&= 1 + \alpha\zeta - \alpha Q\zeta \cdot B + c(\bar{t}\cdot I - K - 1)B.
\end{aligned} \tag{F.3}$$

We perform Q for v_0:

$$\begin{aligned}
Qv_0 &= \alpha Q\zeta + \alpha Q\zeta K + Qc\cdot(\bar{t}\cdot I - K - 1)B - c(\bar{t}\cdot I - K - 1)K \\
&= c(\alpha V(1+K) + \partial c(\bar{t}\cdot I - K - 1)B - (\bar{t}\cdot I - K - 1)K) \\
&\quad + B\gamma^2(\bar{t}\cdot I - K - 1) + \alpha\gamma(1+K) \\
&= c\big(-\det{}_0(1+K) + \partial c(\bar{t}\cdot I - K - 1)B + \bar{t}\cdot I + \alpha^2(1+K)\big) \\
&\quad + B\gamma^2(\bar{t}\cdot I - K - 1) + \alpha\gamma(1+K).
\end{aligned} \tag{F.4}$$

We perform ∂_t for v_0:

$$\partial_t v_0 = q\zeta - q\gamma B + c(-I - qV)B. \tag{F.5}$$

We decompose it as $\partial_t v_0 = \dot{v}^{\mathrm{l}} + \dot{v}_0^{\mathrm{s}}$, depending on whether it is in $\mathcal{H}^{\mathrm{small}}$ or not.

$$\dot{v}^{\mathrm{l}} := q\zeta - qcVB \notin \mathcal{H}^{\mathrm{small}}, \tag{F.6}$$

$$\dot{v}_0^{\mathrm{s}} := -q\gamma B - cIB \in \mathcal{H}^{\mathrm{small}}. \tag{F.7}$$

We write down the explicit form of w and decompose it

$$w = \begin{bmatrix} -\alpha^2 & \alpha \\ -\alpha & 1 \end{bmatrix} = -\alpha^2 Bc - \alpha\zeta + \alpha\gamma B + cB, \tag{F.8}$$

$$w^{\mathrm{l}} := -\alpha\zeta, \tag{F.9}$$

$$w^{\mathrm{s}} := -\alpha^2 Bc + \alpha\gamma B + cB. \tag{F.10}$$

We want to calculate $\tilde{v}_0 Q v_0$, and for that purpose we first calculate wQv_0:

$$\begin{aligned} wQv_0 &= \big((-\alpha^2 B + \alpha\gamma^{-1})c + (\alpha\gamma + c)B\big) \\ &\quad \times (c\phi_0 + B\gamma^2(\bar{t}\cdot I - K - 1) + \alpha\gamma(1+K)) \\ &= -\alpha^2 B\gamma^2(\bar{t}\cdot I - K - 1) + \alpha c B\gamma(\bar{t}\cdot I - K - 1) - \alpha^3 Bc\gamma(1+K) \\ &\quad - \alpha^2 c(1+K) + \alpha\gamma Bc\phi_0 + c\phi_0 + \alpha^2\gamma B\gamma(1+K) + \alpha c B\gamma(1+K) \\ &= \alpha\gamma B(\bar{t}\alpha\gamma\cdot I - \alpha^2 c(1+K) + c\phi_0) \\ &\quad + c(\bar{t}\alpha B\gamma\cdot I - \alpha^2(1+K) + \phi_0). \end{aligned} \tag{F.11}$$

Here we defined

$$\phi_0 := -\mathrm{det}_0(1+K) + \partial c(\bar{t}\cdot I - K - 1)B + \bar{t}\cdot I + \alpha^2(1+K). \tag{F.12}$$

Because $\tilde{v}_0 Q v_0 = D_0 w Q v_0$,

$$\begin{aligned} \tilde{v}_0 Q v_0 &= (\gamma\frac{1}{\mathrm{det}_0}B\zeta + c\frac{1}{\mathrm{det}_0}B) \\ &\quad \times \Big(\alpha\gamma B(\bar{t}\alpha\gamma\cdot I - \alpha^2 c(1+K) + c\phi_0) \\ &\qquad + c(\bar{t}\alpha B\gamma\cdot I - \alpha^2(1+K) + \phi_0)\Big) \\ &= (\alpha\gamma + c)\frac{1}{\mathrm{det}_0}B(\bar{t}\alpha\gamma\cdot I - \alpha^2 c(1+K) + c\phi_0) \\ &= (\alpha\gamma + c)\frac{1}{\mathrm{det}_0}B\Big(\bar{t}\alpha\gamma\cdot I - \alpha^2 c(1+K) \\ &\quad + c\big(-\mathrm{det}_0(1+K) + \partial c(\bar{t}\cdot I - K - 1)B + \bar{t}\cdot I + \alpha^2(1+K)\big)\Big) \\ &= (\alpha\gamma + c)\frac{1}{\mathrm{det}_0}B\big(\bar{t}\alpha\gamma\cdot I - c\mathrm{det}_0(1+K) + \bar{t}c\cdot I - \partial c(\bar{t}\cdot I - K - 1)\big) \\ &= (\alpha\gamma + c)\frac{1}{\mathrm{det}_0}B\Phi_0. \end{aligned} \tag{F.13}$$

Here, we defined

$$\Phi_0 := \bar{t}\alpha\gamma \cdot I - c\det_0(1+K) + \bar{t}c \cdot I - \partial c(\bar{t} \cdot I - K - 1). \tag{F.14}$$

We rewrite (F.2) as

$$\mathrm{Tr}[\eta_0(\tilde{v}_0\partial_t v_0)\tilde{v}_0 Q v_0] = \mathrm{Tr}[\eta_0\big(D_0(w^{\mathrm{l}} + w^{\mathrm{s}})(\dot{v}^{\mathrm{l}} + \dot{v}_0^{\mathrm{s}})\big)\tilde{v}_0 Q v_0]. \tag{F.15}$$

The explicit forms of the terms in $(w^{\mathrm{l}} + w^{\mathrm{s}})(\dot{v}^{\mathrm{l}} + \dot{v}_0^{\mathrm{s}})$ are

$$w^{\mathrm{l}}\dot{v}^{\mathrm{l}} = -\alpha\zeta(q\zeta - qcVB) = 0, \tag{F.16}$$
$$w^{\mathrm{l}}\dot{v}_0^{\mathrm{s}} = -\alpha\zeta(-q\gamma B - cIB) = q\alpha cB, \tag{F.17}$$
$$w^{\mathrm{s}}\dot{v}^{\mathrm{l}} = (-\alpha^2 Bc + (\alpha\gamma + c)B)(q\zeta - qcVB) = q(\alpha\gamma + c)B(\zeta - V), \tag{F.18}$$
$$w^{\mathrm{s}}\dot{v}_0^{\mathrm{s}} = (-\alpha^2 Bc + (\alpha\gamma + c)B)(-q\gamma B - cIB) = (q\alpha^2\gamma - \alpha\gamma I - cI)B \tag{F.19}$$

Then, (F.15) becomes

$$(\mathrm{F.15}) = \mathrm{Tr}[\eta_0 D_0 \cdot w^{\mathrm{l}}\dot{v}_0^{\mathrm{s}}\tilde{v}_0 Q v_0] \tag{F.20}$$
$$+ \mathrm{Tr}[\eta_0(D_0 w^{\mathrm{s}}\dot{v}^{\mathrm{l}})\tilde{v}_0 Q v_0] \tag{F.21}$$
$$+ \mathrm{Tr}[\eta_0 D_0 \cdot w^{\mathrm{s}}\dot{v}_0^{\mathrm{s}}\tilde{v}_0 Q v_0]. \tag{F.22}$$

The factors $\eta_0 D_0$ and $\eta_0(D_0 w^{\mathrm{s}}\dot{v}^{\mathrm{l}})$ are calculated as follows:

$$\begin{aligned}\eta_0 D_0 &= \eta_0\left(\gamma\frac{1}{\det_0}B\zeta + c\frac{1}{\det_0}B\right)\\ &= -\gamma\left(\eta_0\frac{1}{\det_0}\right)B\zeta - \gamma\frac{1}{\det_0}Bc(\eta_0\gamma^{-1}) - c\left(\eta_0\frac{1}{\det_0}\right)B,\end{aligned} \tag{F.23}$$

$$\begin{aligned}\eta_0(D_0 w^{\mathrm{s}}\dot{v}^{\mathrm{l}}) &= \eta_0\left(\left(\gamma\frac{1}{\det_0}B\zeta + c\frac{1}{\det_0}B\right)q(\alpha\gamma + c)B(\zeta - V)\right)\\ &= q\eta_0\left(\alpha\gamma\frac{1}{\det_0}B(\zeta - V) + c\frac{1}{\det_0}B(\zeta - V)\right)\\ &= -q(\alpha\gamma + c)\eta_0\left(\frac{1}{\det_0}B(\zeta - V)\right).\end{aligned} \tag{F.24}$$

The first term (F.20) becomes

$$\begin{aligned}(\mathrm{F.20}) &= \mathrm{Tr}[\eta_0 D_0 \cdot w^{\mathrm{l}}\dot{v}_0^{\mathrm{s}}\tilde{v}_0 Q v_0]\\ &= -\mathrm{Tr}\left[c\left(\eta_0\frac{1}{\det_0}\right)Bq\alpha cB(\alpha\gamma + c)\frac{1}{\det_0}B\Phi_0\right]\\ &= q\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}B\Phi_0 c\right]\end{aligned}$$

$$
\begin{aligned}
&= q\alpha \mathrm{Tr}\Bigg[\left(\eta_0 \frac{1}{\det_0}\right) \frac{1}{\det_0} B \\
&\qquad \times (\bar{t}\alpha\gamma \cdot Ic - c\det_0(1+K)c + \bar{t}c \cdot Ic - \partial c(\bar{t}\cdot I - K - 1)c)\Bigg],
\end{aligned}
\tag{F.25}
$$

where we used the cyclicity of Tr[1], $\mathrm{Tr}[\varphi_1\varphi_2] = (-)^{\epsilon(\varphi_1)\epsilon(\varphi_2)}\mathrm{Tr}[\varphi_2\varphi_1]$. Here,

$$
\begin{aligned}
&- c\det_0(1+K)c + \bar{t}c \cdot Ic - \partial c(\bar{t}\cdot I - K - 1)c) \\
&= -([c, \det_0] + \det_0 c)(1+K)c + \bar{t}c \cdot Ic - \partial c(\bar{t}\cdot I - K - 1)c) \\
&= -([c, \bar{t}I - K - \alpha V + \alpha^2] + \det_0 c)(1+K)c + \bar{t}c \cdot Ic - \partial c(\bar{t}\cdot I - K - 1)c) \\
&= -(\bar{t}\,[c, I] + \partial c + \det_0 c)(1+K)c + \bar{t}c \cdot Ic - \partial c(\bar{t}\cdot I - K - 1)c) \\
&= -\det_0 c\partial c - \bar{t}\,[c, I](1+K)c - \partial c(1+K)c + \bar{t}cIc - \bar{t}\partial c \cdot Ic + \partial c(K+1)c \\
&= -\det_0 c\partial c - \bar{t}\,[c, I]Kc - \bar{t}\partial c \cdot Ic \\
&= -\det_0 c\partial c - \bar{t}(cIKc - Ic\partial c + \partial c \cdot Ic) \\
&= -\det_0 c\partial c + \bar{t}(Ic\partial c - KcIc) \\
&= -\det_0 c\partial c + \bar{t}(Ic\partial c - Kc[I, c]),
\end{aligned}
\tag{F.26}
$$

where we used

$$
[c, \det_0] = \bar{t}\,[c, I] + \partial c. \tag{F.27}
$$

Therefore,

$$
\Phi_0 c = \bar{t}\alpha\gamma \cdot Ic - \det_0 c\partial c + \bar{t}(Ic\partial c - Kc[I, c]). \tag{F.28}
$$

We continue to calculate the term (F.25):

$$
\begin{aligned}
(\mathrm{F.25}) &= q\alpha \mathrm{Tr}\left[\left(\eta_0 \frac{1}{\det_0}\right) \frac{1}{\det_0} B\Phi_0 c\right] \\
&= q\alpha \mathrm{Tr}\left[\left(\eta_0 \frac{1}{\det_0}\right) \frac{1}{\det_0} B\Big(\bar{t}\alpha\gamma \cdot Ic - \det_0 c\partial c + \bar{t}\big(Ic\partial c - Kc[I, c]\big)\Big)\right] \\
&= q\alpha \mathrm{Tr}\left[\left(\eta_0 \frac{1}{\det_0}\right) \frac{1}{\det_0} B\Big(\bar{t}\alpha\gamma \cdot Ic + \bar{t}\big(Ic\partial c - Kc[I, c]\big)\Big)\right].
\end{aligned}
\tag{F.29}
$$

Here, we used the following equation:

$$
\mathrm{Tr}\left[\left(\eta_0 \frac{1}{\det_0}\right) \frac{1}{\det_0} B\det_0 c\partial c\right] = \mathrm{Tr}\left[\eta_0 \left(\frac{1}{\det_0} Bc\partial c\right)\right] = 0, \tag{F.30}
$$

where the first and the second equality comes from $[B, \det_0] = 0$ and $\mathrm{Tr}[\eta_0\varphi] = 0$, respectively.

The second term (F.21) becomes

$$
(\mathrm{F.21}) = \mathrm{Tr}[\eta_0(D_0 w^{\mathrm{s}}\dot{v}^1)\tilde{v}_0 Q v_0]
$$

[1]In the case of the cubic theory, since the non-vanishing ghost number input in the trace is 3, then $\epsilon(\varphi_1)\epsilon(\varphi_2) = 0$. While, in the case of the Berkovits' SFT, since the ghost number is 2, it may appear a minus sign in the cyclicty of the trace.

$$
\begin{aligned}
&= -\mathrm{Tr}\left[q(\alpha\gamma + c)\eta_0\left(\frac{1}{\det_0}B(\zeta - V)\right)(\alpha\gamma + c)\frac{1}{\det_0}B\Phi_0\right] \\
&= -\mathrm{Tr}\left[q(\alpha\gamma + c)\eta_0\left(\frac{1}{\det_0}B(\zeta - V)(\alpha\gamma + c)\right)\frac{1}{\det_0}B\Phi_0\right] \\
&= q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B\Phi_0\gamma\right] \qquad \text{(F.31)} \\
&\quad + q\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B\Phi_0 c\right]. \qquad \text{(F.32)}
\end{aligned}
$$

The first term (F.31) becomes

$$
\begin{aligned}
\text{(F.31)} &= q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B\Phi_0\gamma\right] \\
&= q\alpha\mathrm{Tr}\Bigg[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B \\
&\qquad\qquad \times\left(\bar{t}\alpha\gamma\cdot I - c\det_0(1+K) + \bar{t}c\cdot I - \partial c(\bar{t}\cdot I - K - 1)\right)\gamma\Bigg] \\
&= q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B(-c\det_0(1+K) + \bar{t}c\cdot I)\gamma\right] \\
&= -q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B([c,\det_0] + \det_0 c)(1+K)\gamma\right] \\
&\quad + \bar{t}q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}Bc\cdot I\gamma\right] \\
&= -q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B(\bar{t}\,[c,I] + \partial c + \det_0 c)(1+K)\gamma\right] \\
&\quad + \bar{t}q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}Bc\cdot I\gamma\right] \\
&= -q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)Bc(1+K)\gamma\right)\right] \\
&\quad + \bar{t}q\alpha\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}Bc\cdot I\gamma\right] \\
&= -\bar{t}q\alpha\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha - V)\left(\eta_0\frac{1}{\det_0}\right)Bc\cdot I\gamma\right], \qquad \text{(F.33)}
\end{aligned}
$$

where we used $\mathrm{Tr}[B\varphi] = 0$ for φ s.t. $[B,\varphi\} = 0$ and $\mathrm{Tr}[\eta_0\varphi] = 0$. The second term (F.32) becomes

$$
\begin{aligned}
\text{(F.32)} &= q\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B\Phi_0 c\right] \\
&= q\mathrm{Tr}\Bigg[\eta_0\left(\frac{1}{\det_0}(\alpha - V)\right)\frac{1}{\det_0}B \\
&\qquad\qquad \times\left(\bar{t}\alpha\gamma\cdot Ic - \det_0 c\partial c + \bar{t}(Ic\partial c - Kc[I,c])\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= -q\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha-V)Bc\partial c\right)\right] \\
&\quad + q\mathrm{Tr}\left[\eta_0\left(\frac{1}{\det_0}(\alpha-V)\right)\frac{1}{\det_0}B\Big(\bar{t}\alpha\gamma\cdot Ic+\bar{t}\big(Ic\partial c-Kc[I,c]\big)\Big)\right] \\
&= -q\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha-V)\left(\eta_0\frac{1}{\det_0}\right)B\Big(\bar{t}\alpha\gamma\cdot Ic+\bar{t}\big(Ic\partial c-Kc[I,c]\big)\Big)\right]. \qquad \text{(F.34)}
\end{aligned}
$$

Then, we reach the following form of (F.21):

$$
\begin{aligned}
\text{(F.21)} &= \text{(F.33)} + \text{(F.34)} \\
&= -\bar{t}q\alpha\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha-V)\left(\eta_0\frac{1}{\det_0}\right)Bc\cdot I\gamma\right] \\
&\quad - q\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha-V)\left(\eta_0\frac{1}{\det_0}\right)B\Big(\bar{t}\alpha\gamma\cdot Ic+\bar{t}\big(Ic\partial c-Kc[I,c]\big)\Big)\right] \\
&= -\bar{t}q\alpha\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha-V)\left(\eta_0\frac{1}{\det_0}\right)[I,\gamma]Bc\right] \\
&\quad - \bar{t}q\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha-V)\left(\eta_0\frac{1}{\det_0}\right)B\big(Ic\partial c-Kc[I,c]\big)\right]. \qquad \text{(F.35)}
\end{aligned}
$$

The third term (F.22) becomes

$$
\begin{aligned}
\text{(F.22)} &= \mathrm{Tr}\left[\eta_0 D_0\cdot w^{\mathrm{s}}\dot{v}_0^{\mathrm{s}}\tilde{v}_0 Q v_0\right] \\
&= \mathrm{Tr}\left[\eta_0 D_0(q\alpha^2\gamma-\alpha\gamma I-cI)B(\alpha\gamma+c)\frac{1}{\det_0}B\Phi_0\right] \\
&= \mathrm{Tr}\left[\eta_0 D_0(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\right] \\
&= \mathrm{Tr}\Bigg[\left(-\gamma\left(\eta_0\frac{1}{\det_0}\right)B\zeta-\gamma\frac{1}{\det_0}Bc(\eta_0\gamma^{-1})-c\left(\eta_0\frac{1}{\det_0}\right)B\right) \\
&\qquad\quad \times(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\Bigg] \\
&= -\mathrm{Tr}\left[\gamma\left(\eta_0\frac{1}{\det_0}\right)B\zeta(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\right] \qquad \text{(F.36)} \\
&\quad - \mathrm{Tr}\left[\gamma\frac{1}{\det_0}Bc(\eta_0\gamma^{-1})(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\right] \qquad \text{(F.37)} \\
&\quad - \mathrm{Tr}\left[c\left(\eta_0\frac{1}{\det_0}\right)B(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\right]. \qquad \text{(F.38)}
\end{aligned}
$$

The first term (F.36) becomes

$$
\begin{aligned}
\text{(F.36)} &= -\mathrm{Tr}\left[\gamma\left(\eta_0\frac{1}{\det_0}\right)B\zeta(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\right] \\
&= \mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)(q\alpha^2-\alpha I)\frac{1}{\det_0}B\Phi_0\gamma\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Tr}\Bigg[\left(\eta_0\frac{1}{\det_0}\right)(q\alpha^2-\alpha I)\frac{1}{\det_0}B \\
&\qquad \times \big(\bar{t}\alpha\gamma\cdot I - c\det_0(1+K)+\bar{t}c\cdot I-\partial c(\bar{t}\cdot I-K-1)\big)\gamma\Bigg] \\
&= \mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)(q\alpha^2-\alpha I)\frac{1}{\det_0}B(-c\det_0(1+K)+\bar{t}c\cdot I)\gamma\right] \\
&= \mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)(q\alpha^2-\alpha I)\frac{1}{\det_0}B\Big(-\big([c,\det_0]+\det_0 c\big)(1+K)+\bar{t}c\cdot I\Big)\gamma\right] \\
&= \bar{t}\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)(q\alpha^2-\alpha I)\frac{1}{\det_0}Bc\cdot I\gamma\right]. \qquad \text{(F.39)}
\end{aligned}
$$

The second term (F.37) vanishes

$$
\begin{aligned}
\text{(F.37)} &= -\mathrm{Tr}\left[\gamma\frac{1}{\det_0}Bc(\eta_0\gamma^{-1})(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\right] \\
&= -\mathrm{Tr}\left[\gamma\frac{1}{\det_0}Bc(\eta_0\gamma^{-1})(q\alpha^2\gamma-\alpha\gamma I)\frac{1}{\det_0}B\Phi_0\right] \\
&= -\mathrm{Tr}\left[\gamma\frac{1}{\det_0}Bc\big(\eta_0(\gamma^{-1}\gamma)\big)(q\alpha^2-\alpha I)\frac{1}{\det_0}B\Phi_0\right] \\
&= 0. \qquad \text{(F.40)}
\end{aligned}
$$

The third term (F.38) becomes

$$
\begin{aligned}
\text{(F.38)} &= -\mathrm{Tr}\left[c\left(\eta_0\frac{1}{\det_0}\right)B(q\alpha^2\gamma-\alpha\gamma I-cI)\frac{1}{\det_0}B\Phi_0\right] \\
&= -\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}B\Phi_0 c\right] \\
&= -\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}B\Big(\bar{t}\alpha\gamma\cdot Ic-\det_0 c\partial c+\bar{t}\big(Ic\partial c-Kc[I,c]\big)\Big)\right] \\
&= -\bar{t}\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}B(\alpha\gamma\cdot Ic+Ic\partial c-Kc[I,c])\right]. \qquad \text{(F.41)}
\end{aligned}
$$

Then, we reach the following form of (F.22):

$$
\begin{aligned}
\text{(F.22)} &= \text{(F.39)}+\text{(F.40)}+\text{(F.41)} \\
&= \bar{t}\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)(q\alpha^2-\alpha I)\frac{1}{\det_0}Bc\cdot I\gamma\right] \\
&\quad - \bar{t}\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}B(\alpha\gamma\cdot Ic+Ic\partial c-Kc[I,c])\right] \\
&= \bar{t}q\alpha^2\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}Bc\cdot I\gamma\right] \\
&\quad - \bar{t}\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}[I,\gamma]Bc\right]
\end{aligned}
$$

$$
-\bar{t}\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}B(Ic\partial c-Kc[I,c])\right]. \tag{F.42}
$$

We summarize the above calculations:

$$
\begin{aligned}
&\mathrm{Tr}[\eta_0\big(g_0(t)^{-1}\partial_t g_0(t)\big)g_0(t)^{-1}Qg_0(t)]\\
&=q\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}B\Big(\bar{t}\alpha\gamma\cdot Ic+\bar{t}\big(Ic\partial c-Kc[I,c]\big)\Big)\right]\\
&\quad-\bar{t}q\alpha\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha-V)\left(\eta_0\frac{1}{\det_0}\right)[I,\gamma]Bc\right]\\
&\quad-\bar{t}q\mathrm{Tr}\left[\frac{1}{\det_0}(\alpha-V)\left(\eta_0\frac{1}{\det_0}\right)B(Ic\partial c-Kc[I,c])\right]\\
&\quad+\bar{t}q\alpha^2\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}Bc\cdot I\gamma\right]\\
&\quad-\bar{t}\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}[I,\gamma]Bc\right]\\
&\quad-\bar{t}\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}B(Ic\partial c-Kc[I,c])\right]\\
&=\bar{t}\Bigg\{q\alpha^2\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}B\gamma\cdot Ic\right]+q\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}B(Ic\partial c-Kc[I,c])\right]\\
&\qquad-q\alpha^2\mathrm{Tr}\left[\frac{1}{\det_0}\left(\eta_0\frac{1}{\det_0}\right)[I,\gamma]Bc\right]+q\alpha\mathrm{Tr}\left[\frac{1}{\det_0}V\left(\eta_0\frac{1}{\det_0}\right)[I,\gamma]Bc\right]\\
&\qquad-q\alpha\mathrm{Tr}\left[\frac{1}{\det_0}\left(\eta_0\frac{1}{\det_0}\right)B(Ic\partial c-Kc[I,c])\right]\\
&\qquad+q\mathrm{Tr}\left[\frac{1}{\det_0}V\left(\eta_0\frac{1}{\det_0}\right)B(Ic\partial c-Kc[I,c])\right]\\
&\qquad+q\alpha^2\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}Bc\cdot I\gamma\right]-\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}[I,\gamma]Bc\right]\\
&\qquad-\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}B(Ic\partial c-Kc[I,c])\right]\Bigg\}\\
&=\bar{t}\Bigg\{2q\alpha^2\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}Bc[I,\gamma]\right]\\
&\qquad+2q\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}B(Ic\partial c-Kc[I,c])\right]\\
&\qquad+q\alpha\mathrm{Tr}\left[\frac{1}{\det_0}V\left(\eta_0\frac{1}{\det_0}\right)[I,\gamma]Bc\right]\\
&\qquad+q\mathrm{Tr}\left[\frac{1}{\det_0}V\left(\eta_0\frac{1}{\det_0}\right)B(Ic\partial c-Kc[I,c])\right]\\
&\qquad-\alpha\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)I\frac{1}{\det_0}[I,\gamma]Bc\right]
\end{aligned}
$$

$$- \mathrm{Tr}\left[\left(\eta_0 \frac{1}{\det_0}\right) I \frac{1}{\det_0} B(Ic\partial c - Kc[I,c])\right]\Bigg\}. \tag{F.43}$$

In the tachyon vacuum solution, $I = 1$,

$$\therefore \quad \mathrm{Tr}[\eta_0\big(g_0(t)^{-1}\partial_t g_0(t)\big)g_0(t)^{-1}Qg_0(t)] = \bar{t}\Bigg\{(2q\alpha - 1)\mathrm{Tr}\left[\left(\eta_0 \frac{1}{\det_0}\right)\frac{1}{\det_0}Bc\partial c\right] \tag{F.44}$$

$$+ q\mathrm{Tr}\left[\frac{1}{\det_0}V\left(\eta_0\frac{1}{\det_0}\right)Bc\partial c\right]\Bigg\}. \tag{F.45}$$

Let us calculate the energy of the solution:

$$E(g_0) = -S(g_0) = \int_0^1 dt \mathrm{Tr}[\eta_0\big(g_0(t)^{-1}\partial_t g_0(t)\big)g_0(t)^{-1}Qg_0(t)]. \tag{F.46}$$

We give a definition of the string field $\frac{1}{\det_0}$:

$$\begin{aligned}
\frac{1}{\det_0} &:= (\bar{t} + \alpha^2 - K - \alpha V)^{-1} \\
&= \left(\left(1 - \alpha V\frac{1}{\bar{t}+\alpha^2 - K}\right)\left(\frac{1}{\bar{t}+\alpha^2-K}\right)^{-1}\right)^{-1} \\
&= \frac{1}{\bar{t}+\alpha^2-K}\left(1 - \alpha V \frac{1}{\bar{t}+\alpha^2-K}\right)^{-1} \\
&= \frac{1}{\bar{t}+\alpha^2-K}\left(1 + \alpha V\frac{1}{\bar{t}+\alpha^2-K} + \alpha V \frac{1}{\bar{t}+\alpha^2-K}\alpha V\frac{1}{\bar{t}+\alpha^2-K} + \cdots\right) \\
&= \Big(\frac{1}{\bar{t}+\alpha^2-K} + \frac{1}{\bar{t}+\alpha^2-K}\alpha V \frac{1}{\bar{t}+\alpha^2-K} \\
&\quad + \frac{1}{\bar{t}+\alpha^2-K}\alpha V\frac{1}{\bar{t}+\alpha^2-K}\alpha V\frac{1}{\bar{t}+\alpha^2-K} + \cdots\Big) \\
&= \frac{1}{\beta}\Big(\frac{1}{1-K/\beta} + \frac{1}{1-K/\beta}\frac{\alpha}{\beta}V\frac{1}{1-K/\beta} \\
&\quad + \frac{1}{1-K/\beta}\frac{\alpha}{\beta}V\frac{1}{1-K/\beta}\frac{\alpha}{\beta}V\frac{1}{1-K/\beta} + \cdots\Big),
\end{aligned} \tag{F.47}$$

where $\beta := \bar{t} + \alpha^2$. In calculations, we will pick up suitable terms with bc-ghost number three.

(F.44) becomes

$$\mathrm{Tr}\left[\left(\eta_0\frac{1}{\det_0}\right)\frac{1}{\det_0}Bc\partial c\right] = \frac{\alpha^2}{\beta^4}\mathrm{Tr}\left[\eta_0\left(\frac{1}{1-K/\beta}V\frac{1}{1-K/\beta}\right)\frac{1}{1-K/\beta}V\frac{1}{1-K/\beta}Bc\partial c\right]. \tag{F.48}$$

We rescale K, B, c, γ and γ^{-1}:

$$K \to \beta K, \quad B \to \beta B, \quad c \to \frac{1}{\beta}c, \quad \gamma \to \frac{1}{\sqrt{\beta}}\gamma, \quad \gamma^{-1} \to \sqrt{\beta}\gamma^{-1},$$
$$\zeta \to \frac{1}{\sqrt{\beta}}\zeta, \quad V \to \sqrt{\beta}V. \tag{F.49}$$

Then, (F.48) becomes

$$\begin{aligned}
(\text{F.48}) &= \frac{\alpha^2}{\beta^3 2^2}\mathrm{Tr}[(\eta_0\gamma^{-1})(\Omega_x)^2\partial c\gamma^{-1}\Omega_x Bc\partial c\Omega_x\partial c] \\
&= \frac{\alpha^2}{\beta^3 2^2}\iiiint_0^\infty \prod_{i=1}^4 dx_i\, e^{-x_i}\, \mathrm{eii}[x_1+x_2, x_3+x_4]\cdot \mathrm{Bcddd}[x_4, x_1+x_2, x_3] \\
&\quad \times \frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3\sigma_3\sigma_3\sigma_3] \\
&= \frac{\alpha^2}{\beta^3 2^2}\left(\frac{4}{\pi^2}\right),
\end{aligned} \tag{F.50}$$

where

$$\mathrm{eii}[t_1,t_2] := \langle \oint_0 \frac{dz}{2\pi i}\eta(z)\xi(0)e^{-\phi}(0)\xi(t_1)e^{-\phi}(t_1)\rangle^{\xi\eta\phi}_{C_{t_1+t_2}} = -\frac{\pi}{t_1+t_2}\frac{1}{\sin\theta_{t_1}}. \tag{F.51}$$

We calculate (F.45) as (F.44):

$$\begin{aligned}
\mathrm{Tr}\left[\frac{1}{\det_0}V(\eta_0\frac{1}{\det_0})Bc\partial c\right] &= \mathrm{Tr}\left[\frac{1}{\beta}\frac{1}{1-K/\beta}V\frac{1}{\beta}\eta_0\left(\frac{1}{1-K/\beta}\frac{\alpha}{\beta}V\frac{1}{1-K/\beta}\right)Bc\partial c\right] \\
&= \frac{\alpha}{\beta^2 2^2}\mathrm{Tr}[(\eta_0\gamma^{-1})\Omega_x Bc\partial c\Omega_x\partial c\gamma^{-1}\Omega_x\partial c] \\
&= \frac{\alpha}{\beta^2 2^2}\iiint_0^\infty \prod_{i=1}^3 dx_i\, e^{-x_i}\, \mathrm{eii}[x_1+x_2, x_3]\cdot \mathrm{Bcddd}[x_2, x_3, x_1] \\
&\quad \times \frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3\sigma_3\sigma_3\sigma_3(-i\sigma_2)\sigma_3] \\
&= -\frac{\alpha}{\beta^2 2^2}\left(\frac{8}{\pi^2}\right).
\end{aligned} \tag{F.52}$$

Finally, we obtain the energy of the tachyon vacuum solution:

$$\begin{aligned}
E(g_0) &= \int_0^1 dt\,\left(\bar{t}(2q\alpha-1)\cdot(\text{F.50}) + q\bar{t}\cdot(\text{F.52})\right) \\
&= \int_0^1 dt\left(\bar{t}(2q\alpha-1)\frac{\alpha^2}{\beta^3 2^2}\frac{4}{\pi^2} - \bar{t}q\frac{\alpha}{\beta^2 2^2}\frac{8}{\pi^2}\right) \\
&= -\frac{1}{2\pi^2}.
\end{aligned} \tag{F.53}$$

Appendix G

Detailed Calculations of the Energy of the Double-brane Solution in Berkovits' SFT

We give detailed calculations of the energy of the double-brane solution in Berkovits' SFT. We explicitly write down the omitted terms in the main text.

The term (7.2.26) becomes

$$\begin{aligned}
&\int_0^1 dt\,(7.2.26)|_{q=\frac{1}{2}} \\
&= \frac{1}{2^8}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^5 \mathrm{Tr}[(\eta_0\gamma^{-1})(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon \\
&\qquad\qquad \times \partial c\gamma^{-1}[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c] \\
&= \frac{1}{2^8}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^5\cdot \mathrm{eiiid}[u_1+u_2+n_1+n_2, u_3+n_3, n_4, n_5]' \\
&\quad \times \mathrm{Bcddd}[u_1+u_2+n_1+n_2, u_3+n_3, n_5+n_5]' \\
&\quad \times \frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)i\sigma_2\sigma_3\sigma_3\sigma_3].
\end{aligned} \tag{G.1}$$

Here, we define

$$\begin{aligned}
&\mathrm{eiiig}[t_1,t_2,t_3,t_4] \\
&:= \Big\langle \oint_0 \frac{dz}{2\pi i}\eta(z)\xi e^{-\phi}(0)\xi e^{-\phi}(t_1)\xi e^{-\phi}(t_1+t_2)e^{\phi}\eta(t_1+t_2+t_3)\Big\rangle^{\xi\eta\phi}_{\mathrm{C}_{t_1+t_2+t_3+t_4}} \\
&= -\frac{\pi}{L}\frac{\sin\theta_{t_1+t_2+t_3}}{\sin\theta_{t_1}\sin\theta_{t_1+t_2}},
\end{aligned} \tag{G.2}$$

$$\mathrm{eiiid}[t_1,t_2,t_3,t_4] := \lim_{y\to 0}\partial_y\{\mathrm{eiiig}[t_1,t_2,t_3+y,t_4]-\mathrm{eiiig}[t_1,t_2,t_3,t_4+y]\}. \tag{G.3}$$

We use the abbreviation explained around (E.3); in this case, we use the certain letter n_i, u_i and y_i:

$$[\![F_2]\!]_\epsilon = \frac{1}{k_+-k_-}\int_0^\infty dn_i(e^{-k_-n_i}-e^{-k_+n_i})\Omega^{n_i},$$

$$
\begin{aligned}
1-K_\epsilon &= -\lim_{u_i\to 0}\partial_{u_i}\{e^{-(1+\epsilon)u_i}\Omega^{u_i}\},\\
K_\epsilon &= \lim_{u_i\to 0}\partial_{y_i}\{e^{-\epsilon y_i}\Omega^{y_i}\}.
\end{aligned}
\tag{G.4}
$$

For example, we abbreviate the term:

$$
\begin{aligned}
&\mathrm{Tr}[(\eta_0\gamma^{-1})(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon\partial c\gamma^{-1}[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c]\\
&= \int_0^\infty\prod_{i=1}^{5}dn_i\prod_{j=1}^{3}\lim_{u_j\to 0}(-\partial_{u_j})\left(\frac{1}{k_+-k_-}\right)^5(e^{-k_-n_i}-e^{-k_+n_i})e^{(1+\epsilon)u_i}\\
&\times \mathrm{eiiid}[u_1+u_2+n_1+n_2,u_3+n_3,n_4,n_5]\\
&\times \mathrm{Bcddd}[u_1+u_2+n_1+n_2,u_3+n_3,n_5+n_5]\\
&\times\frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)i\sigma_2\sigma_3\sigma_3\sigma_3],
\end{aligned}
\tag{G.5}
$$

as

$$
\begin{aligned}
&\mathrm{eiiid}[u_1+u_2+n_1+n_2,u_3+n_3,n_4,n_5]'\\
&\times \mathrm{Bcddd}[u_1+u_2+n_1+n_2,u_3+n_3,n_5+n_5]'\\
&\times\frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)i\sigma_2\sigma_3\sigma_3\sigma_3].
\end{aligned}
\tag{G.6}
$$

The Second term (7.2.13) becomes

$$
\begin{aligned}
&(7.2.13)\\
&= -2\lim_{\epsilon\to 0}\bar{t}q\alpha\mathrm{Tr}\left[\left[\!\!\left[\left(\eta_0\frac{1}{\det_2}\right)\frac{1}{\det_2}B\Omega' c\partial c\Omega'\right]\!\!\right]_\epsilon\right]\\
&= -2\lim_{\epsilon\to 0}\bar{t}q\alpha\mathrm{Tr}[(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha(\eta_0V)(1-K_\epsilon)[\![F_2]\!]_\epsilon\\
&\qquad\times(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon B[\![\Omega']\!]_\epsilon c\partial c[\![\Omega']\!]_\epsilon]\\
&= -2\lim_{\epsilon\to 0}\bar{t}q\alpha^3\mathrm{Tr}[[\![F_2]\!]_\epsilon(\eta_0V)(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2V[\![F_2]\!]_\epsilon BK_\epsilon c\partial cK_\epsilon]\\
&= -\lim_{\epsilon\to 0}\frac{\bar{t}q\alpha^3}{2}\mathrm{Tr}[(\eta_0\gamma^{-1})(1-K_\epsilon)^2[\![F_2]\!]_\epsilon{}^2\partial c\gamma^{-1}[\![F_2]\!]_\epsilon K_\epsilon Bc\partial cK_\epsilon[\![F_2]\!]_\epsilon\partial c].
\end{aligned}
\tag{G.7}
$$

Then (G.7) becomes

$$
\begin{aligned}
\int_0^1 dt\,(\mathrm{G.7})|_{q=\frac{1}{2}} &= -\frac{1}{2^5}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^3\cdot\mathrm{eii}[u_1+u_2+n_1+n_2,n_3+y_1+y_2+n_4]'\\
&\times\mathrm{Bcddd}[y_2+n_4,u_1+u_2+n_1+n_2,n_3+y_1]'\\
&\times\frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3\sigma_3\sigma_3\sigma_3].
\end{aligned}
\tag{G.8}
$$

The third term (7.2.14) becomes

(7.2.14)

$$
\begin{aligned}
&= \lim_{\epsilon\to 0} \bar{t} q \alpha \mathrm{Tr}\left[\left[\!\left[\frac{1}{\det_2} V \left(\eta_0 \frac{1}{\det_2}\right) \Omega_x \partial\gamma \Omega_x Bc\right]\!\right]_\epsilon\right] \\
&= \lim_{\epsilon\to 0} \bar{t} q \alpha \mathrm{Tr}[(1-K_\epsilon)[\![F_2]\!]_\epsilon V \eta_0\big((1-K_\epsilon)[\![F_2]\!]_\epsilon \alpha V (1-K_\epsilon)[\![F_2]\!]_\epsilon \alpha V (1-K_\epsilon)[\![F_2]\!]_\epsilon\big) \\
&\qquad\qquad \times \Omega_\epsilon \partial\gamma \Omega_\epsilon Bc] \\
&\quad + \lim_{\epsilon\to 0} \bar{t} q \alpha \mathrm{Tr}[(1-K_\epsilon)[\![F_2]\!]_\epsilon \alpha V (1-K_\epsilon)[\![F_2]\!]_\epsilon V \eta_0\big((1-K_\epsilon)[\![F_2]\!]_\epsilon \alpha V (1-K_\epsilon)[\![F_2]\!]_\epsilon\big) \\
&\qquad\qquad \times \Omega_\epsilon \partial\gamma \Omega_\epsilon Bc] \\
&= -\lim_{\epsilon\to 0} \bar{t} q \alpha^3 \mathrm{Tr}[[\![F_2]\!]_\epsilon (\eta_0 V)(1-K_\epsilon)[\![F_2]\!]_\epsilon V (1-K_\epsilon)[\![F_2]\!]_\epsilon V [\![F_2]\!]_\epsilon \partial\gamma Bc] \\
&\quad + \lim_{\epsilon\to 0} \bar{t} q \alpha^3 \mathrm{Tr}[[\![F_2]\!]_\epsilon V (1-K_\epsilon)[\![F_2]\!]_\epsilon V (1-K_\epsilon)[\![F_2]\!]_\epsilon (\eta_0 V)[\![F_2]\!]_\epsilon \partial\gamma Bc] \\
&= -\lim_{\epsilon\to 0} \frac{\bar{t} q \alpha^3}{2^3} \mathrm{Tr}[(\eta_0\gamma^{-1})(1-K_\epsilon)[\![F_2]\!]_\epsilon \partial c \gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon \partial c \gamma^{-1}[\![F_2]\!]_\epsilon \partial\gamma [\![F_2]\!]_\epsilon Bc\partial c] \qquad \text{(G.9)} \\
&\quad + \lim_{\epsilon\to 0} \frac{\bar{t} q \alpha^3}{2^3} \mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon \partial\gamma [\![F_2]\!]_\epsilon Bc\partial c \gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon \partial c \gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon \partial c]. \qquad \text{(G.10)}
\end{aligned}
$$

Then (G.9) and (G.10) become

$$
\begin{aligned}
&\int_0^1 dt\, (\mathrm{G.9})|_{q=\frac{1}{2}} \\
&= -\frac{1}{2^7} \lim_{\epsilon\to 0} \int_0^1 dt\, \bar{t} t^3 \mathrm{Tr}[(\eta_0\gamma^{-1})(1-K_\epsilon)[\![F_2]\!]_\epsilon \\
&\qquad\qquad \times \partial c \gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon \partial c \gamma^{-1}[\![F_2]\!]_\epsilon \partial\gamma [\![F_2]\!]_\epsilon Bc\partial c] \\
&= -\frac{1}{2^7} \lim_{\epsilon\to 0} \int_0^1 dt\, \bar{t} t^3 \cdot \mathrm{eiiid}[u_1+n_1, u_2+n_2, n_3, n_4]' \times \mathrm{Bcddd}[u_1+n_1, u_2+n_2, n_3+n_4]' \\
&\quad \times \frac{1}{2} \mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2) i\sigma_2 \sigma_3\sigma_3\sigma_3], \qquad \text{(G.11)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 dt\, (\mathrm{G.10})|_{q=\frac{1}{2}} \\
&= \frac{1}{2^7} \lim_{\epsilon\to 0} \int_0^1 dt\, \bar{t} t^3 \mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon \partial\gamma [\![F_2]\!]_\epsilon Bc\partial c \gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon \partial c \gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon \partial c] \\
&= \frac{1}{2^7} \lim_{\epsilon\to 0} \int_0^1 dt\, \bar{t} t^3 \cdot \mathrm{eidii}[n_1, n_2, u_1+n_3, u_2+n_4]' \times \mathrm{Bcddd}[u_1+n_3, u_2+n_4, n_1+n_2]' \\
&\quad \times \frac{1}{2} \mathrm{Tr}[\sigma_3(-i\sigma_2) i\sigma_2 \sigma_3\sigma_3\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3]. \qquad \text{(G.12)}
\end{aligned}
$$

The fourth term (7.2.15) becomes

$$
(7.2.15) = -\lim_{\epsilon\to 0} \bar{t} q \mathrm{Tr}\left[\left[\!\left[\frac{1}{\det_2} V \left(\eta_0 \frac{1}{\det_2}\right) B\Omega' c\partial c \Omega'\right]\!\right]_\epsilon\right]
$$

$$
\begin{aligned}
&= -\lim_{\epsilon\to 0}\bar{t}q\mathrm{Tr}[(1-K_\epsilon)[\![F_2]\!]_\epsilon V\eta_0\big((1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\big)\\
&\qquad\qquad \times B[\![\Omega']\!]_\epsilon c\partial c[\![\Omega']\!]_\epsilon]\\
&= -\lim_{\epsilon\to 0}\bar{t}q\mathrm{Tr}[[\![F_2]\!]_\epsilon V(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha(\eta_0 V)[\![F_2]\!]_\epsilon BK_\epsilon c\partial cK_\epsilon]\\
&= -\lim_{\epsilon\to 0}\frac{\bar{t}q\alpha}{2^2}\mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon K_\epsilon Bc\partial cK_\epsilon[\![F_2]\!]_\epsilon\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon\partial c]. \qquad \text{(G.13)}
\end{aligned}
$$

Then, (G.13) becomes

$$
\begin{aligned}
&\int_0^1 dt\,(\text{G.13})|_{q=\frac{1}{2}}\\
&= -\frac{1}{2^4}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t\mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon K_\epsilon Bc\partial cK_\epsilon[\![F_2]\!]_\epsilon\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon\partial c]\\
&= -\frac{1}{2^4}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t\cdot\mathrm{eii}[n_1+y_1+y_2+n_2,u_1+n_3]'\times\mathrm{Bcddd}[y_2+n_2,u_1+n_3,n_1+y_1]'\\
&\times\frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3\sigma_3\sigma_3\sigma_3(-i\sigma_2)\sigma_3]. \qquad \text{(G.14)}
\end{aligned}
$$

The fifth term (7.2.16) becomes

$$
\begin{aligned}
(7.2.16) &= -\lim_{\epsilon\to 0}\bar{t}\alpha\mathrm{Tr}\left[\left[\!\!\left[\left(\eta_0\frac{1}{\det_2}\right)\Omega'\frac{1}{\det_2}\Omega_x\partial\gamma\Omega_x Bc\right]\!\!\right]_\epsilon\right]\\
&= -\lim_{\epsilon\to 0}\bar{t}\alpha\mathrm{Tr}[\eta_0\big((1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\big)\\
&\qquad\times[\![\Omega']\!]_\epsilon(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\Omega_\epsilon\partial\gamma\Omega_\epsilon Bc]\\
&\quad -\lim_{\epsilon\to 0}\bar{t}\alpha\mathrm{Tr}[\eta_0\big((1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\big)\\
&\qquad\times[\![\Omega']\!]_\epsilon(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\Omega_\epsilon\partial\gamma\Omega_\epsilon Bc]\\
&= -\lim_{\epsilon\to 0}\bar{t}\alpha^4\mathrm{Tr}[[\![F_2]\!]_\epsilon(\eta_0 V)[\![F_2]\!]_\epsilon K_\epsilon(1-K_\epsilon)[\![F_2]\!]_\epsilon V(1-K_\epsilon)[\![F_2]\!]_\epsilon V[\![F_2]\!]_\epsilon\partial\gamma Bc]\\
&\quad +\lim_{\epsilon\to 0}\bar{t}\alpha^4\mathrm{Tr}[[\![F_2]\!]_\epsilon V(1-K_\epsilon)[\![F_2]\!]_\epsilon V[\![F_2]\!]_\epsilon K_\epsilon(1-K_\epsilon)[\![F_2]\!]_\epsilon(\eta_0 V)[\![F_2]\!]_\epsilon\partial\gamma Bc]\\
&= -\lim_{\epsilon\to 0}\frac{\bar{t}\alpha^4}{2^3}\mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon{}^2K_\epsilon(1-K_\epsilon)\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon\\
&\qquad\times\partial c\gamma^{-1}[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c] \qquad \text{(G.15)}\\
&\quad +\lim_{\epsilon\to 0}\frac{\bar{t}\alpha^4}{2^3}\mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c\gamma^{-1}(1-K_\epsilon)[\![F_2]\!]_\epsilon\\
&\qquad\times\partial c\gamma^{-1}[\![F_2]\!]_\epsilon{}^2K_\epsilon(1-K_\epsilon)\partial c]. \qquad \text{(G.16)}
\end{aligned}
$$

Then (G.15) and (G.16) become

$$
\begin{aligned}
&\int_0^1 dt\,(\text{G.15})|_{q=\frac{1}{2}}\\
&= -\frac{1}{2^7}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^4\mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon{}^2K_\epsilon(1-K_\epsilon)\partial c\gamma^{-1}\\
&\qquad\qquad\times(1-K_\epsilon)[\![F_2]\!]_\epsilon\partial c\gamma^{-1}[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c]
\end{aligned}
$$

$$= -\frac{1}{2^7}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^4\cdot \mathrm{eiiid}[n_1+n_2+y_1+u_1, u_2+n_3, n_4, n_5]'$$
$$\times\,\mathrm{Bcddd}[n_1+n_2+y_1+u_1, u_2+n_3, n_4+n_5]'$$
$$\times\,\frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)i\sigma_2\sigma_3\sigma_3\sigma_3], \tag{G.17}$$

and

$$\int_0^1 dt\,(\mathrm{G.16})|_{q=\frac{1}{2}}$$
$$= \frac{1}{2^7}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^4\mathrm{Tr}[(\eta_0\gamma^{-1})[\![F_2]\!]_\epsilon\partial\gamma[\![F_2]\!]_\epsilon Bc\partial c\gamma^{-1}$$
$$\times(1-K_\epsilon)[\![F_2]\!]_\epsilon\partial c\gamma^{-1}[\![F_2]\!]_\epsilon{}^2K_\epsilon(1-K_\epsilon)\partial c]$$
$$= \frac{1}{2^7}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^4\cdot \mathrm{eidii}[n_1, n_2, u_1+n_3, n_4+n_5+y_1+u_2]'$$
$$\times\,\mathrm{Bcddd}[u_1+n_3, n_4+n_5+y_1+u_2, n_1+n_2]'$$
$$\times\,\frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)i\sigma_2\sigma_3\sigma_3\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3]. \tag{G.18}$$

Here, we define

$$\mathrm{eigii}[t_1,t_2,t_3,t_4]$$
$$:= \Big\langle \oint_0 \frac{dz}{2\pi i}\eta(z)\xi e^{-\phi}(0)e^{\phi}\eta(t_1)\xi e^{-\phi}(t_1+t_2)\xi e^{-\phi}(t_1+t_2+t_3)\Big\rangle^{\xi\eta\phi}_{C_{t_1+t_2+t_3+t_4}}$$
$$= -\frac{\pi}{L}\frac{\sin\theta_{t_1}}{\sin\theta_{t_1+t_2}\sin\theta_{t_1+t_2+t_3}}, \tag{G.19}$$

$$\mathrm{eidii}[t_1,t_2,t_3,t_4] := \lim_{y\to 0}\partial_y\{\mathrm{eiiig}[t_1,t_2,t_3+y,t_4]-\mathrm{eiiig}[t_1,t_2,t_3,t_4+y]\}. \tag{G.20}$$

The sixth term (7.2.17) becomes

$$(7.2.17) = \lim_{\epsilon\to 0}\bar{t}\mathrm{Tr}\left[\left[\!\!\left[\left(\eta_0\frac{1}{\det_2}\right)\Omega'\frac{1}{\det_2}B\Omega'c\partial c\Omega'\right]\!\!\right]_\epsilon\right]$$
$$= \lim_{\epsilon\to 0}\bar{t}\mathrm{Tr}[\eta_0\big((1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon\big)$$
$$\times[\![\Omega']\!]_\epsilon(1-K_\epsilon)[\![F_2]\!]_\epsilon\alpha V(1-K_\epsilon)[\![F_2]\!]_\epsilon B[\![\Omega']\!]_\epsilon c\partial c[\![\Omega']\!]_\epsilon]$$
$$= \lim_{\epsilon\to 0}\bar{t}\alpha^2\mathrm{Tr}[[\![F_2]\!]_\epsilon(\eta_0V)(1-K_\epsilon)[\![F_2]\!]_\epsilon^2K_\epsilon V[\![F_2]\!]_\epsilon BK_\epsilon c\partial cK_\epsilon]$$
$$= \lim_{\epsilon\to 0}\frac{\bar{t}\alpha^2}{2^2}\mathrm{Tr}[(\eta_0\gamma^{-1})(1-K_\epsilon)[\![F_2]\!]_\epsilon^2K_\epsilon\partial c\gamma^{-1}[\![F_2]\!]_\epsilon K_\epsilon Bc\partial cK_\epsilon[\![F_2]\!]_\epsilon\partial c]. \tag{G.21}$$

Then, (G.21) becomes

$$\int_0^1 dt\,(\mathrm{G.21})|_{q=\frac{1}{2}}$$
$$= \frac{1}{2^4}\lim_{\epsilon\to 0}\int_0^1 dt\,\bar{t}t^2\mathrm{Tr}[(\eta_0\gamma^{-1})(1-K_\epsilon)[\![F_2]\!]_\epsilon^2K_\epsilon\partial c\gamma^{-1}[\![F_2]\!]_\epsilon K_\epsilon Bc\partial cK_\epsilon[\![F_2]\!]_\epsilon\partial c]$$

$$= \frac{1}{2^4} \lim_{\epsilon\to 0} \int_0^1 dt\, \bar{t}t^2 \cdot \mathrm{eii}[u_1+n_1+n_2+y_1, n_3+y_2+y_3+n_4]'$$
$$\cdot \mathrm{Bcddd}[y_3+n_4, u_1+n_1+n_2+y_1, n_3+y_2]'$$
$$\times \frac{1}{2}\mathrm{Tr}[\sigma_3(-i\sigma_2)\sigma_3(-i\sigma_2)\sigma_3\sigma_3\sigma_3\sigma_3]. \tag{G.22}$$

Then, we have the expression for the energy of the solution g_2 with $q=1/2$:

$$\lim_{\epsilon\to 0} [\![E(g_2)]\!]_\epsilon = \lim_{\epsilon\to 0} \int_0^1 dt\, \mathrm{Tr}[\eta_0 \big([\![g_2(t)^{-1}\partial_t g_2(t)]\!]_\epsilon \big) [\![g_2(t)^{-1} Q g_2(t)]\!]_\epsilon]$$
$$= (7.2.27) + (\mathrm{G}.8) + (\mathrm{G}.11) + (\mathrm{G}.12)$$
$$+ (\mathrm{G}.14) + (\mathrm{G}.17) + (\mathrm{G}.18) + (\mathrm{G}.22). \tag{G.23}$$

If we complete this calculation, we obtain the energy of the solution of the solution g_2.

Let us derive the correlators in Berkovits' SFT. First, we note the normalization in this thesis:

$$\langle\, \xi c\partial c \partial^2 c e^{-2\phi} \,\rangle^{\mathrm{gh}}_{\mathrm{UHP}} := 2, \tag{G.24}$$

or

$$\langle\, \xi \,\rangle^{\xi\eta}_{\mathrm{UHP}} \langle\, e^{-2\phi} \,\rangle^{\phi}_{\mathrm{UHP}} := -1, \quad \langle\, c\partial c \partial^2 c \,\rangle^{bc}_{\mathrm{UHP}} := -2. \tag{G.25}$$

Let us derive the following correlator:

$$\mathrm{eii}[t_1, t_2] = \langle \oint_0 \frac{dz}{2\pi i} \eta(z) \xi e^{-\phi}(0) \xi e^{-\phi}(t_1) \,\rangle^{\xi\eta\phi}_{\mathrm{C}_L}, \tag{G.26}$$

from $\mathrm{Tr}[\eta_0 \gamma^{-1}\Omega^{t_1}\gamma^{-1}\Omega^{t_2}]$;

$$\langle \oint_0 \frac{dz}{2\pi i} \eta(z) \xi e^{-\phi}(0) \xi e^{-\phi}(t_1) \,\rangle^{\xi\eta\phi}_{\mathrm{C}_L}$$
$$= -\langle\, \xi(t_1) \rangle^{\xi\eta}_{\mathrm{UHP}} \langle e^{-\phi}(0) e^{-\phi}(t_1) \,\rangle^{\phi}_{\mathrm{UHP}}$$
$$= -\langle\, f_{\mathrm{s}}^{-1} \circ f_{L\to 2} \circ \xi(t_1) \,\rangle^{\xi\eta}_{\mathrm{UHP}} \langle\, f_{\mathrm{s}}^{-1} \circ f_{L\to 2} \circ e^{-\phi}(0) f_{\mathrm{s}}^{-1} \circ f_{L\to 2} \circ e^{-\phi}(t_1) \,\rangle^{\phi}_{\mathrm{UHP}}$$
$$= -\langle\, \xi(\tan\theta_{t_1}) \,\rangle^{\xi\eta}_{\mathrm{UHP}} \left(\frac{\pi}{L}\right)^{\frac{1}{2}\cdot 2} \left(\frac{1}{\cos^2\theta_{t_1}}\right)^{\frac{1}{2}} \langle\, e^{-\phi}(0) e^{-\phi}(\tan\theta_{t_1}) \,\rangle^{\phi}_{\mathrm{UHP}}$$
$$= -\frac{\pi}{L} \langle\, \xi \,\rangle^{\xi\eta}_{\mathrm{UHP}} \langle e^{-2\phi} \,\rangle^{\phi}_{\mathrm{UHP}} \frac{1}{\cos\theta_{t_1}} (0 - \tan\theta_{t_1})^{-1}$$
$$= -\frac{\pi}{L} \frac{1}{\sin\theta_{t_1}}. \tag{G.27}$$

Here, $h(\xi) = 0$, $h(e^{-\phi}) = 1/2$ and $\phi(z)\phi(0) \sim -\ln z$. Similarly, we have the following correlator:

$$\mathrm{eiiig}[t_1, t_2, t_3, t_4] = \langle \oint_0 \frac{dz}{2\pi i} \eta(z) \xi e^{-\phi}(0) \xi e^{-\phi}(t_1) \xi e^{-\phi}(t_1+t_2) e^{\phi} \eta(t_1+t_2+t_3) \,\rangle^{\xi\eta\phi}_{\mathrm{UHP}}, \tag{G.28}$$

from $\mathrm{Tr}[\eta_0\gamma^{-1}\Omega^{t_1}\gamma^{-1}\Omega^{t_2}\gamma^{-1}\Omega^{t_3}\gamma\Omega^{t_4}]$;

$$
\begin{aligned}
&\Big\langle \oint_0 \frac{dz}{2\pi i}\eta(z)\xi e^{-\phi}(0)\xi e^{-\phi}(t_1)\xi e^{-\phi}(t_1+t_2)e^{\phi}\eta(t_1+t_2+t_3)\Big\rangle^{\xi\eta\phi}_{\mathrm{UHP}} \\
&= -\frac{\pi}{L}\left(\frac{1}{\cos^2\theta_{t_1}}\right)^{\frac{1}{2}}\left(\frac{1}{\cos^2\theta_{t_1+t_2}}\right)^{\frac{1}{2}}\left(\frac{1}{\cos^2\theta_{t_1+t_2+t_3}}\right)^{-\frac{1}{2}} \\
&\quad\times\Big\langle e^{-\chi}(\tan\theta_{t_1})e^{-\chi}(\tan\theta_{t_1+t_2})e^{-\chi}(\tan\theta_{t_1+t_2+t_3})\Big\rangle^{\xi}_{\mathrm{UHP}} \\
&\quad\times\Big\langle e^{-\phi}(0)e^{-\phi}(\tan\theta_{t_1})e^{-\phi}(\tan\theta_{t_1+t_2})e^{\phi}(\tan\theta_{t_1+t_2+t_3})\Big\rangle^{\phi}_{\mathrm{UHP}} \\
&= \frac{\pi}{L}\frac{\cos\theta_{t_1+t_2+t_3}}{\cos\theta_{t_1}\cos\theta_{t_1+t_2}}(0-\tan\theta_{t_1})^{-1}(0-\tan\theta_{t_1+t_2})^{-1}(0-\tan\theta_{t_1+t_2+t_3}) \\
&= -\frac{\pi}{L}\frac{\sin\theta_{t_1+t_2+t_3}}{\sin\theta_{t_1}\sin\theta_{t_1+t_2}}. \qquad (G.29)
\end{aligned}
$$

Here, $h(\eta)=1$, $h(e^{\phi})=-3/2$, $\chi(z)\chi(0)\sim\ln z$. By performing the same procedure, we have

$$
\mathrm{eigii}[t_1,t_2,t_3,t_4] = -\frac{\pi}{L}\frac{\sin\theta_{t_1}}{\sin\theta_{t_1+t_2}\sin\theta_{t_1+t_2+t_3}}. \qquad (G.30)
$$

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