Twistor formulation of a massive spinning particle

(スピンをもつ有質量粒子のツイスター形式)

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Chapter 1

Introduction

Twistor theory was first proposed by Penrose in 1967 as a novel approach to finding a unified framework for general relativity and quantum physics, aiming at establishing a theory of quantum gravity [1]. In twistor theory [2, 3, 4, 5, 6, 7, 11], a complex space called twistor space is considered to be a primary object for expressing physics, while 4-dimensional space-time is treated as a secondary object. One of the common motivations in early studies on twistor theory is thus to describe 4-dimensional space-time, gravity and even the elementary particles in an equal footing on the basis of the complex geometry of twistor space. Such an ambitious attempt in twistor theory has been summarized by Penrose himself as the twistor programme [6, 7].

Twistor theory is basically appropriate for describing massless systems with conformal symmetry [2, 3, 4]. Nevertheless, there have been some approaches to formulating massive particle systems in terms of twistors [6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. For describing a massive particle, it is common to use two or more independent twistors. In fact, introducing two twistors has been considered until recently [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], and introducing more than two twistors was considered in some earlier studies [6, 8, 9, 10, 11]. By virtue of using two or more independent twistors, an extra symmetry between the twistors occurs naturally in the system. Penrose, Perjés, and Hughston proposed the idea of identifying this symmetry with an internal symmetry in particle physics, such as the symmetry for leptons or that for hadrons, toward explaining internal symmetries of elementary particles on the basis of twistor theory [6, 8, 9, 10, 11]. Although this idea is quite interesting, it seems that its detailed investigations have been made from neither a mechanical point of view nor a dynamical point of view.

Therefore we would have to say that the idea is still poorly understood.

Long after Penrose, Perjés, and Hughston proposed their own idea, Lagrangian mechanics of a massive spinning particle formulated in terms of two twistors has been studied in Refs. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Most of these papers begin with generalization of the Shirafuji action that describes a free massless spinning particle in four dimensions in terms of a twistor [23]. In fact, various generalizations of the Shirafuji action have been presented to specify twistorial models of massive spinning particles. The generalized Shirafuji actions are constructed by incorporating a mass-shell condition of a particle and certain other conditions for the twistor variables. The canonical formalism based on each generalized Shirafuji action and its subsequent quantization were also studied in Refs. [12, 14, 15, 17, 18, 19, 20]. It was shown that the canonical quantization of each twistorial model leads to generalized Dirac equations or the Dirac-Fierz-Pauli (DFP) equations for massive spinor fields of arbitrary rank [24, 25, 26].

In the present thesis, we first prove that the $n(\geq 3)$-twistor expression of a particle's four-momentum vector enables us to describe only a massless particle. Therefore the n-twistor description of a massive particle is not valid for the case $n \geq 3$. Taking into account this fact, we consider a generalization of the Shirafuji action to define a new twistor model of a free massive spinning particle in four dimensions by using two twistors. Our formulation is precisely a non-Abelian extension of the gauged twistor formulation of a free massless spinning particle in four dimensions [27, 28, 29]. In the gauged twistor formulation, the Shirafuji action is modified in accordance with the gauge principle so that it can become invariant under the local $U(1)$ (phase) transformation of twistor variables. Here "local" means that the transformation parameter depends on a worldline parameter along the particle's worldline. This modification is accomplished by gauging the Shirafuji action with the aid of a $U(1)$ gauge field on the one-dimensional (1D) parameter space of the worldline and by adding the 1D Chern-Simons term consisting of the $U(1)$ gauge field. The modified action, named the *gauged* Shirafuju action, includes a helicity constraint term due to the modification. Hence it follows that this action describes a free massless spinning particle with a fixed value of helicity. The Shirafuji action can furthermore be modified so as to be invariant under the local scale transformation of twistor variables with the aid of another gauge field on the 1D parameter space. From the point of view of twistor theory, it is desirable that the modified action remains invariant under the combination of the local $U(1)$

and local scale transformations, which is referred to in Refs. [28, 29, 30] as the complexified local scale transformation. In actuality, the gauge field for the local scale transformation can be gauged away by a scaling of the twistor variables. Therefore it turns out that only the local $U(1)$ transformation is essential and one does not need to consider the local scale transformation in practice.

As will be seen later, we set up a generalized Shirafuji action that consists of two twistors and involves a mass-shell condition. Here, for convenience, we exploit the mass-shell condition with a complexified mass parameter introduced in Refs. [18, 19]. The generalized Shirafuji action remains invariant under the global $U(1)$ transformation of twistor variables supplemented with that of auxiliary fields on the 1D parameter space. In addition, the generalized Shirafuji action remains invariant under the global $SU(2)$ transformation defined for a doublet of twistors. In accordance with the gauge principle, we modify the generalized Shirafuji action in such a way that the modified action remains invariant under the local $U(1)$ and $SU(2)$ transformations of twistor variables. The modification is performed by gauging the generalized Shirafuji action with the aid of $U(1)$ and $SU(2)$ gauge fields on the 1D parameter space and by adding the 1D $U(1)$ and $SU(2)$ Chern-Simons terms. The 1D $SU(2)$ Chern-Simons term, however, vanishes owing to the traceless property of the $SU(2)$ gauge field. For this reason, the variation of the modified action with respect to the $SU(2)$ gauge field yields too strong constraints that, after quantizing the model, permit us to have only massive spinless fields in four dimensions. A similar consequence has been found by Fedoruk and Lukierski in their twistorial model of a massive particle [18]. To overcome such an undesirable situation, they modified the model by incorporating the Souriau-Wess-Zumino term, following the successful argument for a twistorial model of a massive spinning particle in three dimensions [17]. In the present thesis, we consider an alternative approach based on a nonlinear realization of $SU(2)$ to eventually obtain massive spinor fields of arbitrary rank. This approach makes it possible to define the 1D $U(1)$ Chern-Simons term consisting of the third (or diagonal) component of the $SU(2)$ gauge field in a particular gauge. In addition, this approach can provide a novel gauge-invariant term consisting of the first and second (or off-diagonal) components of the same $SU(2)$ gauge field. With the new terms, we furthermore modify the generalized Shirafuji action by adding these terms to the modified action mentioned above. The completely modified action is thus the sum of the gauged twistorial part, the two 1D $U(1)$ Chern-Simons terms, and the novel term.

This action, hereafter referred to as the *gauged* generalized Shirafuji (GGS) action, remains invariant under reparametrization of the worldline parameter and under the local $U(1)$ and $SU(2)$ transformations. The GGS action yields just sufficient constraints for the twistor variables in a systematic and consistent manner. All the constraints except for the mass-shell condition are derived on the basis of the gauge symmetry. This is an advantage of our gauged twistor model.

Having obtained the GGS action, we study the canonical Hamiltonian formalism based on it by completely following the Dirac algorithm for Hamiltonian systems with constraints [36, 37, 38]. In the present thesis, the canonical Hamiltonian formalism is investigated in two different ways. One of these ways treats the twistor variables as fundamental dynamical variables. Another way adopts the space-time and spinor variables as fundamental variables, after being decomposed the twistor variables into the space-time and spinor variables. In this approach, the mass-shell condition included in the GGS action is slightly modified so as to have a real mass parameter. We can expect that this approach clarifies relations between the twistor and ordinary space-time formulations of a massive spinning particle and makes it possible to consider coupling to external fields. In both the twistor and spinor formulations, some of the first-class constraints eventually turn into simultaneous differential equations for a function of half the twistor of spinor variables. Each solution of the simultaneous differential equations is characterized by the three quantum numbers that originate from the $U(1)$ and $SU(2)$ symmetries inherent in the GGS action.

In the twistor formulation, we consider the Penrose transform of the twistor function to define a four-dimensional spinor field of arbitrary rank. The spinor field defined in this manner has extra upper and lower $SU(2)$ indices in addition to dotted and undotted spinor indices. Because of the structure of the Penrose transform, the number of upper (lower) $SU(2)$ indices is equal to the number of undotted (dotted) spinor indices. We demonstrate that the present spinor field satisfies generalized DFP equations with $SU(2)$ indices. In the simplest case, the generalized DFP equations reduce to the ordinary Dirac equations for particle and antiparticle spinor fields. Investigating properties of these fields, we clarify the physical meanings of the $U(1)$ and $SU(2)$ symmetries; ultimately, we see that the $U(1)$ symmetry is a gauge symmetry concerning the chiralities of the particle and antiparticle spinor fields, while the $SU(2)$ symmetry is a gauge symmetry realized in a doublet consisting of the particle and antiparticle spinor fields. Therefore it turns out that the idea proposed by Penrose, Perjés, and Hughston, in which the $SU(2)$ symmetry is identified with the weak isospin symmetry, is not valid in our gauged twistor formulation.

The present thesis is organized as follows. In Chapter 2, we briefly explain the twistor description of a massive particle and prove a related no-go theorem. In Chapter 3, we elaborate the GGS action, after making some preliminary arrangements. In Chapter 4, the canonical formalism based on the GGS action is considered within the framework of the twistor formulation. Then the subsequent canonical quantization is performed. We here define a massive spinor field of arbitrary rank by the Penrose transform of a twistor function and demonstrate that this spinor field satisfies the generalized DFP equations. Furthermore, we particularly investigate the rank-one spinor fields to clarify the physical meaning of the $U(1)$ and $SU(2)$ symmetries. In Chapter 5, we rewrite the GGS action in terms of the space-time and spinor variables. In this process, an alternative form of the mass-shell condition is adopted instead of the one used in earlier chapters. The canonical formalism based on the GGS action is considered within the framework of the spinor formulation, and the subsequent canonical quantization is performed in the usual manner. From solutions of the simultaneous differential equations obtained in the quantization, we define positive- and negative-frequency spinor fields of arbitrary rank satisfying the generalized DFP equations. Also, we express the spinor fields in the form of the Penrose transforms. Furthermore, we define the exponential generating function for the spinor fields and derive a novel representation for each of the spinor fields. The physical meaning of the $U(1)$ and $SU(2)$ symmetries is clarified again. Chapter 6 is devoted to a summary and discussion. In Appendix A, we give a theorem useful for proving the no-go theorem in Chapter 2. In Appendix B, we treat the Poincaré symmetry and the Pauli-Lubanski pseudovector written in terms of the twistor variables.

Chapter 2

A free massive particle in twistor theory

In this chapter, we first briefly review the twistor description of a free massive particle and then we prove a related no-go theorem.

2.1 The n-twistor description of a free massive particle

To describe a massive particle in four dimensions, Penrose, Perjés, and Hughston introduced two or more [i.e., $n(\in \mathbb{N}+1)$] independent twistors and their dual twistors

$$Z_i^A = (\omega_i^\alpha, \pi_{i\dot{\alpha}})\,, \qquad \bar{Z}_A^i = (\bar{\pi}_\alpha^i, \bar{\omega}^{i\dot{\alpha}}) \tag{2.1.1}$$

($A = 0,1,2,3;\, \alpha = 0,1;\, \dot{\alpha} = \dot{0},\dot{1}$) distinguished by the index i ($i = 1,2,\ldots n$). Here, $\bar{\pi}_\alpha^i$ and $\bar{\omega}^{i\dot{\alpha}}$ denote the complex conjugates of the two-component spinors $\pi_{i\dot{\alpha}}$ and ω_i^α, respectively: $\bar{\pi}_\alpha^i := \overline{\pi_{i\dot{\alpha}}}$, $\bar{\omega}^{i\dot{\alpha}} := \overline{\omega_i^\alpha}$. The spinors ω_i^α and $\pi_{i\dot{\alpha}}$ are related by

$$\omega_i^\alpha = iz^{\alpha\dot{\alpha}}\pi_{i\dot{\alpha}}\,, \tag{2.1.2}$$

where $z^{\alpha\dot{\alpha}}$ are coordinates of a point in complexified Minkowski space.

As can be seen in Refs. [6, 8, 9, 10, 11], the N-twistor expression of four-momentum is given by

$$p_{\alpha\dot{\alpha}} = \sum_{i=1}^{N} \bar{\pi}_\alpha^i \pi_{i\dot{\alpha}} \equiv \bar{\pi}_\alpha^i \pi_{i\dot{\alpha}}\,. \tag{2.1.3}$$

Then the squared mass $m^2 = p_{\alpha\dot{\alpha}} p^{\alpha\dot{\alpha}}$ can be written as

$$m^2 = \bar{\pi}^i_{\alpha} \pi_{i\dot{\alpha}} \bar{\pi}^{j\alpha} \pi^{\dot{\alpha}}_j, \tag{2.1.4}$$

where $\bar{\pi}^{i\alpha} := \epsilon^{\alpha\beta} \bar{\pi}^i_{\beta}$ and $\pi^{\dot{\alpha}}_i := \epsilon^{\dot{\alpha}\dot{\beta}} \pi_{i\dot{\beta}}$. In Lagrangian mechanics of a massive spinning particle, Eq. (2.1.4) with $n = 2$, or its equivalent expression, is incorporated into a generalization of the Shirafuji action [23] with the aid of a Lagrange multiplier.

2.2 A No-Go theorem for the twistor description

In this section, we present the following theorem:

Theorem: In the case $n \geq 3$, the four-momentum defined by Eq. (2.1.3) satisfies the null-vector condition $p_{\alpha\dot{\alpha}} p^{\alpha\dot{\alpha}} = 0$, so that $m = 0$.

Hence the $n(\geq 3)$-twistor system turns out to be a massless system. The purpose of the present thesis is to prove this theorem. The theorem leads to the fact that in actuality, the n-twistor description of a massive particle is not valid for the case $n \geq 3$. For this reason, the above-mentioned idea for the $SU(3)$ [or $ISU(3)$] symmetry cannot be accepted. In this sense, the theorem given here can be said to be a no-go theorem. Also, the theorem justifies the fact that only the two-twistor description (i.e., the case $n = 2$) has been considered in Lagrangian mechanics of a massive spinning particle formulated in terms of twistors.

To prove the theorem, it is necessary to provide the following lemma.

Lemma: Let A be an arbitrary $n \times n$ complex antisymmetric matrix. Then A can be transformed into its normal form, $\tilde{A}$, according to

$$\tilde{A} = U A U^{\mathrm{T}}, \tag{2.2.1}$$

where U is an $n \times n$ unitary matrix. If n is even, then the normal form $\tilde{A}$ is given by

$$\tilde{A} = \begin{pmatrix} 0 & \sqrt{a_1} & 0 & 0 & \cdots & 0 & 0 \\ -\sqrt{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{a_2} & \cdots & 0 & 0 \\ 0 & 0 & -\sqrt{a_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{a_{n/2}} \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{a_{n/2}} & 0 \end{pmatrix}, \qquad (2.2.2)$$

and if n is odd, then $\tilde{A}$ is given by

$$\tilde{A} = \begin{pmatrix} 0 & \sqrt{a_1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\sqrt{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{a_2} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{a_2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{a_{(n-1)/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{a_{(n-1)/2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (2.2.3)$$

Here, $a_1, a_2, \ldots, a_{n/2}$ [or $a_{(n-1)/2}$] are eigenvalues of the Hermitian matrix $AA^\dagger$, and hence it follows that these eigenvalues are non-negative real numbers.

In this thesis, we do not give a proof of this lemma, because it can be seen in Refs. [53, 54, 55, 56].

Proof of the theorem: Hereafter, we treat the case $n \geq 3$. In order to prove the theorem, let us consider the $n \times n$ complex matrix $\mathit{\Pi}$ consisting of elements

$$\mathit{\Pi}_{ij} := \pi_{i\dot{\alpha}} \pi_j^{\dot{\alpha}}. \qquad (2.2.4)$$

Because $\pi_{i\dot{\alpha}} \pi_j^{\dot{\alpha}} = -\pi_i^{\dot{\alpha}} \pi_{j\dot{\alpha}}$ holds, $\mathit{\Pi}$ turns out to be antisymmetric. According to the lemma, the matrix $\mathit{\Pi}$ can be transformed into its normal form

$$\tilde{\mathit{\Pi}} = U \mathit{\Pi} U^{\mathrm{T}} \qquad (2.2.5)$$

by means of an appropriate $n \times n$ unitary matrix $U = (U_i{}^j)$. Expressions corresponding to Eqs. (3.1.2) and (3.1.3) are concisely given by

$$\tilde{\Pi}_{2n-1,j} = \delta_{2n,j}\tilde{\Pi}_{2n-1,2n}\,, \tag{2.2.6a}$$

$$\tilde{\Pi}_{2n,j} = \delta_{2n,j+1}\tilde{\Pi}_{2n,2n-1}\,, \tag{2.2.6b}$$

$$\begin{cases} n = 1, 2, \ldots, n/2\,, & \text{for } n \text{ even}\,, \\ n = 1, 2, \ldots, (n+1)/2\,, & \text{for } n \text{ odd}\,, \end{cases}$$

where $\tilde{\Pi}_{2n-1,2n}$ is the square root of an eigenvalue of $\Pi\Pi^{\dagger}$, being a non-negative real number.

Substituting Eq. (2.4) into Eq. (2.5), we can express the elements of $\tilde{\Pi}$ as

$$\tilde{\Pi}_{ij} = \tilde{\pi}_{i\dot{\alpha}}\tilde{\pi}_{j}^{\dot{\alpha}}, \tag{2.2.7}$$

with the two-component spinor

$$\tilde{\pi}_{i\dot{\alpha}} := U_{i}^{\ j}\pi_{j\dot{\alpha}}\,. \tag{2.2.8}$$

From Eqs. (3.1.6a) and (3.2.1), we see $\tilde{\pi}_{1\dot{\alpha}}\tilde{\pi}_{k}^{\dot{\alpha}} = 0$ ($k = 3, 4, \ldots, n$). This implies that $\tilde{\pi}_{k\dot{\alpha}}$ is proportional to $\tilde{\pi}_{1\dot{\alpha}}$, i.e.,

$$\tilde{\pi}_{k\dot{\alpha}} = \rho_{k1}\tilde{\pi}_{1\dot{\alpha}}\,, \qquad \rho_{k1} \in \mathbb{C}\,. \tag{2.2.9}$$

Here, it is assumed that the proportional constants ρ_{k1} do not vanish simultaneously, because we now treat the case $n \geq 3$. Substituting Eq. (3.2.3) into Eq. (3.2.1) and noting the property $\pi_{i\dot{\alpha}}\pi_{i}^{\dot{\alpha}} = 0$ (no sum with respect to i), we obtain

$$\tilde{\Pi}_{kl} = \tilde{\pi}_{k\dot{\alpha}}\tilde{\pi}_{l}^{\dot{\alpha}} = \rho_{k1}\rho_{l1}\tilde{\pi}_{1\dot{\alpha}}\tilde{\pi}_{1}^{\dot{\alpha}} = 0\,, \qquad k, l = 3, 4, \ldots, n\,. \tag{2.2.10}$$

By using Eq. (3.2.3), $\tilde{\Pi}_{2k} = \tilde{\pi}_{2\dot{\alpha}}\tilde{\pi}_{k}^{\dot{\alpha}}$ can be written as

$$\tilde{\Pi}_{2k} = \rho_{k1}\tilde{\pi}_{2\dot{\alpha}}\tilde{\pi}_{1}^{\dot{\alpha}} = \rho_{k1}\tilde{\Pi}_{21}\,. \tag{2.2.11}$$

Equations (3.1.6b) and (3.2.5) give $\rho_{k1}\tilde{\Pi}_{21} = 0$ for any k. Since the constants ρ_{k1} do not vanish simultaneously, it follows that $\tilde{\Pi}_{21} = 0$. Combining this, Eq. (3.2.4), and the $\tilde{\Pi}_{1k} = \tilde{\Pi}_{2k} = 0$ included in Eq. (3.1.6) together, we eventually have $\tilde{\Pi}_{ij} = 0$. As a result, $\tilde{\Pi}$ turns out to be the $n \times n$ zero matrix. Then the use of Eq. (3.1.5) immediately leads to $\Pi = 0$, or equivalently, $\Pi_{ij} = 0$, because U is unitary and hence invertible. With this result, the squared mass $m^2 = p_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}}$ can be evaluated as follows:

$$m^2 = p_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}} = \bar{\pi}_{\alpha}^{i}\bar{\pi}^{j\alpha}\pi_{i\dot{\alpha}}\pi_{j}^{\dot{\alpha}} = \bar{\Pi}^{ij}\Pi_{ij} = 0\,. \tag{2.2.12}$$

Thus, the null-vector condition $p_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}} = 0$ is found, so that the proof of the theorem is complete. ■

Because $\Pi_{ij} = \pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} = 0$, the spinors $\pi_{i\dot{\alpha}}$ are proportional to each other. Then, using Eq. (2.1.2), it can be shown that the $n(\geq 3)$ twistors Z_i^A are proportional to each other in actuality. This fact implies that all the twistors Z_i^A correspond to a single projective twistor defined as the proportionality class $[Z_1^A] := \{\rho Z_1^A \,|\, \rho \in \mathbb{C} \setminus \{0\}\}$. Therefore it turns out that the present system is essentially described by $[Z_1^A]$. As is well known in twistor theory, a projective twistor precisely specifies the configuration of a massless particle. From this fact, we see once again that the $n(\geq 3)$-twistor system is a massless system.

Since the $n(\geq 3)$-twistor system is a massless system, the associated $SU(n)$ [or $ISU(n)$] symmetry cannot be identified with the internal symmetry of a massive physical system consisting of e.g. hadorns. For this reason, the idea proposed by Penrose, Perjés, and Hughston fails in the case $n \geq 3$. Of course, there still remains a possibility that the $SU(n)$ [or $ISU(n)$] symmetry will be identified with the internal symmetry of a massless system.

Chapter 3

Gauged twistor formulation of a massive spinning particle

In this chapter, we construct a gauged twistor model. We begin with setting up a generalized Shirafuji action that consists of two twistors and involves a mass-shell condition. For convenience, we exploit the mass-shell condition with a complexified mass parameter. The generalized Shirafuji action remains invariant under the global $U(1)$ and $SU(2)$ transformations of twistor variables. In accordance with the gauge principle, we modify the generalized Shirafuji action in such a way that the modified action remains invariant under the local $U(1)$ and $SU(2)$ transformations of twistor variables. The modification is performed by gauging the generalized Shirafuji action with the aid of $U(1)$ and $SU(2)$ gauge fields on the 1D parameter space and by adding the 1D $U(1)$ Chern-Simons terms. However, this modified action governs only massive spinless fields, owing to a fact that the $SU(2)$ gauge field yields too strong constraints for the twistor variables. Thus we consider a further modification of the action based on a nonlinear realization of $SU(2)$ to eventually obtain massive spinor fields of arbitrary rank. This approach makes it possible to define new gauge-invariant terms consisting of components of $SU(2)$ gauge field. With the new terms, we completely modify the generalized Shirafuji action by adding these terms to the modified action mentioned above. In this way, we obtain the GGS action, which yields just sufficient constraints for the twistor variables. The GGS action remains invariant under reparametrization of the worldline parameter and under the local $U(1)$ and $SU(2)$ transformations.

3.1 Generalization of the Shirafuji action to a massive particle

In this section, we construct the GGS action for a free massive spinning particle in four-dimensional Minkowski space.

In order to describe a massive particle in terms of twistors, we introduce two twistors $Z_i^A = (\omega_i^\alpha, \pi_{i\dot\alpha})$ $(A = 0, 1, 2, 3; \alpha = 0, 1; \dot\alpha = \dot 0, \dot 1)$ distinguished by the extra index i $(i = 1, 2)$ and their dual twistors $\bar Z_A^i = (\bar\pi_\alpha^i, \bar\omega^{i\dot\alpha})$. Here, $\bar\pi_\alpha^i$ and $\bar\omega^{i\dot\alpha}$ denote the complex conjugates of $\pi_{i\dot\alpha}$ and ω_i^α, respectively: $\bar\pi_\alpha^i := \overline{\pi_{i\dot\alpha}}$, $\bar\omega^{i\dot\alpha} := \overline{\omega_i^\alpha}$. It is assumed that Z_1^A and Z_2^A are not proportional to each other: $Z_1^A \neq c Z_2^A$ $(c \in \mathbb{C})$, so that $\bar Z_A^1 \neq \bar c \bar Z_A^2$. The 2-component spinors ω_i^α and $\pi_{i\dot\alpha}$ are related by

$$\omega_i^\alpha = i z^{\alpha\dot\alpha} \pi_{i\dot\alpha} , \tag{3.1.1}$$

where $z^{\alpha\dot\alpha}$ are coordinates of a point in complexified Minkowski space, $\mathbb{C}\mathbf{M}$, with the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. As can be seen in the literature on twistor theory [6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], the four-momentum of a massive particle is expressed as $p_{\alpha\dot\alpha} = \bar\pi_\alpha^1 \pi_{1\dot\alpha} + \bar\pi_\alpha^2 \pi_{2\dot\alpha} \equiv \bar\pi_\alpha^i \pi_{i\dot\alpha}$. (For this reason, $\pi_{i\dot\alpha}$ and $\bar\pi_\alpha^i$ are named as momentum spinors.) The squared norm of $p_{\alpha\dot\alpha}$ remains nonvanishing even after using the formula $\pi_{i\dot\alpha}\pi_i^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\pi_{i\dot\alpha}\pi_{i\dot\beta} = 0$ (no sum w.r.t. i) and its complex conjugate,[1] because the cross terms provided from different twistors still survive: $p_{\alpha\dot\alpha}p^{\alpha\dot\alpha} = \bar\pi_\alpha^i \pi_{i\dot\alpha} \bar\pi^{j\alpha} \pi_j^{\dot\alpha} = 2\big|\pi_{1\dot\alpha}\pi_2^{\dot\alpha}\big|^2$. Thus the mass-shell condition $p_{\alpha\dot\alpha}p^{\alpha\dot\alpha} = m^2$ with a mass parameter m can be written as

$$\bar\pi_\alpha^i \pi_{i\dot\alpha} \bar\pi^{j\alpha} \pi_j^{\dot\alpha} = m^2. \tag{3.1.2}$$

It is easy to see that this condition is equivalent to

$$\epsilon^{ij}\pi_{i\dot\alpha}\pi_j^{\dot\alpha} - \sqrt{2} m e^{i\varphi} = 0 , \tag{3.1.3a}$$

$$\epsilon_{ij}\bar\pi_\alpha^i \bar\pi^{j\alpha} - \sqrt{2} m e^{-i\varphi} = 0 , \tag{3.1.3b}$$

where φ is a real parameter. These equations have been incorporated in twistorial models of massive spinning particles [18, 19], in which $me^{i\varphi}/\sqrt{2}$ is called a com-

[1] The 2-dimensional Levi-Civita symbols $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$, $\epsilon_{\dot\alpha\dot\beta}$, ϵ^{ij}, and ϵ_{ij} are defined as $\epsilon^{01} = \epsilon_{01} = \epsilon^{\dot 0\dot 1} = \epsilon_{\dot 0\dot 1} = \epsilon^{12} = \epsilon_{12} = 1$ and conform to the rules $\overline{\epsilon^{\alpha\beta}} = \epsilon^{\dot\alpha\dot\beta}$, $\overline{\epsilon_{\alpha\beta}} = \epsilon_{\dot\alpha\dot\beta}$, $\overline{\epsilon^{ij}} = \epsilon_{ij}$, and $\overline{\epsilon_{ij}} = \epsilon^{ij}$. The contravariant spinors $\pi_i^{\dot\alpha}$ and $\bar\pi^{i\alpha}$ are defined by $\pi_i^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\pi_{i\dot\beta}$ and $\bar\pi^{i\alpha} = \epsilon^{\alpha\beta}\bar\pi_\beta^i$, respectively. These relations can be expressed as $\pi_{i\dot\alpha} = \pi_i^{\dot\beta}\epsilon_{\dot\beta\dot\alpha}$ and $\bar\pi_\alpha^i = \bar\pi^{i\beta}\epsilon_{\beta\alpha}$.

plexified mass parameter. In this thesis, we also adopt a pair of Eqs. (3.1.3a) and (3.1.3b) as the mass-shell condition because of the convenience for our formulation.

The Shirafuji action of a free massless spinning particle[2] can be generalized to describe a free spinning particle of mass m propagating in four-dimensional Minkowski space $\mathbf{M}$. A generalized Shirafuji action is indeed given by

$$S_m = \int_{\tau_0}^{\tau_1} d\tau \biggl[\frac{i}{2} \Bigl(\bar{Z}^i_A \dot{Z}^A_i - Z^A_i \dot{\bar{Z}}^i_A \Bigr) + h \Bigl(\epsilon^{ij} \pi_{i\dot{\alpha}} \pi^{\dot{\alpha}}_j - \sqrt{2} m e^{i\varphi} \Bigr) + \bar{h} \Bigl(\epsilon_{ij} \bar{\pi}^i_\alpha \bar{\pi}^{j\alpha} - \sqrt{2} m e^{-i\varphi} \Bigr) \biggr] , \tag{3.1.4}$$

where $Z^A_i = Z^A_i(\tau)$ and $\bar{Z}^i_A = \bar{Z}^i_A(\tau)$ are understood as complex scalar fields on the one-dimensional parameter space $\mathcal{T} := \{\tau \,|\, \tau_0 \leq \tau \leq \tau_1\}$ of a particle's world-line, and $h = h(\tau)$ is treated as a complex scalar-density field of weight 1 on $\mathcal{T}$. [That is, h transforms as $h(\tau) \to h'(\tau') = (d\tau/d\tau')h(\tau)$ under the proper reparametrization $\tau \to \tau' = \tau'(\tau)$ $(d\tau'/d\tau > 0)$.] The exponent φ is now considered a real scalar field on $\mathcal{T}$ and hence is treated as a real function $\varphi = \varphi(\tau)$. This setting is different from that in Refs. [18, 19], in which the complexified mass parameter is regarded as a constant. A dot over a variable denotes its derivative w.r.t. τ. The variation of S_m w.r.t. h and $\bar{h}$ yields the mass-shell condition (3.1.3).[3]

The generalized Shirafuji action S_m remains invariant under the reparametrization $\tau \to \tau' = \tau'(\tau)$. In addition, S_m remains invariant under the global $U(1)$ transformation

$$Z^A_i \to Z'^A_i = e^{i\theta} Z^A_i \,, \quad \bar{Z}^i_A \to \bar{Z}'^i_A = e^{-i\theta} \bar{Z}^i_A \,, \tag{3.1.5a}$$

$$h \to h' = e^{-2i\theta} h \,, \qquad \bar{h} \to \bar{h}' = e^{2i\theta} \bar{h} \,, \tag{3.1.5b}$$

$$\varphi \to \varphi' = \varphi + 2\theta \,, \tag{3.1.5c}$$

[2]With a twistor Z^A and its dual twistor $\bar{Z}_A$, the Shirafuji action is defined by [23]

$$S_0 = \int_{\tau_0}^{\tau_1} d\tau \frac{i}{2} \Bigl(\bar{Z}_A \dot{Z}^A - Z^A \dot{\bar{Z}}_A \Bigr) .$$

[3]Instead of the action S_m, we can consider an alternative action

$$S'_m = \int_{\tau_0}^{\tau_1} d\tau \biggl[\frac{i}{2} \Bigl(\bar{Z}^i_A \dot{Z}^A_i - Z^A_i \dot{\bar{Z}}^i_A \Bigr) + \frac{1}{2} f \bigl(\bar{\pi}^i_\alpha \pi_{i\dot{\alpha}} \bar{\pi}^{j\alpha} \pi^{\dot{\alpha}}_j - m^2 \bigr) \biggr] ,$$

where $f = f(\tau)$ is a real scalar-density field of weight 1 on $\mathcal{T}$. The variation of S'_m w.r.t. f yields the mass-shell condition (3.1.2).

with a real constant parameter θ and under the global $SU(2)$ transformation

$$Z_i^A \to Z_i'^A = U_i{}^j Z_j^A \,, \quad \bar{Z}_A^i \to \bar{Z}_A'^i = \bar{Z}_A^j U^\dagger{}_j{}^i \,, \tag{3.1.6a}$$

$$h \to h' = h \,, \qquad \bar{h} \to \bar{h}' = \bar{h} \,, \tag{3.1.6b}$$

$$\varphi \to \varphi' = \varphi \,, \tag{3.1.6c}$$

with a constant matrix U belonging to $SU(2)$. The $SU(2)$ invariance of S_m can be verified using $\epsilon^{ij} U_i{}^k U_j{}^l = \epsilon^{kl}$ and $\epsilon_{ij} U^\dagger{}_k{}^i U^\dagger{}_l{}^j = \epsilon_{kl}$ together with the unitarity property of U. We thus see that S_m possesses two global internal symmetries specified by $U(1)$ and $SU(2)$. We also see that the two terms $\bar{Z}_A^i \dot{Z}_i^A$ and $Z_i^A \dot{\bar{Z}}_A^i$ in Eq. (3.1.4) are invariant under the global $SU(2,2)$ transformation (or more simply, the global conformal transformation) $Z_i^A \to Z_i'^A = \mathcal{U}^A{}_B Z_i^B$, $\bar{Z}_A^i \to \bar{Z}_A'^i = \bar{Z}_B^i \mathcal{U}^{\dagger B}{}_A$, with a constant matrix $\mathcal{U}$ belonging to $SU(2,2)$. In contrast, the two terms $\epsilon^{ij} \pi_{i\dot{\alpha}} \pi_j^{\dot{\alpha}}$ and $\epsilon_{ij} \bar{\pi}_\alpha^i \bar{\pi}^{j\alpha}$ in Eq. (3.1.4) are invariant only under the global $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ transformation (or more simply, the global Poincaré transformation). Hence it turns out that the symmetry reduction from $SU(2,2)$ to $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ occurs in S_m as a result of adding the term proportional to h and its complex conjugate term.

3.2 Gauging the internal symmetries of the generalized Shirafuji action

Now, we perform a gauging of the global $U(1)$ and $SU(2)$ symmetries in such a way that the gauged action remains invariant under the local $U(1)$ and $SU(2)$ transformations that depend on τ. That is, we consider a $U(1) \times SU(2)$ gauge theory on the parameter space $\mathcal{T}$. To this end, in accordance with the gauge principle, we introduce a $U(1)$ gauge field, $a = a(\tau)$, and an $SU(2)$ gauge field, $b = b(\tau)$. The field a is assumed to be a real scalar-density field of weight 1 on $\mathcal{T}$, while b is assumed to be a 2×2 traceless Hermitian matrix that behaves as a scalar-density field of weight 1 on $\mathcal{T}$. The field b can be represented as $(b_i{}^j)$ with its matrix elements $b_i{}^j$ and can be expanded in terms of the Pauli matrices σ_r $(r = 1, 2, 3)$, satisfying $[\sigma_r, \sigma_s] = 2i\epsilon_{rst}\sigma_t$, as $b = b^r \sigma_r$. Here, $b^r = b^r(\tau)$ are real scalar-density fields of weight 1 on $\mathcal{T}$. The (primitive) gauged action, S_{mg}, can be obtained by replacing $d/d\tau$ in S_m with a covariant derivative operator as follows:

$$S_{m\mathrm{g}} = \int_{\tau_0}^{\tau_1} d\tau \biggl[\frac{i}{2}\bigl(\bar{Z}^i_A D Z^A_i - Z^A_i \bar{D}\bar{Z}^i_A\bigr) + h\Bigl(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi}\Bigr) + \bar{h}\Bigl(\epsilon_{ij}\bar{\pi}^i_{\alpha}\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\Bigr)\biggr], \tag{3.2.1}$$

where

$$DZ^A_i := \dot{Z}^A_i - iaZ^A_i - ib_i{}^j Z^A_j\,, \tag{3.2.2a}$$

$$\bar{D}\bar{Z}^i_A := \dot{\bar{Z}}{}^i_A + ia\bar{Z}^i_A + i\bar{Z}^j_A b_j{}^i\,. \tag{3.2.2b}$$

We see that the action $S_{m\mathrm{g}}$ is reparametrization invariant. It can easily be verified that $S_{m\mathrm{g}}$ remains invariant under the local $U(1)$ transformation

$$Z^A_i \to Z'^A_i = e^{i\theta(\tau)}Z^A_i\,, \tag{3.2.3a}$$

$$\bar{Z}^i_A \to \bar{Z}'^i_A = e^{-i\theta(\tau)}\bar{Z}^i_A\,, \tag{3.2.3b}$$

$$h \to h' = e^{-2i\theta(\tau)}h\,, \tag{3.2.3c}$$

$$\bar{h} \to \bar{h}' = e^{2i\theta(\tau)}\bar{h}\,, \tag{3.2.3d}$$

$$\varphi \to \varphi' = \varphi + 2\theta(\tau)\,, \tag{3.2.3e}$$

$$a \to a' = a + \dot{\theta}\,, \tag{3.2.3f}$$

$$b \to b' = b\,, \tag{3.2.3g}$$

with a real gauge function $\theta = \theta(\tau)$ and under the local $SU(2)$ transformation

$$Z^A_i \to Z'^A_i = U_i{}^j(\tau)Z^A_j\,, \tag{3.2.4a}$$

$$\bar{Z}^i_A \to \bar{Z}'^i_A = \bar{Z}^j_A U^\dagger{}_j{}^i(\tau)\,, \tag{3.2.4b}$$

$$h \to h' = h\,, \tag{3.2.4c}$$

$$\bar{h} \to \bar{h}' = \bar{h}\,, \tag{3.2.4d}$$

$$\varphi \to \varphi' = \varphi\,, \tag{3.2.4e}$$

$$a \to a' = a\,, \tag{3.2.4f}$$

$$b \to b' = UbU^\dagger - i\dot{U}U^\dagger, \tag{3.2.4g}$$

with a gauge function $U = U(\tau)$ taking its value in $SU(2)$. Because each of a and b^r is a single-component gauge field associated with $d/d\tau$, we cannot define their field strengths. For this reason, there exists neither the Maxwell action for a nor the Yang-Mills action for b. As for a, it is possible to define the (non-vanishing) 1D $U(1)$ Chern-Simons term

$$S_a = -2s \int_{\tau_0}^{\tau_1} d\tau a \,, \tag{3.2.5}$$

where s is a real constant. The 1D $SU(2)$ Chern-Simons term for b, i.e., $S_b = -2t \int_{\tau_0}^{\tau_1} d\tau \mathrm{Tr} b$ vanishes by the reason of $\mathrm{Tr} b = 0$. Since a is a scalar-density field of weight 1, S_a is reparametrization invariant. Also, S_a remains invariant under the gauge transformation (3.2.3f), provided that θ satisfies an appropriate boundary condition such as $\theta(\tau_1) = \theta(\tau_0)$. The $SU(2)$ invariance of S_a is evident from Eq. (3.2.4f). Therefore we can consider the reparametrization-invariant and gauge-invariant action $\tilde{S}_{m\mathrm{g}} := S_{m\mathrm{g}} + S_a$.[4] However, $\tilde{S}_{mg}$ eventually turns out to govern only massive *spinless* fields in four dimensions owing to the too strong constraints $\bar{Z}^i_A \sigma_{ri}{}^j Z^A_j = 0$ $(r = 1, 2, 3)$ that are derived by varying $\tilde{S}_{m\mathrm{g}}$ w.r.t. b^r.[5] (Here, $\sigma_{rj}{}^k$ denotes the (j, k) entry of the Pauli matrix σ_r.) To avoid such an undesirable situation, next we perform a modification of $\tilde{S}_{mg}$ with the aid of a nonlinear realization of $SU(2)$.

3.3 A model with the nonlinearly realized internal $SU(2)$ gauge symmetry

Let us now consider the coset space $SU(2)/U(1) (\cong \mathbb{C}\mathbf{P}^1)$ and representative elements, $V(\xi, \bar{\xi})$ $(V \in SU(2), \xi \in \mathbb{C})$, that are chosen one by one from each left coset

[4]The action $\tilde{S}_{mg}$ is a simple and natural generalization of the *gauged* Shirafuji action (without invariance under the local scale transformation of Z^A and $\bar{Z}_A$)

$$\tilde{S}_{0\mathrm{g}} = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A D Z^A - Z^A \bar{D} \bar{Z}_A) - 2sa \right],$$

where $D := d/d\tau - ia$. This action describes a free massless spinning particle of helicity s [27, 28, 29] and is equivalent to the action for a massless particle with rigidity at least at the classical mechanical level [30].

[5]From the action $\tilde{S}_{mg}$, the Pauli-Lubanski spin vector $W^{\alpha\dot{\alpha}}$ is found to be

$$W^{\alpha\dot{\alpha}} = T_r \sigma_{ri}{}^j \bar{\pi}^{i\alpha} \pi^{\dot{\alpha}}_j \,, \quad T_r := \frac{1}{2} \bar{Z}^k_B \sigma_{rk}{}^l Z^B_l$$

(see Appendix). Using the mass-shell condition (3.1.3), we can show that $W_{\alpha\dot{\alpha}} W^{\alpha\dot{\alpha}} = -m^2 T_r T_r$. Obviously, $T_r = 0$ $(r = 1, 2, 3)$ leads to $W_{\alpha\dot{\alpha}} W^{\alpha\dot{\alpha}} = 0$. Hence, it follows that only massive *spinless* particles are admissible in the model defined by $\tilde{S}_{m\mathrm{g}}$. Accordingly, it turns out that only massive *spinless* fields are provided after quantizing the model.

of $U(1)$ in $SU(2)$. Here, ξ labels the cosets in a way of one-to-one correspondence and can be regarded as an inhomogeneous coordinate of a point on $SU(2)/U(1)$. (To completely coordinatize $SU(2)/U(1)$, it is necessary to use ξ^{-1} in addition to ξ.) The representative elements $V(\xi, \bar{\xi})$ are assumed to constitute a smooth function of ξ and $\bar{\xi}$ so that we can simply treat $V(\xi, \bar{\xi})$ as an $SU(2)$-valued smooth function. We consider ξ to be a complex scalar field $\xi = \xi(\tau)$ on $\mathcal{T}$. The left action of U on $V(\xi, \bar{\xi})$ generates a nonlinear transformation $\xi \to \xi' = \xi'(\xi)$ in accordance with

$$V(\xi, \bar{\xi}) \to V(\xi', \bar{\xi}') = U(\tau) V(\xi, \bar{\xi}) \Theta^{-1}(\tau) , \tag{3.3.1}$$

where $\Theta(\tau) := \exp\{i\vartheta(\tau)\sigma_3\}$, and $\vartheta = \vartheta(\tau)$ is a real gauge function [31, 32, 33]. Note here that ϑ is determined depending on $(\xi, \bar{\xi})$ as well as U. Using $V = V(\xi, \bar{\xi})$, we define the following new fields on $\mathcal{T}$:

$$\mathsf{Z}_i^A := V^{\dagger}{}_i{}^j Z_j^A , \quad \bar{\mathsf{Z}}_A^i := \bar{Z}_A^j V_j{}^i , \tag{3.3.2a}$$

$$\mathsf{b} := V^{\dagger} b V - i \dot{V}^{\dagger} V . \tag{3.3.2b}$$

The field b can be expanded as $\mathsf{b} = \mathsf{b}^r \sigma_r$, where $\mathsf{b}^r = \mathsf{b}^r(\tau)$ are real fields. Clearly, b^r behave as scalar-density fields of weight 1 on $\mathcal{T}$. With the new fields, the local $U(1)$ transformation (3.2.3) reads

$$\mathsf{Z}_i^A \to \mathsf{Z}_i'^A = e^{i\theta(\tau)} \mathsf{Z}_i^A , \tag{3.3.3a}$$

$$\bar{\mathsf{Z}}_A^i \to \bar{\mathsf{Z}}_A'^i = e^{-i\theta(\tau)} \bar{\mathsf{Z}}_A^i , \tag{3.3.3b}$$

$$h \to h' = e^{-2i\theta(\tau)} h , \tag{3.3.3c}$$

$$\bar{h} \to \bar{h}' = e^{2i\theta(\tau)} \bar{h} , \tag{3.3.3d}$$

$$\varphi \to \varphi' = \varphi + 2\theta(\tau) , \tag{3.3.3e}$$

$$a \to a' = a + \dot{\theta} , \tag{3.3.3f}$$

$$\mathsf{b} \to \mathsf{b}' = \mathsf{b} . \tag{3.3.3g}$$

On the other hand, from Eqs. (3.2.4) and (3.3.1), we have

$$\mathsf{Z}_i^A \to \mathsf{Z}_i'^A = \Theta_i{}^j(\tau)\mathsf{Z}_j^A\,, \tag{3.3.4a}$$

$$\bar{\mathsf{Z}}_A^i \to \bar{\mathsf{Z}}_A'^i = \bar{\mathsf{Z}}_A^j \Theta^\dagger{}_j{}^i(\tau)\,, \tag{3.3.4b}$$

$$h \to h' = h\,, \tag{3.3.4c}$$

$$\bar{h} \to \bar{h}' = \bar{h}\,, \tag{3.3.4d}$$

$$\varphi \to \varphi' = \varphi\,, \tag{3.3.4e}$$

$$a \to a' = a\,, \tag{3.3.4f}$$

$$\mathsf{b} \to \mathsf{b}' = \Theta \mathsf{b} \Theta^\dagger + \dot{\vartheta}\sigma_3\,. \tag{3.3.4g}$$

Equation (3.3.4) is precisely a local $U(1)$ transformation. Hereafter, we refer to the local $U(1)$ transformation specified by Eq. (3.2.3), or Eq. (3.3.3), as the $U(1)_a$ transformation and refer to that specified by Eq. (3.3.4) as the $U(1)_\mathsf{b}$ transformation. Their corresponding gauge groups are simply denoted as $U(1)_a$ and $U(1)_\mathsf{b}$. The local SU(2) transformation is not manifestly seen in Eq. (3.3.4); instead, it is realized as a nonlinear transformation of ξ. We may say that the function V converts the local $SU(2)$ transformation into the $U(1)_\mathsf{b}$ transformation while ξ undergoes a nonlinear transformation. Equation (3.3.4g) defines the transformation rules of the fields b^r:

$$\mathsf{b}^1 \to \mathsf{b}'^1 = \mathsf{b}^1 \cos 2\vartheta + \mathsf{b}^2 \sin 2\vartheta\,, \tag{3.3.5a}$$

$$\mathsf{b}^2 \to \mathsf{b}'^2 = -\mathsf{b}^1 \sin 2\vartheta + \mathsf{b}^2 \cos 2\vartheta\,, \tag{3.3.5b}$$

$$\mathsf{b}^3 \to \mathsf{b}'^3 = \mathsf{b}^3 + \dot{\vartheta}\,. \tag{3.3.5c}$$

We see that $\mathsf{b}^{\hat{\imath}}$ ($\hat{\imath} = 1, 2$) transform homogeneously, obeying together an $SO(2)$ rotation, while b^3 transforms inhomogeneously as a $U(1)$ gauge field.

Now, we can provide the following two terms:

$$S_{\mathsf{b}12} = -k \int_{\tau_0}^{\tau_1} d\tau \sqrt{\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}}}\,, \tag{3.3.6}$$

with $\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}} := (\mathsf{b}^1)^2 + (\mathsf{b}^2)^2$, and

$$S_{\mathsf{b}3} = -2t \int_{\tau_0}^{\tau_1} d\tau\, \mathsf{b}^3\,. \tag{3.3.7}$$

Here, k is a positive constant and t is a real constant. Since b^r are scalar-density fields of weight 1 on $\mathcal{T}$, both $S_{\mathsf{b}12}$ and $S_{\mathsf{b}3}$ are reparametrization invariant. It is obvious that $S_{\mathsf{b}12}$ remains invariant under the $SO(2)$ rotation defined by Eqs. (3.3.5a) and (3.3.5b). Also, $S_{\mathsf{b}3}$, which is the 1D Chern-Simons term for b^3, remains invariant under the gauge transformation (3.3.5c), provided that ϑ satisfies an appropriate boundary condition such as $\vartheta(\tau_1) = \vartheta(\tau_0)$. We thus see that both $S_{\mathsf{b}12}$ and $S_{\mathsf{b}3}$ possess the $U(1)_{\mathsf{b}}$ symmetry. The $U(1)_a$ invariance of $S_{\mathsf{b}12}$ and $S_{\mathsf{b}3}$ is evident from Eq. (3.3.3g). For our investigation, it is convenient to express $S_{\mathsf{b}12}$ as

$$S_{\mathsf{be}} = -\int_{\tau_0}^{\tau_1} d\tau \left(\frac{1}{2\mathsf{e}} \mathsf{b}^{\hat{\imath}} \mathsf{b}^{\hat{\imath}} + \frac{k^2}{2} \mathsf{e} \right) \tag{3.3.8}$$

with the aid of $\mathsf{e} = \mathsf{e}(\tau)$ being a positive scalar-density field of weight 1 on $\mathcal{T}$. It is assumed that e does not change under the $U(1)_a$ and $U(1)_{\mathsf{b}}$ transformations. (At this stage, we should include the transformation rule $\mathsf{e} \to \mathsf{e}' = \mathsf{e}$ in each of Eqs. (3.3.3) and (3.3.4).) The action S_{mg} can be rewritten in terms of Z_i^A, $\bar{\mathsf{Z}}_A^i$, h, $\bar{h}$, φ, a, and b. The resulting rewritten expression of S_{mg} is precisely what is obtained by replacing Z_i^A, $\bar{Z}_A^i$, and b in Eq. (3.2.1) with Z_i^A, $\bar{\mathsf{Z}}_A^i$, and b, respectively. With this expression, we modify $\tilde{S}_{mg} = S_{mg} + S_a$ by adding S_{be} and $S_{\mathsf{b}3}$ to it. That is, we consider the modified action $S := S_{mg} + S_a + S_{\mathsf{be}} + S_{\mathsf{b}3}$, or more precisely,

$$\begin{aligned} S = \int_{\tau_0}^{\tau_1} d\tau \bigg[& \frac{i}{2} \big(\bar{\mathsf{Z}}_A^i \mathsf{D} \mathsf{Z}_i^A - \mathsf{Z}_i^A \bar{\mathsf{D}} \bar{\mathsf{Z}}_A^i \big) - 2sa - 2t\mathsf{b}^3 - \frac{1}{2\mathsf{e}} \mathsf{b}^{\hat{\imath}} \mathsf{b}^{\hat{\imath}} - \frac{k^2}{2} \mathsf{e} \\ & + h \Big(\epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} - \sqrt{2} m e^{i\varphi} \Big) + \bar{h} \Big(\epsilon_{ij} \bar{\varpi}_\alpha^i \bar{\varpi}^{j\alpha} - \sqrt{2} m e^{-i\varphi} \Big) \bigg] , \end{aligned} \tag{3.3.9}$$

with

$$\mathsf{D}\mathsf{Z}_i^A := \dot{\mathsf{Z}}_i^A - ia\mathsf{Z}_i^A - i\mathsf{b}_i{}^j \mathsf{Z}_j^A , \tag{3.3.10a}$$

$$\bar{\mathsf{D}}\bar{\mathsf{Z}}_A^i := \dot{\bar{\mathsf{Z}}}_A^i + ia\bar{\mathsf{Z}}_A^i + i\bar{\mathsf{Z}}_A^j \mathsf{b}_j{}^i . \tag{3.3.10b}$$

Here, $\varpi_{i\dot{\alpha}}$ and $\bar{\varpi}_\alpha^i$ are momentum-spinor components of the twistors $\mathsf{Z}_i^A = (\varrho_i^\alpha, \varpi_{i\dot{\alpha}})$ and $\bar{\mathsf{Z}}_A^i = (\bar{\varpi}_\alpha^i, \bar{\varrho}^{i\dot{\alpha}})$, respectively. We refer to S as the *gauged* generalized Shirafuji (GGS) action. From Eq. (3.3.2a), it follows that $\varpi_{i\dot{\alpha}} = V^{\dagger}{}_i{}^j \pi_{j\dot{\alpha}}$ and $\bar{\varpi}_\alpha^i = \bar{\pi}_\alpha^j V_j{}^i$. The other components are given by $\varrho_i^\alpha = V^{\dagger}{}_i{}^j \omega_j^\alpha$ and $\bar{\varrho}^{i\dot{\alpha}} = \bar{\omega}^{j\dot{\alpha}} V_j{}^i$. It is now obvious that $\overline{\varpi_{i\dot{\alpha}}} = \bar{\varpi}_\alpha^i$ and $\overline{\varrho_i^\alpha} = \bar{\varrho}^{i\dot{\alpha}}$. In terms of Z_i^A, Eq. (3.1.1) can be written as

$$\varrho_i^\alpha = iz^{\alpha\dot{\alpha}} \varpi_{i\dot{\alpha}} . \tag{3.3.11}$$

It is clear from (3.3.9) that the GGS action S remains invariant under the reparametrization and the $U(1)_a$ and $U(1)_{\mathsf{b}}$ transformations. However, in actuality, S remains invariant under the reparametrization and the $U(1)_a$ and local $SU(2)$ transformations, because the $U(1)_{\mathsf{b}}$ transformation is induced by the local $SU(2)$ transformation in accordance with Eq. (3.3.1). In fact, we can express S in a manifestly $SU(2)$ invariant form as follows:

$$S = \int_{\tau_0}^{\tau_1} d\tau \bigg[\frac{i}{2}\big(\bar{Z}_A^i D Z_i^A - Z_i^A \bar{D}\bar{Z}_A^i\big) - 2sa - 2t\Big(b^r \mathcal{V}_r{}^3 - \dot{\xi} e_\xi{}^3 - \dot{\bar{\xi}} e_{\bar{\xi}}{}^3\Big) - \frac{1}{\mathsf{e}} g_{\xi\bar{\xi}} D\xi D\bar{\xi} - \frac{k^2}{2}\mathsf{e} + h\Big(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi}\Big) + \bar{h}\Big(\epsilon_{ij}\bar{\pi}_\alpha^i \bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\Big)\bigg], \tag{3.3.12}$$

with $g_{\xi\bar{\xi}} := e_\xi{}^{\hat{\imath}} e_{\bar{\xi}}{}^{\hat{\imath}}$, $D\xi := \dot{\xi} - b^r K_r{}^\xi$, and $D\bar{\xi} := \dot{\bar{\xi}} - b^r K_r{}^{\bar{\xi}}$. Here, $g_{\xi\bar{\xi}}$ is a metric on $SU(2)/U(1)$, $(K_r{}^\xi, K_r{}^{\bar{\xi}})$ $(r = 1, 2, 3)$ are the $SU(2)$ Killing vectors on this coset space, and $e_\xi{}^r$ and $e_{\bar{\xi}}{}^r$ $(r = \hat{\imath}, 3)$ are defined by $e_\xi{}^r \sigma_r = -iV^\dagger(\partial V/\partial\xi)$ and $e_{\bar{\xi}}{}^r \sigma_r = -iV^\dagger(\partial V/\partial\bar{\xi})$, respectively. Also, $\mathcal{V}_r{}^3$ is defined according to $V^\dagger \sigma_r V = \mathcal{V}_r{}^{\hat{\imath}} \sigma_{\hat{\imath}} + \mathcal{V}_r{}^3 \sigma_3$. Using the transformation rule (3.3.1), we can show that $\mathcal{V}_r{}^{\hat{\imath}} = K_r{}^\xi e_\xi{}^{\hat{\imath}} + K_r{}^{\bar{\xi}} e_{\bar{\xi}}{}^{\hat{\imath}}$. In addition, it can be verified that $K_r := K_r{}^\xi \partial/\partial\xi + K_r{}^{\bar{\xi}} \partial/\partial\bar{\xi}$ satisfy the $SU(2)$ commutation relations. In the expression (3.3.9), we should understand that the local $SU(2)$ symmetry of S is hidden rather than is broken, because no symmetry breaking mechanisms are incorporated in the model. The action (3.3.9) can be regarded as the action (3.3.12) in a particular gauge $\xi(\tau) = \xi_0$, where ξ_0 is a constant such that $V(\xi_0, \bar{\xi}_0) = 1$. We term this gauge the unitary gauge, because it corresponds to the so-called unitary gauge in massive Yang-Mills theory [34, 35]. Then b can be said to be the $SU(2)$ gauge field in the unitary gauge. The action (3.3.9) can be written as

$$S = \int_{\tau_0}^{\tau_1} d\tau \bigg[\frac{i}{2}\Big(\bar{\mathsf{Z}}_A^i \dot{\mathsf{Z}}_i^A - \mathsf{Z}_i^A \dot{\bar{\mathsf{Z}}}_A^i\Big) + a\big(\bar{\mathsf{Z}}_A^i \mathsf{Z}_i^A - 2s\big) + \mathsf{b}^3\big(\bar{\mathsf{Z}}_A^j \sigma_{3j}{}^k \mathsf{Z}_k^A - 2t\big) + \mathsf{b}^{\hat{\imath}} \bar{\mathsf{Z}}_A^j \sigma_{\hat{\imath}j}{}^k \mathsf{Z}_k^A - \frac{1}{2\mathsf{e}}\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}} - \frac{k^2}{2}\mathsf{e} + h\Big(\epsilon^{ij}\varpi_{i\dot{\alpha}}\varpi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi}\Big) + \bar{h}\Big(\epsilon_{ij}\bar{\varpi}_\alpha^i \bar{\varpi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\Big)\bigg]. \tag{3.3.13}$$

Chapter 4

Canonical quantization in the gauged twistor formulation

In this chapter, we study the canonical Hamiltonian formalism based on the GGS action obtained in previous chapter, by completely following the Dirac algorithm for Hamiltonian systems with constraints. We see that most of the Dirac brackets between the twistor variables take on complicated forms. Fortunately, these Dirac brackets can be reduced to simple Dirac brackets for new twistor variables that are in one-to-one correspondence with the old ones. Also, all the constraints for the (old) twistor variables can be written completely in terms of the new twistor variables. The canonical quantization of the twistor model governed by the GGS action is performed with the commutation relations between the operators that correspond to the new twistor variables or the other canonical variables. Some of the first-class constraints eventually turn into simultaneous differential equations for a holomorphic function of half the new twistor variables. Each solution of the simultaneous differential equations, referred to here as a twistor function, is characterized by the three quantum numbers that originate from the $U(1)$ and $SU(2)$ symmetries inherent in the GGS action. We also consider the Penrose transform of the twistor function to define a spinor field of arbitrary rank with $SU(2)$ indices. Because of the structure of the Penrose transform, the number of $SU(2)$ indices is equal to the number of spinor indices. We demonstrate that the present spinor field satisfies generalized DFP equations with $SU(2)$ indices. To clarify the physical meanings of the $U(1)$ and $SU(2)$ symmetries, we investigate properties of the rank-one spinor fields and the generalized DFP equations satisfied by them.

4.1 Canonical formalism

In this section, we study the canonical Hamiltonian formalism of the model governed by the GGS action in the unitary gauge.

Let L be the Lagrangian defined in Eq. (3.3.13) as the integrand of the GGS action S. We treat the variables $\big(\mathsf{Z}_i^A, \bar{\mathsf{Z}}_A^i, a, \mathsf{b}^r, \mathsf{e}, h, \bar{h}, \varphi\big)$ as canonical coordinates. Their canonical conjugate momenta are found to be

$$P_A^i := \frac{\partial L}{\partial \dot{\mathsf{Z}}_i^A} = \frac{i}{2}\bar{\mathsf{Z}}_A^i\,, \tag{4.1.1a}$$

$$\bar{P}_i^A := \frac{\partial L}{\partial \dot{\bar{\mathsf{Z}}}_A^i} = -\frac{i}{2}\mathsf{Z}_i^A, \tag{4.1.1b}$$

$$P^{(a)} := \frac{\partial L}{\partial \dot{a}} = 0\,, \tag{4.1.1c}$$

$$P_r^{(\mathsf{b})} := \frac{\partial L}{\partial \dot{\mathsf{b}}^r} = 0\,, \tag{4.1.1d}$$

$$P^{(\mathsf{e})} := \frac{\partial L}{\partial \dot{\mathsf{e}}} = 0\,, \tag{4.1.1e}$$

$$P^{(h)} := \frac{\partial L}{\partial \dot{h}} = 0\,, \tag{4.1.1f}$$

$$P^{(\bar{h})} := \frac{\partial L}{\partial \dot{\bar{h}}} = 0\,, \tag{4.1.1g}$$

$$P^{(\varphi)} := \frac{\partial L}{\partial \dot{\varphi}} = 0\,. \tag{4.1.1h}$$

The canonical Hamiltonian corresponding to L is defined by the Legendre transform of L:

$$\begin{aligned} H_{\mathrm{C}} &:= \dot{\mathsf{Z}}_i^A P_A^i + \dot{\bar{\mathsf{Z}}}_A^i \bar{P}_i^A + \dot{a}P^{(a)} + \dot{\mathsf{b}}^r P_r^{(\mathsf{b})} + \dot{\mathsf{e}}P^{(\mathsf{e})} + \dot{h}P^{(h)} + \dot{\bar{h}}P^{(\bar{h})} + \dot{\varphi}P^{(\varphi)} - L \\ &= -a\big(\bar{\mathsf{Z}}_A^i \mathsf{Z}_i^A - 2s\big) - \mathsf{b}^3\big(\bar{\mathsf{Z}}_A^j \sigma_{3j}{}^k \mathsf{Z}_k^A - 2t\big) - \mathsf{b}^{\hat{\imath}}\bar{\mathsf{Z}}_A^j \sigma_{\hat{\imath}j}{}^k \mathsf{Z}_k^A + \frac{1}{2\mathsf{e}}\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}} + \frac{k^2}{2}\mathsf{e} \\ &\quad - h\Big(\epsilon^{ij}\varpi_{i\dot{\alpha}}\varpi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi}\Big) - \bar{h}\Big(\epsilon_{ij}\bar{\varpi}_\alpha^i \bar{\varpi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\Big). \end{aligned} \tag{4.1.2}$$

The equal-time Poisson brackets between the canonical variables are given by

$$
\begin{aligned}
&\left\{\mathsf{Z}_i^A, P_B^j\right\} = \delta_i^j \delta_B^A\,, \quad \left\{\bar{\mathsf{Z}}_A^i, \bar{P}_j^B\right\} = \delta_j^i \delta_A^B\,,\\
&\left\{a, P^{(a)}\right\} = 1\,, \qquad \left\{\mathsf{b}^r, P_s^{(\mathsf{b})}\right\} = \delta_s^r\,,\\
&\left\{\mathsf{e}, P^{(\mathsf{e})}\right\} = 1\,, \qquad \left\{h, P^{(h)}\right\} = 1\,,\\
&\left\{\bar{h}, P^{(\bar{h})}\right\} = 1\,, \qquad \left\{\varphi, P^{(\varphi)}\right\} = 1\,,\\
&\text{all others} = 0\,,
\end{aligned}
\tag{4.1.3}
$$

which can be used for calculating the Poisson bracket between two arbitrary analytic functions of the canonical variables.

Equations (4.1.1a)–(4.1.1h) are read as the primary constraints

$$\phi_A^i := P_A^i - \frac{i}{2}\bar{\mathsf{Z}}_A^i \approx 0\,, \tag{4.1.4a}$$

$$\bar{\phi}_i^A := \bar{P}_i^A + \frac{i}{2}\mathsf{Z}_i^A \approx 0\,, \tag{4.1.4b}$$

$$\phi^{(a)} := P^{(a)} \approx 0\,, \tag{4.1.4c}$$

$$\phi_r^{(\mathsf{b})} := P_r^{(\mathsf{b})} \approx 0\,, \tag{4.1.4d}$$

$$\phi^{(\mathsf{e})} := P^{(\mathsf{e})} \approx 0\,, \tag{4.1.4e}$$

$$\phi^{(h)} := P^{(h)} \approx 0\,, \tag{4.1.4f}$$

$$\phi^{(\bar{h})} := P^{(\bar{h})} \approx 0\,, \tag{4.1.4g}$$

$$\phi^{(\varphi)} := P^{(\varphi)} \approx 0\,, \tag{4.1.4h}$$

where the symbol "$\approx$" denotes the weak equality. Now, we follow the Dirac algorithm for constrained Hamiltonian systems [36, 37, 38] to establish the canonical formalism of the present model. We see that the Poisson brackets between the primary constraint functions ϕ's are summarized in

$$\left\{\phi_A^i, \bar{\phi}_j^B\right\} = -i\delta_j^i \delta_A^B\,, \quad \text{all others} = 0\,. \tag{4.1.5}$$

The Poisson brackets between H_{C} and the primary constraint functions are found to be

$$
\begin{aligned}
\{\phi^i_A, H_{\rm C}\} &= a\bar{\mathsf{Z}}^i_A + \mathsf{b}^r\sigma_{rj}{}^i\bar{\mathsf{Z}}^j_A + 2h\epsilon^{ij}I_{AB}\mathsf{Z}^B_j\,, &(4.1.6a)\\
\{\bar{\phi}^A_i, H_{\rm C}\} &= a\mathsf{Z}^A_i + \mathsf{b}^r\sigma_{ri}{}^j\mathsf{Z}^A_j + 2\bar{h}\epsilon_{ij}I^{AB}\bar{\mathsf{Z}}^j_B\,, &(4.1.6b)\\
\{\phi^{(a)}, H_{\rm C}\} &= \bar{\mathsf{Z}}^i_A\mathsf{Z}^A_i - 2s\,, &(4.1.6c)\\
\left\{\phi^{(\mathsf{b})}_{\hat{\imath}}, H_{\rm C}\right\} &= \bar{\mathsf{Z}}^j_A\sigma_{\hat{\imath}j}{}^k\mathsf{Z}^A_k - \frac{1}{\mathsf{e}}\mathsf{b}_{\hat{\imath}}\,, &(4.1.6d)\\
\left\{\phi^{(\mathsf{b})}_3, H_{\rm C}\right\} &= \bar{\mathsf{Z}}^j_A\sigma_{3j}{}^k\mathsf{Z}^A_k - 2t\,, &(4.1.6e)\\
\{\phi^{(\mathsf{e})}, H_{\rm C}\} &= \frac{1}{2\mathsf{e}^2}\big(\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}} - k^2\mathsf{e}^2\big)\,, &(4.1.6f)\\
\{\phi^{(h)}, H_{\rm C}\} &= \epsilon^{ij}\varpi_{i\dot{\alpha}}\varpi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi}, &(4.1.6g)\\
\left\{\phi^{(\bar{h})}, H_{\rm C}\right\} &= \epsilon_{ij}\bar{\varpi}^i_\alpha\bar{\varpi}^{j\alpha} - \sqrt{2}me^{-i\varphi}, &(4.1.6h)\\
\{\phi^{(\varphi)}, H_{\rm C}\} &= -i\sqrt{2}m\big(he^{i\varphi} - \bar{h}e^{-i\varphi}\big)\,. &(4.1.6i)
\end{aligned}
$$

where I_{AB} and I^{AB} are the so-called infinity twistors [2, 3, 11], defined by

$$
I_{AB} := \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad I^{AB} := \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}.
$$

With $H_{\rm C}$ and the primary constraint functions, we define the total Hamiltonian

$$
\begin{aligned}
H_{\rm T} := H_{\rm C} &+ u^A_i\phi^i_A + \bar{u}^i_A\bar{\phi}^A_i + u_{(a)}\phi^{(a)} + u^r_{(\mathsf{b})}\phi^{(\mathsf{b})}_r \\
&+ u_{(\mathsf{e})}\phi^{(\mathsf{e})} + u_{(h)}\phi^{(h)} + u_{(\bar{h})}\phi^{(\bar{h})} + u_{(\varphi)}\phi^{(\varphi)}, \qquad (4.1.7)
\end{aligned}
$$

where u^A_i, $\bar{u}^i_A$, $u_{(a)}$, $u^r_{(\mathsf{b})}$, $u_{(\mathsf{e})}$, $u_{(h)}$, $u_{(\bar{h})}$, and $u_{(\varphi)}$ are Lagrange multipliers. The time evolution of a function f of the canonical variables is governed by the canonical equation

$$
\dot{f} = \{f, H_{\rm T}\}\,. \qquad (4.1.8)
$$

Using this equation together with Eqs. (4.1.4)–(4.1.7), we can evaluate the time evolution of the primary constraint functions. Because the primary constraints (4.1.4a)–(4.1.4h) are valid at any time, they must be preserved in time. This fact leads to the consistency conditions

$$
\begin{aligned}
\dot{\phi}^i_A &= \left\{\phi^i_A, H_{\rm T}\right\} \approx a\bar{\mathsf{Z}}^i_A + \mathsf{b}^r\sigma_{rj}{}^i\bar{\mathsf{Z}}^j_A + 2h\epsilon^{ij}I_{AB}\mathsf{Z}^B_j - i\bar{u}^i_A \approx 0\,, && (4.1.9a)\\
\dot{\bar{\phi}}^A_i &= \left\{\bar{\phi}^A_i, H_{\rm T}\right\} \approx a\mathsf{Z}^A_i + \mathsf{b}^r\sigma_{ri}{}^j\mathsf{Z}^A_j + 2\bar{h}\epsilon_{ij}I^{AB}\bar{\mathsf{Z}}^j_B + iu^A_i \approx 0\,, && (4.1.9b)\\
\dot{\phi}^{(a)} &= \left\{\phi^{(a)}, H_{\rm T}\right\} \approx \bar{\mathsf{Z}}^i_A\mathsf{Z}^A_i - 2s \approx 0\,, && (4.1.9c)\\
\dot{\phi}^{(\mathsf{b})}_{\hat{i}} &= \left\{\phi^{(\mathsf{b})}_{\hat{i}}, H_{\rm T}\right\} \approx \bar{\mathsf{Z}}^j_A\sigma_{\hat{i}j}{}^k\mathsf{Z}^A_k - \frac{1}{\mathsf{e}}\mathsf{b}_{\hat{i}} \approx 0\,, && (4.1.9d)\\
\dot{\phi}^{(\mathsf{b})}_3 &= \left\{\phi^{(\mathsf{b})}_3, H_{\rm T}\right\} \approx \bar{\mathsf{Z}}^j_A\sigma_{3j}{}^k\mathsf{Z}^A_k - 2t \approx 0\,, && (4.1.9e)\\
\dot{\phi}^{(\mathsf{e})} &= \left\{\phi^{(\mathsf{e})}, H_{\rm T}\right\} \approx \frac{1}{2\mathsf{e}^2}\big(\mathsf{b}^{\hat{i}}\mathsf{b}^{\hat{i}} - k^2\mathsf{e}^2\big) \approx 0\,, && (4.1.9f)\\
\dot{\phi}^{(h)} &= \left\{\phi^{(h)}, H_{\rm T}\right\} \approx \epsilon^{ij}\varpi_{i\dot{\alpha}}\varpi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi} \approx 0\,, && (4.1.9g)\\
\dot{\phi}^{(\bar{h})} &= \left\{\phi^{(\bar{h})}, H_{\rm T}\right\} \approx \epsilon_{ij}\bar{\varpi}^i_\alpha\bar{\varpi}^{j\alpha} - \sqrt{2}me^{-i\varphi} \approx 0\,, && (4.1.9h)\\
\dot{\phi}^{(\varphi)} &= \left\{\phi^{(\varphi)}, H_{\rm T}\right\} \approx -i\sqrt{2}m\big(he^{i\varphi} - \bar{h}e^{-i\varphi}\big) \approx 0\,. && (4.1.9i)
\end{aligned}
$$

Equations (4.1.9a) and (4.1.9b) determine $\bar{u}^i_A$ and u^A_i, respectively, as follows:

$$
\begin{aligned}
\bar{u}^i_A &= -ia\bar{\mathsf{Z}}^i_A - i\mathsf{b}^r\sigma_{rj}{}^i\bar{\mathsf{Z}}^j_A - 2ih\epsilon^{ij}I_{AB}\mathsf{Z}^B_j\,, && (4.1.10a)\\
u^A_i &= ia\mathsf{Z}^A_i + i\mathsf{b}^r\sigma_{ri}{}^j\mathsf{Z}^A_j + 2i\bar{h}\epsilon_{ij}I^{AB}\bar{\mathsf{Z}}^j_B\,. && (4.1.10b)
\end{aligned}
$$

In contrast, Eqs. (4.1.9c)–(4.1.9i) give rise to the secondary constraints

$$
\begin{aligned}
\chi^{(a)} &:= \bar{\mathsf{Z}}^i_A\mathsf{Z}^A_i - 2s \approx 0\,, && (4.1.11a)\\
\chi^{(\mathsf{b})}_{\hat{i}} &:= \bar{\mathsf{Z}}^j_A\sigma_{\hat{i}j}{}^k\mathsf{Z}^A_k - \frac{1}{\mathsf{e}}\mathsf{b}_{\hat{i}} \approx 0\,, && (4.1.11b)\\
\chi^{(\mathsf{b})}_3 &:= \bar{\mathsf{Z}}^j_A\sigma_{3j}{}^k\mathsf{Z}^A_k - 2t \approx 0\,, && (4.1.11c)\\
\chi^{(\mathsf{e})} &:= \frac{1}{2}\big(\mathsf{b}^{\hat{i}}\mathsf{b}^{\hat{i}} - k^2\mathsf{e}^2\big) \approx 0\,, && (4.1.11d)\\
\chi^{(h)} &:= \epsilon^{ij}\varpi_{i\dot{\alpha}}\varpi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi} \approx 0\,, && (4.1.11e)\\
\chi^{(\bar{h})} &:= \epsilon_{ij}\bar{\varpi}^i_\alpha\bar{\varpi}^{j\alpha} - \sqrt{2}me^{-i\varphi} \approx 0\,, && (4.1.11f)\\
\chi^{(\varphi)} &:= i\big(he^{i\varphi} - \bar{h}e^{-i\varphi}\big) \approx 0\,. && (4.1.11g)
\end{aligned}
$$

All the Poisson brackets between $H_{\rm C}$ and the secondary constraint functions χ's vanish. The Poisson brackets between the primary and secondary constraint functions are found to be

$$
\begin{aligned}
\{\chi^{(a)},\phi^i_A\} &= \bar{\mathsf{Z}}^i_A\,, & \Big\{\chi^{(a)},\bar{\phi}^A_i\Big\} &= \mathsf{Z}^A_i,\\
\{\chi^{(\mathsf{b})}_r,\phi^i_A\} &= \sigma_{rj}{}^i\bar{\mathsf{Z}}^j_A\,, & \Big\{\chi^{(\mathsf{b})}_r,\bar{\phi}^A_i\Big\} &= \sigma_{ri}{}^j\mathsf{Z}^A_j,\\
\Big\{\chi^{(\mathsf{b})}_{\hat{\imath}},\phi^{(\mathsf{b})}_{\hat{\jmath}}\Big\} &= -\frac{1}{\mathsf{e}}\delta_{\hat{\imath}\hat{\jmath}}\,, & \Big\{\chi^{(\mathsf{b})}_{\hat{\imath}},\phi^{(\mathsf{e})}\Big\} &= \frac{1}{\mathsf{e}^2}\mathsf{b}_{\hat{\imath}}\,,\\
\Big\{\chi^{(\mathsf{e})},\phi^{(\mathsf{b})}_{\hat{\jmath}}\Big\} &= \mathsf{b}_{\hat{\jmath}}\,, & \{\chi^{(\mathsf{e})},\phi^{(\mathsf{e})}\} &= -k^2\mathsf{e}\,,\\
\{\chi^{(h)},\phi^{i\dot{\alpha}}\} &= 2\epsilon^{ij}\varpi^{\dot{\alpha}}_j, & \{\chi^{(h)},\phi^{(\varphi)}\} &= -i\sqrt{2}me^{i\varphi},\\
\Big\{\chi^{(\bar{h})},\bar{\phi}^{\alpha}_i\Big\} &= 2\epsilon_{ij}\bar{\varpi}^{j\alpha}, & \Big\{\chi^{(\bar{h})},\phi^{(\varphi)}\Big\} &= i\sqrt{2}me^{-i\varphi},\\
\{\chi^{(\varphi)},\phi^{(h)}\} &= ie^{i\varphi}, & \Big\{\chi^{(\varphi)},\phi^{(\bar{h})}\Big\} &= -ie^{-i\varphi},\\
\{\chi^{(\varphi)},\phi^{(\varphi)}\} &= -\big(he^{i\varphi}+\bar{h}e^{-i\varphi}\big)\,, & &\\
&\text{all others} = 0\,, & & \qquad (4.1.12)
\end{aligned}
$$

All the Poisson brackets between the secondary constraint functions vanish.

Next we investigate the time evolution of the secondary constraint functions using Eqs. (4.1.8) and (4.1.12). The time evolution of $\chi^{(a)}$ is evaluated as

$$\dot{\chi}^{(a)} = \{\chi^{(a)},H_{\mathrm{T}}\} \approx u^A_i\bar{\mathsf{Z}}^i_A + \bar{u}^i_A\mathsf{Z}^A_i. \qquad (4.1.13)$$

The condition $\dot{\chi}^{(a)} \approx 0$ is identically fulfilled with the aid of Eqs. (4.1.10a), (4.1.10b), (4.1.11e), (4.1.11f), and (4.1.11g), and hence no new constraints are obtained from $\dot{\chi}^{(a)} \approx 0$. The time evolution of $\chi^{(\mathsf{b})}_r$ is evaluated as

$$
\begin{aligned}
\dot{\chi}^{(\mathsf{b})}_r &= \{\chi^{(\mathsf{b})}_r,H_{\mathrm{T}}\}\\
&\approx u^A_i\sigma_{rj}{}^i\bar{\mathsf{Z}}^j_A + \bar{u}^i_A\sigma_{ri}{}^j\mathsf{Z}^A_j + u^s_{(\mathsf{b})}\Big\{\chi^{(\mathsf{b})}_r,\phi^{(\mathsf{b})}_s\Big\} + u_{(\mathsf{e})}\Big\{\chi^{(\mathsf{b})}_r,\phi^{(\mathsf{e})}\Big\}\\
&= -2\epsilon_{rst}\mathsf{b}^s\bar{\mathsf{Z}}^j_A\sigma_{tj}{}^k\mathsf{Z}^A_k + u^s_{(\mathsf{b})}\Big\{\chi^{(\mathsf{b})}_r,\phi^{(\mathsf{b})}_s\Big\} + u_{(\mathsf{e})}\Big\{\chi^{(\mathsf{b})}_r,\phi^{(\mathsf{e})}\Big\}\\
&\approx -\frac{2}{\mathsf{e}}\epsilon_{rs\hat{\jmath}}\mathsf{b}^s\mathsf{b}^{\hat{\jmath}} - 4t\epsilon_{rs3}\mathsf{b}^s + u^s_{(\mathsf{b})}\Big\{\chi^{(\mathsf{b})}_r,\phi^{(\mathsf{b})}_s\Big\} + u_{(\mathsf{e})}\Big\{\chi^{(\mathsf{b})}_r,\phi^{(\mathsf{e})}\Big\} \qquad (4.1.14)
\end{aligned}
$$

by using Eqs. (3.10a), (3.10b), (3.11b), and (3.11c), together with the formulas $\sigma_{rk}{}^i\epsilon^{kj} = \sigma_{rk}{}^j\epsilon^{ki}$ and $\sigma_{ri}{}^k\epsilon_{kj} = \sigma_{rj}{}^k\epsilon_{ki}$. Then we see that the condition $\dot{\chi}^{(\mathsf{b})}_{\hat{\imath}} \approx 0$ determines $u^{\hat{\imath}}_{(\mathsf{b})}$ as follows:

$$u^{\hat{\imath}}_{(\mathsf{b})} = 2\epsilon^{\hat{\imath}\hat{\jmath}}\mathsf{b}_{\hat{\jmath}}\big(\mathsf{b}^3 - 2t\mathsf{e}\big) + \frac{1}{\mathsf{e}}\mathsf{b}^{\hat{\imath}}u_{(\mathsf{e})}\,, \qquad (4.1.15)$$

while $\dot{\chi}_3^{(\mathsf{b})} \approx 0$ is identically satisfied. The time evolution of $\chi^{(\mathsf{e})}$ is calculated as

$$\begin{aligned}\dot{\chi}^{(\mathsf{e})} &= \{\chi^{(\mathsf{e})}, H_{\mathrm{T}}\} \\ &\approx \mathsf{b}_{\hat{\imath}} u^{\hat{\imath}}_{(\mathsf{b})} - k^2 \mathsf{e} u_{(\mathsf{e})} = \frac{2}{\mathsf{e}} \chi^{(\mathsf{e})} u_{(\mathsf{e})} \approx 0 \end{aligned} \tag{4.1.16}$$

by using Eqs. (4.1.15) and (4.1.11d). Hence $\dot{\chi}^{(\mathsf{e})} \approx 0$ is identically satisfied. The time evolution of $\chi^{(h)}$ is evaluated as

$$\begin{aligned}\dot{\chi}^{(h)} &= \{\chi^{(h)}, H_{\mathrm{T}}\} \\ &\approx 2\epsilon^{ij} u_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} - i\sqrt{2} m e^{i\varphi} u_{(\varphi)} \\ &= 2ia\epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} - i\sqrt{2} m e^{i\varphi} u_{(\varphi)} \\ &\approx i\sqrt{2} m e^{i\varphi} (2a - u_{(\varphi)}) \end{aligned} \tag{4.1.17}$$

by using Eqs. (4.1.10b), (4.1.11e) and the formula $\sigma_{rk}{}^{i} \epsilon^{kj} = \sigma_{rk}{}^{j} \epsilon^{ki}$. (Its associated formula $\sigma_{ri}{}^{k} \epsilon_{kj} = \sigma_{rj}{}^{k} \epsilon_{ki}$ is also valid.) From the condition $\dot{\chi}^{(h)} \approx 0$, the Lagrange multiplier $u_{(\varphi)}$ is determined to be $u_{(\varphi)} = 2a$. Similarly, $\dot{\chi}^{(\bar{h})} \approx -i\sqrt{2} m e^{-i\varphi} (2a - u_{(\varphi)}) \approx 0$ leads to $u_{(\varphi)} = 2a$. The time evolution of $\chi^{(\varphi)}$ is found to be

$$\begin{aligned}\dot{\chi}^{(\varphi)} &= \{\chi^{(\varphi)}, H_{\mathrm{T}}\} \\ &\approx i\big(u_{(h)} - u_{(\bar{h})}\big) - 2a\big(h e^{i\varphi} + \bar{h} e^{-i\varphi}\big)\,, \end{aligned} \tag{4.1.18}$$

so that the condition $\dot{\chi}^{(\varphi)} \approx 0$ gives $u_{(h)} - u_{(\bar{h})} = -2ia\big(h e^{i\varphi} + \bar{h} e^{-i\varphi}\big)$. From the above analysis, we see that no further constraints are derived anymore; thus, the procedure for deriving constraints is now completed. We also see that u_i^A, $\bar{u}_A^i$, $u^{\hat{\imath}}_{(\mathsf{b})}$, $u_{(h)} - u_{(\bar{h})}$, and $u_{(\varphi)}$ are determined to be what are written in terms of other variables such as the canonical coordinates, while $u_{(a)}$, $u^3_{(\mathsf{b})}$, $u_{(\mathsf{e})}$, and $u_{(h)} + u_{(\bar{h})}$ still remain as arbitrary functions of τ.

We have obtained all the Poisson brackets between the constraint functions, as in Eqs. (4.1.5) and (4.1.12). However, it is difficult to classify the constraints in Eqs. (4.1.4) and (4.1.11) into first and second classes on the basis of Eqs. (4.1.5) and (4.1.12) together with the vanishing Poisson brackets between the secondary constraint functions. To find simpler forms of the relevant Poisson brackets, we first define

$$\tilde{\phi}^{(\mathsf{e})} := \phi^{(\mathsf{e})} + \frac{1}{\mathsf{e}}\mathsf{b}^{\hat{\imath}}\phi^{(\mathsf{b})}_{\hat{\imath}}, \tag{4.1.19a}$$

$$\tilde{\phi}^{(\varphi)} := \phi^{(\varphi)} - ih\phi^{(h)} + i\bar{h}\phi^{(\bar{h})}, \tag{4.1.19b}$$

$$\tilde{\chi}^{(a)} := \chi^{(a)} + i\bar{\mathsf{Z}}^{j}_{A}\bar{\phi}^{A}_{j} - i\phi^{j}_{A}\mathsf{Z}^{A}_{j}, \tag{4.1.19c}$$

$$\tilde{\chi}^{(\mathsf{b})}_{\hat{\imath}} := \chi^{(\mathsf{b})}_{\hat{\imath}} + i\bar{\mathsf{Z}}^{j}_{A}\sigma_{\hat{\imath}j}{}^{k}\bar{\phi}^{A}_{k} - i\phi^{j}_{A}\sigma_{\hat{\imath}j}{}^{k}\mathsf{Z}^{A}_{k} - 2t\mathsf{e}\epsilon_{\hat{\imath}\hat{\jmath}}\phi^{(\mathsf{b})}_{\hat{\jmath}}, \tag{4.1.19d}$$

$$\tilde{\chi}^{(\mathsf{b})}_{3} := \chi^{(\mathsf{b})}_{3} + i\bar{\mathsf{Z}}^{j}_{A}\sigma_{3j}{}^{k}\bar{\phi}^{A}_{k} - i\phi^{j}_{A}\sigma_{3j}{}^{k}\mathsf{Z}^{A}_{k} + 2\epsilon_{\hat{\imath}\hat{\jmath}}\mathsf{b}^{\hat{\imath}}\phi^{(\mathsf{b})}_{\hat{\jmath}}, \tag{4.1.19e}$$

$$\tilde{\chi}^{(\mathsf{e})} := \chi^{(\mathsf{e})} + \mathsf{e}\mathsf{b}^{\hat{\imath}}\left(\tilde{\chi}^{(\mathsf{b})}_{\hat{\imath}} - 2t\mathsf{e}\epsilon_{\hat{\imath}\hat{\jmath}}\phi^{(\mathsf{b})}_{\hat{\jmath}}\right), \tag{4.1.19f}$$

$$\tilde{\chi}^{(h)} := \chi^{(h)} + 2i\epsilon^{jk}\bar{\phi}_{j\dot{\alpha}}\varpi^{\dot{\alpha}}_{k}\,, \tag{4.1.19g}$$

$$\tilde{\chi}^{(\bar{h})} := \chi^{(\bar{h})} - 2i\epsilon_{jk}\phi^{j}_{\alpha}\bar{\varpi}^{k\alpha}, \tag{4.1.19h}$$

where $\bar{\phi}_{i\dot{\alpha}}$ and ϕ^{i}_{α} are spinor components of $\bar{\phi}^{A}_{i} = \left(\bar{\phi}^{\alpha}_{i}, \bar{\phi}_{i\dot{\alpha}}\right)$ and $\phi^{i}_{A} = \left(\phi^{i}_{\alpha}, \phi^{i\dot{\alpha}}\right)$, respectively. Furthermore, it is convenient to define

$$\upsilon^{(\pm)} := \frac{1}{2\sqrt{2}m}\left(\tilde{\phi}^{(\varphi)} \pm \frac{1}{2}\tilde{\chi}^{(a)}\right), \tag{4.1.20a}$$

$$\phi^{(+)} := \frac{1}{2}\left(e^{-i\varphi}\phi^{(h)} + e^{i\varphi}\phi^{(\bar{h})}\right), \tag{4.1.20b}$$

$$\phi^{(-)} := \frac{1}{2i}\left(e^{-i\varphi}\phi^{(h)} - e^{i\varphi}\phi^{(\bar{h})}\right), \tag{4.1.20c}$$

$$\tilde{\chi}^{(+)} := \frac{1}{2}\left(e^{-i\varphi}\tilde{\chi}^{(h)} + e^{i\varphi}\tilde{\chi}^{(\bar{h})}\right), \tag{4.1.20d}$$

$$\tilde{\chi}^{(-)} := \frac{1}{2i}\left(e^{-i\varphi}\tilde{\chi}^{(h)} - e^{i\varphi}\tilde{\chi}^{(\bar{h})}\right). \tag{4.1.20e}$$

It can readily be seen that the set of all the constraints given in Eqs. (4.1.4) and (4.1.11), i.e.,

$$\left(\phi^{i}_{A}, \bar{\phi}^{A}_{i}, \phi^{(a)}, \phi^{(\mathsf{b})}_{r}, \phi^{(\mathsf{e})}, \phi^{(h)}, \phi^{(\bar{h})}, \phi^{(\varphi)}, \chi^{(a)}, \chi^{(\mathsf{b})}_{\hat{\imath}}, \chi^{(\mathsf{b})}_{3}, \chi^{(\mathsf{e})}, \chi^{(h)}, \chi^{(\bar{h})}, \chi^{(\varphi)}\right) \approx 0\,, \tag{4.1.21}$$

is equivalent to the new set of constraints

$$\left(\phi^{i}_{A}, \bar{\phi}^{A}_{i}, \phi^{(a)}, \phi^{(\mathsf{b})}_{r}, \tilde{\phi}^{(\mathsf{e})}, \phi^{(+)}, \phi^{(-)}, \upsilon^{(+)}, \upsilon^{(-)}, \tilde{\chi}^{(\mathsf{b})}_{\hat{\imath}}, \tilde{\chi}^{(\mathsf{b})}_{3}, \tilde{\chi}^{(\mathsf{e})}, \tilde{\chi}^{(+)}, \tilde{\chi}^{(-)}, \chi^{(\varphi)}\right) \approx 0\,. \tag{4.1.22}$$

We can show that except for

$$\left\{\phi^{i}_{A}, \bar{\phi}^{B}_{j}\right\} = -i\delta^{i}_{j}\delta^{B}_{A}\,, \qquad \left\{\phi^{(\mathsf{b})}_{\hat{\imath}}, \tilde{\chi}^{(\mathsf{b})}_{\hat{\jmath}}\right\} = \frac{1}{\mathsf{e}}\delta_{\hat{\imath}\hat{\jmath}}\,,$$
$$\left\{\upsilon^{(+)}, \tilde{\chi}^{(-)}\right\} = 1\,, \qquad \left\{\chi^{(\varphi)}, \phi^{(-)}\right\} = 1\,, \tag{4.1.23}$$

all other Poisson brackets between the constraint functions in Eq. (4.1.22) vanish. In this way, the relevant Poisson brackets are simplified with the aid of the new constraint functions. The Poisson brackets between the constraint functions are summarized in a matrix form as

$$
\begin{array}{c|cccccccccccccccc|}
 & \phi^{j}_{B} & \bar{\phi}^{B}_{j} & \phi^{(a)} & \phi^{(\mathsf{b})}_{\hat{j}} & \phi^{(\mathsf{b})}_{3} & \tilde{\phi}^{(\mathsf{e})} & \phi^{(+)} & \phi^{(\ )} & \upsilon^{(+)} & \upsilon^{(\ )} & \tilde{\chi}^{(\mathsf{b})}_{\hat{j}} & \tilde{\chi}^{(\mathsf{b})}_{3} & \tilde{\chi}^{(\mathsf{e})} & \tilde{\chi}^{(+)} & \tilde{\chi}^{(\ )} & \chi^{(\varphi)} \\
\phi^{i}_{A} & 0 & -i\delta^{i}_{j}\delta^{B}_{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\phi}^{A}_{i} & i\delta^{j}_{i}\delta^{A}_{B} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi^{(a)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi^{(\mathsf{b})}_{\hat{i}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{e}\ ^{1}\delta_{\hat{i}\hat{j}} & 0 & 0 & 0 & 0 & 0 \\
\phi^{(\mathsf{b})}_{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\phi}^{(\mathsf{e})} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi^{(+)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi^{(\ )} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\upsilon^{(+)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\upsilon^{(\ )} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\chi}^{(\mathsf{b})}_{\hat{i}} & 0 & 0 & 0 & -\mathsf{e}\ ^{1}\delta_{\hat{i}\hat{j}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\chi}^{(\mathsf{b})}_{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\chi}^{(\mathsf{e})} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\chi}^{(+)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\chi}^{(\ )} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\chi^{(\varphi)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} . \tag{4.1.24}
$$

We can immediately see from this matrix that $\phi^{(a)} \approx 0$, $\phi^{(\mathsf{b})}_{3} \approx 0$, $\tilde{\phi}^{(\mathsf{e})} \approx 0$, $\phi^{(+)} \approx 0$, $\upsilon^{(-)} \approx 0$, $\tilde{\chi}^{(\mathsf{b})}_{3} \approx 0$, $\tilde{\chi}^{(\mathsf{e})} \approx 0$, and $\tilde{\chi}^{(+)} \approx 0$ are first-class constraints, while $\phi^{i}_{A} \approx 0$, $\bar{\phi}^{A}_{i} \approx 0$, $\phi^{(\mathsf{b})}_{\hat{i}} \approx 0$, $\phi^{(-)} \approx 0$, $\upsilon^{(+)} \approx 0$, $\tilde{\chi}^{(\mathsf{b})}_{\hat{i}} \approx 0$, $\tilde{\chi}^{(-)} \approx 0$, and $\chi^{(\varphi)} \approx 0$ are second-class constraints. Following Dirac's approach to second-class constraints, we define the Dirac bracket by using the largest invertible submatrix of the matrix (4.1.24). For arbitrary smooth functions f and g of the canonical variables, the Dirac bracket is defined by

$$
\begin{aligned}
\{f,g\}_{\mathrm{D}} := \{f,g\} &+ i\left\{f,\phi^{i}_{A}\right\}\left\{\bar{\phi}^{A}_{i},g\right\} - i\left\{f,\bar{\phi}^{A}_{i}\right\}\left\{\phi^{i}_{A},g\right\} \\
&- \mathsf{e}\left\{f,\tilde{\chi}^{(\mathsf{b})}_{\hat{i}}\right\}\left\{\phi^{(\mathsf{b})}_{\hat{i}},g\right\} + \mathsf{e}\left\{f,\phi^{(\mathsf{b})}_{\hat{i}}\right\}\left\{\tilde{\chi}^{(\mathsf{b})}_{\hat{i}},g\right\} \\
&+ \left\{f,\chi^{(\varphi)}\right\}\left\{\phi^{(-)},g\right\} - \left\{f,\phi^{(-)}\right\}\left\{\chi^{(\varphi)},g\right\} \\
&+ \left\{f,\upsilon^{(+)}\right\}\left\{\tilde{\chi}^{(-)},g\right\} - \left\{f,\tilde{\chi}^{(-)}\right\}\left\{\upsilon^{(+)},g\right\} .
\end{aligned} \tag{4.1.25}
$$

The Dirac bracket between f and each of the constraint functions ϕ^{i}_{A}, $\bar{\phi}^{A}_{i}$, $\phi^{(\mathsf{b})}_{\hat{i}}$, $\phi^{(-)}$, $\upsilon^{(+)}$, $\tilde{\chi}^{(\mathsf{b})}_{\hat{i}}$, $\tilde{\chi}^{(-)}$, and $\chi^{(\varphi)}$ vanishes identically. For this reason, the second-class constraints can be set strongly equal to zero and may be expressed as $\phi^{i}_{A} = 0$, $\bar{\phi}^{A}_{i} = 0$, $\phi^{(\mathsf{b})}_{\hat{i}} = 0$, $\phi^{(-)} = 0$, $\upsilon^{(+)} = 0$, $\tilde{\chi}^{(\mathsf{b})}_{\hat{i}} = 0$, $\tilde{\chi}^{(-)} = 0$, and $\chi^{(\varphi)} = 0$, as long as the Dirac bracket $\{f,g\}_{\mathrm{D}}$ is adopted. We see that the second-class constraints lead to

$$
\begin{aligned}
P^i_A &= \frac{i}{2}\bar{\mathsf{Z}}^i_A\,, & \bar{P}^A_i &= -\frac{i}{2}\mathsf{Z}^A_i\,, && \text{(4.1.26a)}\\
\mathsf{b}_{\hat{\imath}} &= \mathsf{e}\bar{\mathsf{Z}}^j_A \sigma_{\hat{\imath}j}{}^k \mathsf{Z}^A_k\,, & P^{(\mathsf{b})}_{\hat{\imath}} &= 0 && \text{(4.1.26b)}\\
h &= \mathsf{h}e^{-i\varphi}\,, & P^{(h)} &= e^{i\varphi}P^{(\mathsf{h})}\,, && \text{(4.1.26c)}\\
\bar{h} &= \mathsf{h}e^{i\varphi}\,, & P^{(\bar{h})} &= e^{-i\varphi}P^{(\mathsf{h})}\,, && \text{(4.1.26d)}\\
\varphi &= -\frac{i}{2}\ln\!\left(\frac{\epsilon^{ij}\varpi_{i\dot{\alpha}}\varpi^{\dot{\alpha}}_j}{\epsilon_{ij}\bar{\varpi}^i_\alpha\bar{\varpi}^{j\alpha}}\right), & P^{(\varphi)} &= -\frac{1}{2}\chi^{(a)}\,, && \text{(4.1.26e)}
\end{aligned}
$$

where $\mathsf{h}=\mathsf{h}(\tau)$ is a real scalar-density field of weight 1 on $\mathcal{T}$, and $P^{(\mathsf{h})}$ its associated momentum variable. At this stage, P^i_A, $\bar{P}^A_i$, $\mathsf{b}^{\hat{\imath}}$, $P^{(\mathsf{b})}_{\hat{\imath}}$, h, $P^{(h)}$, $\bar{h}$, $P^{(\bar{h})}$, φ, and $P^{(\varphi)}$ are treated as dependent variables specified by Eq. (4.1.26), while the other canonical variables Z^A_i, $\bar{\mathsf{Z}}^i_A$, a, $P^{(a)}$, b^3, $P^{(\mathsf{b})}_3$, e, $P^{(\mathsf{e})}$, h, and $P^{(\mathsf{h})}$ are treated as independent variables. By virtue of the strong equalities of the second-class constraints, the set of all the first-class constraints, i.e,

$$
\left(\phi^{(a)}, \phi^{(\mathsf{b})}_3, \tilde{\phi}^{(\mathsf{e})}, \phi^{(+)}, \upsilon^{(-)}, \tilde{\chi}^{(\mathsf{b})}_3, \tilde{\chi}^{(\mathsf{e})}, \tilde{\chi}^{(+)}\right) \approx 0\,, \tag{4.1.27}
$$

turns out to be equivalent to the set consisting of

$$
\begin{aligned}
&\phi^{(a)} \approx 0\,, && \text{(4.1.28a)}\\
&\phi^{(\mathsf{b})}_3 \approx 0\,, && \text{(4.1.28b)}\\
&\phi^{(\mathsf{e})} \approx 0\,, && \text{(4.1.28c)}\\
&\phi^{(\mathsf{h})} := P^{(\mathsf{h})} \approx 0\,, && \text{(4.1.28d)}\\
&\chi^{(a)} \approx 0\,, && \text{(4.1.28e)}\\
&\chi^{(\mathsf{b})}_3 \approx 0\,, && \text{(4.1.28f)}\\
&\chi^{(\mathsf{e})} \approx 0\,, && \text{(4.1.28g)}\\
&\chi^{(h)} \approx 0\,, && \text{(4.1.28h)}\\
&\chi^{(\bar{h})} \approx 0\,. && \text{(4.1.28i)}
\end{aligned}
$$

Here we have taken into account both of Eqs. (4.1.28h) and (4.1.28i) for later convenience, although it is sufficient to consider one of them in actuality.

The Dirac brackets between the spinor components of Z^A_i and $\bar{\mathsf{Z}}^i_A$ are found from Eq. (4.1.25) to be

$$
\begin{aligned}
\left\{\varrho_i^\alpha, \varrho_j^\beta\right\}_{\mathrm{D}} &= \frac{i}{4\sqrt{2}m} e^{i\varphi}\left(\varrho_i^\alpha \epsilon_{jk}\bar{\varpi}^{k\beta} - \varrho_j^\beta \epsilon_{ik}\bar{\varpi}^{k\alpha}\right), \\
\left\{\varrho_i^\alpha, \varpi_{j\dot{\beta}}\right\}_{\mathrm{D}} &= -\frac{i}{4\sqrt{2}m} e^{i\varphi}\epsilon_{ik}\bar{\varpi}^{k\alpha}\varpi_{j\dot{\beta}}, \\
\left\{\varpi_{i\dot{\alpha}}, \varpi_{j\dot{\beta}}\right\}_{\mathrm{D}} &= 0, \\
\left\{\varrho_i^\alpha, \bar{\varrho}^{j\dot{\beta}}\right\}_{\mathrm{D}} &= \frac{i}{4\sqrt{2}m}\left(e^{i\varphi}\epsilon_{ik}\bar{\varpi}^{k\alpha}\bar{\varrho}^{j\dot{\beta}} + e^{-i\varphi}\varrho_i^\alpha \epsilon^{jk}\varpi_k^{\dot{\beta}}\right), \\
\left\{\varrho_i^\alpha, \bar{\varpi}_\beta^j\right\}_{\mathrm{D}} &= -i\delta_i^j\delta_\beta^\alpha + \frac{i}{4\sqrt{2}m} e^{i\varphi}\epsilon_{ik}\bar{\varpi}^{k\alpha}\bar{\varpi}_\beta^j, \\
\left\{\varpi_{i\dot{\alpha}}, \bar{\varrho}^{j\dot{\beta}}\right\}_{\mathrm{D}} &= -i\delta_i^j\delta_{\dot{\alpha}}^{\dot{\beta}} + \frac{i}{4\sqrt{2}m} e^{-i\varphi}\varpi_{i\dot{\alpha}}\epsilon^{jk}\varpi_k^{\dot{\beta}}, \\
\left\{\varpi_{i\dot{\alpha}}, \bar{\varpi}_\beta^j\right\}_{\mathrm{D}} &= 0, \\
\left\{\bar{\varrho}^{i\dot{\alpha}}, \bar{\varrho}^{j\dot{\beta}}\right\}_{\mathrm{D}} &= -\frac{i}{4\sqrt{2}m} e^{-i\varphi}\left(\bar{\varrho}^{i\dot{\alpha}}\epsilon^{jk}\varpi_k^{\dot{\beta}} - \bar{\varrho}^{j\dot{\beta}}\epsilon^{ik}\varpi_k^{\dot{\alpha}}\right), \\
\left\{\bar{\varrho}^{i\dot{\alpha}}, \bar{\varpi}_\beta^j\right\}_{\mathrm{D}} &= \frac{i}{4\sqrt{2}m} e^{-i\varphi}\epsilon^{ik}\varpi_k^{\dot{\alpha}}\bar{\varpi}_\beta^j, \\
\left\{\bar{\varpi}_\alpha^i, \bar{\varpi}_\beta^j\right\}_{\mathrm{D}} &= 0.
\end{aligned}
\tag{4.1.29}
$$

Using Eq. (4.1.29) and taking into account Eqs. (4.1.28h) and (4.1.28i), we can show that

$$
\begin{aligned}
\left\{\chi^{(a)}, \varrho_i^\alpha\right\}_{\mathrm{D}} &= \frac{i}{2}\varrho_i^\alpha, \qquad \left\{\chi^{(a)}, \varpi_{i\dot{\alpha}}\right\}_{\mathrm{D}} = \frac{i}{2}\varpi_{i\dot{\alpha}}, \\
\left\{\chi^{(a)}, \bar{\varpi}_\alpha^i\right\}_{\mathrm{D}} &= -\frac{i}{2}\bar{\varpi}_\alpha^i, \quad \left\{\chi^{(a)}, \bar{\varrho}^{i\dot{\alpha}}\right\}_{\mathrm{D}} = -\frac{i}{2}\bar{\varrho}^{i\dot{\alpha}}.
\end{aligned}
\tag{4.1.30}
$$

Many of the Dirac brackets in Eq. (4.1.29) are rather complicated. Fortunately, however, Eq. (4.1.29) can be expressed in the form of simple canonical brackets as

$$
\begin{aligned}
&\left\{\rho_i^\alpha, \bar{\varpi}_\beta^j\right\}_{\mathrm{D}} = -i\delta_i^j\delta_\beta^\alpha, \quad \left\{\varpi_{i\dot{\alpha}}, \bar{\rho}^{j\dot{\beta}}\right\}_{\mathrm{D}} = -i\delta_i^j\delta_{\dot{\alpha}}^{\dot{\beta}}, \\
&\text{all others} = 0,
\end{aligned}
\tag{4.1.31}
$$

in terms of $\varpi_{i\dot{\alpha}}$, $\bar{\varpi}_\alpha^i$, and

$$
\rho_i^\alpha := \varrho_i^\alpha + \frac{1}{2\sqrt{2}m} e^{i\varphi}\epsilon_{ij}\bar{\varpi}^{j\alpha}\chi^{(a)}, \tag{4.1.32a}
$$

$$
\bar{\rho}^{i\dot{\alpha}} := \bar{\varrho}^{i\dot{\alpha}} + \frac{1}{2\sqrt{2}m} e^{-i\varphi}\epsilon^{ij}\varpi_j^{\dot{\alpha}}\chi^{(a)}. \tag{4.1.32b}
$$

In showing this fact, it is convenient to use Eqs. (4.1.28e) and (4.1.30). Note here that the weak equalities $\rho_i^\alpha \approx \varrho_i^\alpha$, $\bar{\rho}^{i\dot{\alpha}} \approx \bar{\varrho}^{i\dot{\alpha}}$ hold owing to Eq. (4.1.28e). Now we define the new twistors $\mathsf{W}_i^A := (\rho_i^\alpha, \varpi_{i\dot{\alpha}})$ and $\bar{\mathsf{W}}_A^i := (\bar{\varpi}_\alpha^i, \bar{\rho}^{i\dot{\alpha}})$, with which Eq. (4.1.31) can concisely be written as

$$\begin{aligned} &\{\mathsf{W}_i^A, \bar{\mathsf{W}}_B^j\}_{\mathrm{D}} = -i\delta_i^j\delta_B^A\,, \\ &\left\{\mathsf{W}_i^A, \mathsf{W}_j^B\right\}_{\mathrm{D}} = 0\,, \quad \{\bar{\mathsf{W}}_A^i, \bar{\mathsf{W}}_B^j\}_{\mathrm{D}} = 0\,. \end{aligned} \tag{4.1.33}$$

Using Eqs. (4.1.28h), (4.1.28i), and the formulas given under Eq. (4.1.17), we can show for

$$\check{\chi}^{(a)} := \bar{\mathsf{W}}_A^i \mathsf{W}_i^A - 2s\,, \tag{4.1.34a}$$

$$\check{\chi}_3^{(\mathsf{b})} := \bar{\mathsf{W}}_A^j \sigma_{3j}{}^k \mathsf{W}_k^A - 2t \tag{4.1.34b}$$

that

$$\check{\chi}^{(a)} = 2\chi^{(a)}, \tag{4.1.35a}$$

$$\check{\chi}_3^{(\mathsf{b})} = \chi_3^{(\mathsf{b})}. \tag{4.1.35b}$$

Accordingly, the first-class constraints (4.1.28e) and (4.1.28f) read

$$\check{\chi}^{(a)} \approx 0\,, \tag{4.1.36a}$$

$$\check{\chi}_3^{(\mathsf{b})} \approx 0\,. \tag{4.1.36b}$$

With Eq. (4.1.35a), Eqs. (4.1.32a) and (4.1.32b) can be solved inversely as

$$\varrho_i^\alpha = \rho_i^\alpha - \frac{1}{4\sqrt{2}m} e^{i\varphi} \epsilon_{ij} \bar{\varpi}^{j\alpha} \check{\chi}^{(a)}, \tag{4.1.37a}$$

$$\bar{\varrho}^{i\dot{\alpha}} = \bar{\rho}^{i\dot{\alpha}} - \frac{1}{4\sqrt{2}m} e^{-i\varphi} \epsilon^{ij} \varpi_j^{\dot{\alpha}} \check{\chi}^{(a)}. \tag{4.1.37b}$$

Hence it follows that there is a one-to-one correspondence between $(\mathsf{Z}_i^A, \bar{\mathsf{Z}}_A^i)$ and $(\mathsf{W}_i^A, \bar{\mathsf{W}}_A^i)$. Taking into account this fact, we hereafter adopt W_i^A and $\bar{\mathsf{W}}_A^i$ as canonical variables instead of Z_i^A and $\bar{\mathsf{Z}}_A^i$. The first equation in Eq. (4.1.26b) can be written as $\mathsf{b}_{\hat{\imath}} = \mathsf{e}\bar{\mathsf{W}}_A^j \sigma_{\hat{\imath}j}{}^k \mathsf{W}_k^A$. Substituting this into Eq. (4.1.11d), we see that the first-class constraint $\chi^{(\mathsf{e})} \approx 0$ can be expressed as

$$\check{\chi}^{(\mathsf{e})} := \mathsf{T}_{\hat{\imath}} \mathsf{T}_{\hat{\imath}} - \frac{1}{4}k^2 \approx 0\,, \tag{4.1.38}$$

where $\mathsf{T}_{\hat{\imath}}$ ($\hat{\imath} = 1, 2$) are defined in

$$\mathsf{T}_0 := \frac{1}{2}\bar{\mathsf{W}}^i_A \mathsf{W}^A_i\,, \quad \mathsf{T}_r := \frac{1}{2}\bar{\mathsf{W}}^j_A \sigma_{rj}{}^k \mathsf{W}^A_k\,. \tag{4.1.39}$$

Using Eq. (4.1.33), we can readily verify that T_0 and T_r constitute a bases of the $U(1)_a \times SU(2)$ Lie algebra in the following sense:

$$\{\mathsf{T}_0, \mathsf{T}_r\}_{\mathrm{D}} = 0\,, \quad \{\mathsf{T}_r, \mathsf{T}_s\}_{\mathrm{D}} = \epsilon_{rst}\mathsf{T}_t\,. \tag{4.1.40}$$

The canonical variables that we need to consider at the present stage are W^A_i, $\bar{\mathsf{W}}^i_A$, a, $P^{(a)}$, b^3, $P^{(\mathsf{b})}_3$, e, $P^{(\mathsf{e})}$, h, and $P^{(\mathsf{h})}$. All the Dirac brackets between these variables are given in Eq. (4.1.33) and

$$\begin{aligned} &\left\{a, P^{(a)}\right\}_{\mathrm{D}} = 1\,, \quad \left\{\mathsf{b}^3, P^{(\mathsf{b})}_3\right\}_{\mathrm{D}} = 1\,, \\ &\left\{\mathsf{e}, P^{(\mathsf{e})}\right\}_{\mathrm{D}} = 1\,, \quad \left\{\mathsf{h}, P^{(\mathsf{h})}\right\}_{\mathrm{D}} = \frac{1}{2}\,, \\ &\text{all others} = 0\,. \end{aligned} \tag{4.1.41}$$

We also need to consider the first class constraints (4.1.28a)–(4.1.28d), (4.1.36a), (4.1.36b), (4.1.38), (4.1.28h), and (4.1.28i).

4.2 Canonical quantization

In this section, we perform the canonical quantization of the Hamiltonian system studied in Sec. III. To this end, in accordance with Dirac's method of quantization, we introduce the operators $\hat{f}$ and $\hat{g}$ corresponding, respectively, to the functions f and g, and set the commutation relation

$$\left[\hat{f}, \hat{g}\right] = i\,\widehat{\{f, g\}}_{\mathrm{D}} \tag{4.2.1}$$

in units such that $\hbar = 1$. Here, $\widehat{\{f, g\}}_{\mathrm{D}}$ denotes the operator corresponding to the Dirac bracket $\{f, g\}_{\mathrm{D}}$. From Eqs. (4.1.33), (4.1.41), and (4.2.1), we have the canonical commutation relations

$$\left[\hat{\mathsf{W}}_i^A, \hat{\bar{\mathsf{W}}}_B^j\right] = \delta_i^j \delta_B^A \,, \tag{4.2.2a}$$

$$\left[\hat{\mathsf{W}}_i^A, \hat{\mathsf{W}}_j^B\right] = 0 \,, \qquad \left[\hat{\bar{\mathsf{W}}}_A^i, \hat{\bar{\mathsf{W}}}_B^j\right] = 0 \,, \tag{4.2.2b}$$

$$\left[\hat{a}, \hat{P}^{(a)}\right] = i \,, \qquad \left[\hat{\mathsf{b}}^3, \hat{P}_3^{(\mathsf{b})}\right] = i \,, \tag{4.2.2c}$$

$$\left[\hat{\mathsf{e}}, \hat{P}^{(\mathsf{e})}\right] = i \,, \qquad \left[\hat{\mathsf{h}}, \hat{P}^{(\mathsf{h})}\right] = \frac{i}{2} \,, \tag{4.2.2d}$$

$$\text{all others} = 0 \,. \tag{4.2.2e}$$

The commutation relations (4.2.2a) and (4.2.2b) govern together so-called twistor quantization [2, 3].

In the procedure of canonical quantization, the first-class constraints are treated as conditions imposed on the physical states, after the replacement of the first-class constraint functions by the corresponding operators. In the present model, the physical state conditions are found from Eqs. (4.1.28a)–(4.1.28d), (4.1.36a), (4.1.36b), (4.1.38), (4.1.28h), and (4.1.28i) to be

$$\hat{\phi}^{(a)}|F\rangle = \hat{P}^{(a)}|F\rangle = 0 \,, \tag{4.2.3a}$$

$$\hat{\phi}_3^{(\mathsf{b})}|F\rangle = \hat{P}_3^{(\mathsf{b})}|F\rangle = 0 \,, \tag{4.2.3b}$$

$$\hat{\phi}^{(\mathsf{e})}|F\rangle = \hat{P}^{(\mathsf{e})}|F\rangle = 0 \,, \tag{4.2.3c}$$

$$\hat{\phi}^{(\mathsf{h})}|F\rangle = \hat{P}^{(\mathsf{h})}|F\rangle = 0 \,, \tag{4.2.3d}$$

$$\begin{aligned} \hat{\chi}^{(a)}|F\rangle &= \left[\frac{1}{2}\Big(\hat{\bar{\mathsf{W}}}_A^i \hat{\mathsf{W}}_i^A + \hat{\mathsf{W}}_i^A \hat{\bar{\mathsf{W}}}_A^i\Big) - 2s\right]|F\rangle \\ &= 2\Big(\hat{\mathsf{T}}_0 - s - 2\Big)|F\rangle = 0 \,, \end{aligned} \tag{4.2.3e}$$

$$\begin{aligned} \hat{\chi}_3^{(\mathsf{b})}|F\rangle &= \left[\frac{1}{2}\Big(\hat{\bar{\mathsf{W}}}_A^j \sigma_{3j}{}^k \hat{\mathsf{W}}_k^A + \hat{\mathsf{W}}_k^A \sigma_{3j}{}^k \hat{\bar{\mathsf{W}}}_A^j\Big) - 2t\right]|F\rangle \\ &= 2\Big(\hat{\mathsf{T}}_3 - t\Big)|F\rangle = 0 \,, \end{aligned} \tag{4.2.3f}$$

$$\hat{\chi}^{(\mathsf{e})}|F\rangle = \left(\hat{\mathsf{T}}_{\hat{\imath}}\hat{\mathsf{T}}_{\hat{\imath}} - \frac{1}{4}k^2\right)|F\rangle = 0 \,, \tag{4.2.3g}$$

$$\hat{\chi}^{(h)}|F\rangle = \left(\epsilon^{ij}\hat{\varpi}_{i\dot{\alpha}}\hat{\varpi}_j^{\dot{\alpha}} - \sqrt{2}me^{i\hat{\varphi}}\right)|F\rangle = 0 \,, \tag{4.2.3h}$$

$$\hat{\chi}^{(\bar{h})}|F\rangle = \left(\epsilon_{ij}\hat{\bar{\varpi}}_\alpha^i \hat{\bar{\varpi}}^{j\alpha} - \sqrt{2}me^{-i\hat{\varphi}}\right)|F\rangle = 0 \,. \tag{4.2.3i}$$

Here, $|F\rangle$ denotes a physical state, $\hat{\mathsf{T}}_0$ and $\hat{\mathsf{T}}_r$ $(r = \hat{\imath}, 3)$ are defined by

$$\hat{\mathsf{T}}_0 := \frac{1}{2}\hat{\mathsf{W}}_i^A \hat{\bar{\mathsf{W}}}_A^i \,, \quad \hat{\mathsf{T}}_r := \frac{1}{2}\sigma_{rj}{}^k \hat{\mathsf{W}}_k^A \hat{\bar{\mathsf{W}}}_A^j \,, \tag{4.2.4}$$

and $\hat{\varphi}$ is defined according to the first equation in Eq. (4.1.26e) as

$$\hat{\varphi} := -\frac{i}{2}\left[\ln\!\left(\epsilon^{ij}\hat{\varpi}_{i\dot{\alpha}}\hat{\varpi}_j^{\dot{\alpha}}\right) - \ln\!\left(\epsilon_{ij}\hat{\bar{\varpi}}_\alpha^i \hat{\bar{\varpi}}^{j\alpha}\right)\right] . \tag{4.2.5}$$

In defining the operators $\hat{\check{\chi}}^{(a)}$, $\hat{\check{\chi}}_3^{(\mathsf{b})}$, and $\hat{\check{\chi}}^{(\mathsf{e})}$, we have obeyed the Weyl ordering rule and have used the commutation relation (4.2.2a) to simplify the Weyl ordered operators. Using Eqs. (4.2.2a) and (4.2.2b), we can easily show that

$$\Big[\hat{\mathsf{T}}_0, \hat{\mathsf{T}}_r\Big] = 0 \,, \quad \Big[\hat{\mathsf{T}}_r, \hat{\mathsf{T}}_s\Big] = i\epsilon_{rst}\hat{\mathsf{T}}_t \,, \tag{4.2.6}$$

which is precisely the quantum mechanical counterpart of Eq. (4.1.40). It is evident that $\hat{\mathsf{T}}_0$ is the generator of $U(1)_a$ and $\hat{\mathsf{T}}_r$ $(r = 1, 2, 3)$ are the generators of $SU(2)$. In particular, $\hat{\mathsf{T}}_3$ is the generator of $U(1)_{\mathsf{b}}$.

Now we introduce the bra-vector

$$\begin{aligned} &\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \\ &\quad := \langle 0|\exp\!\Big(-\mathsf{W}_i^A \hat{\bar{\mathsf{W}}}_A^i + ia\hat{P}^{(a)} + i\mathsf{b}^3\hat{P}_3^{(\mathsf{b})} + i\mathsf{e}\hat{P}^{(\mathsf{e})} + 2i\mathsf{h}\hat{P}^{(\mathsf{h})}\Big) \end{aligned} \tag{4.2.7}$$

with a reference bra-vector $\langle 0|$ satisfying

$$\langle 0|\hat{\mathsf{W}}_i^A = \langle 0|\hat{a} = \langle 0|\hat{\mathsf{b}}^3 = \langle 0|\hat{\mathsf{e}} = \langle 0|\hat{\mathsf{h}} = 0 \,. \tag{4.2.8}$$

Using the commutation relations (4.2.2a)–(4.2.2e), we can show that

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,|\hat{\mathsf{W}}_i^A = \mathsf{W}_i^A \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \,, \tag{4.2.9a}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,|\hat{a} = a\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \,, \tag{4.2.9b}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,|\hat{\mathsf{b}}^3 = \mathsf{b}^3\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \,, \tag{4.2.9c}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,|\hat{\mathsf{e}} = \mathsf{e}\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \,, \tag{4.2.9d}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,|\hat{\mathsf{h}} = \mathsf{h}\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \,. \tag{4.2.9e}$$

Equation (4.2.9a) can be decomposed into two parts:

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,|\hat{\rho}_i^\alpha = \rho_i^\alpha \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \,, \tag{4.2.10a}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,|\hat{\varpi}_{i\dot{\alpha}} = \varpi_{i\dot{\alpha}} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}\,| \,. \tag{4.2.10b}$$

Also, it is easy to see that

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, \hat{\hat{\mathsf{W}}}{}^i_A = -\frac{\partial}{\partial \mathsf{W}_i^A} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,| \,, \tag{4.2.11a}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, \hat{P}^{(a)} = -i\frac{\partial}{\partial a} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,| \,, \tag{4.2.11b}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, \hat{P}^{(\mathsf{b})}_3 = -i\frac{\partial}{\partial \mathsf{b}^3} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,| \,, \tag{4.2.11c}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, \hat{P}^{(\mathsf{e})} = -i\frac{\partial}{\partial \mathsf{e}} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,| \,, \tag{4.2.11d}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, \hat{P}^{(\mathsf{h})} = -\frac{i}{2}\frac{\partial}{\partial \mathsf{h}} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,| \,. \tag{4.2.11e}$$

Equation (4.2.11a) can be decomposed into two parts:

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, \hat{\hat{\varpi}}{}^i_\alpha = -\frac{\partial}{\partial \rho_i^\alpha} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,| \,, \tag{4.2.12a}$$

$$\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, \hat{\hat{\rho}}{}^{i\dot{\alpha}} = -\frac{\partial}{\partial \varpi_{i\dot{\alpha}}} \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,| \,. \tag{4.2.12b}$$

Multiplying each of Eqs. (4.2.3a)–(4.2.3i) by $\langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|$ on the left and using Eqs. (4.2.9)–(4.2.12), we obtain a set of simultaneous differential equations for $F(\mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h}) := \langle \mathsf{W}, a, \mathsf{b}^3, \mathsf{e}, \mathsf{h} \,|\, F \rangle$ as follows:

$$\frac{\partial}{\partial a} F = 0 \,, \tag{4.2.13a}$$

$$\frac{\partial}{\partial \mathsf{b}^3} F = 0 \,, \tag{4.2.13b}$$

$$\frac{\partial}{\partial \mathsf{e}} F = 0 \,, \tag{4.2.13c}$$

$$\frac{\partial}{\partial \mathsf{h}} F = 0 \,, \tag{4.2.13d}$$

$$\check{\mathsf{T}}_0 F = (s+2) F \,, \tag{4.2.13e}$$

$$\check{\mathsf{T}}_3 F = t F \,, \tag{4.2.13f}$$

$$\check{\mathsf{T}}_{\hat{\imath}} \check{\mathsf{T}}_{\hat{\imath}} F = \frac{1}{4} k^2 F \,, \tag{4.2.13g}$$

$$\epsilon^{ij} \varpi_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} F = \sqrt{2} m e^{i\check{\varphi}} F \,, \tag{4.2.13h}$$

$$\epsilon_{ij} \epsilon^{\alpha\beta} \frac{\partial}{\partial \rho_i^\alpha} \frac{\partial}{\partial \rho_j^\beta} F = \sqrt{2} m e^{-i\check{\varphi}} F \,. \tag{4.2.13i}$$

Here, $\check{\mathsf{T}}_0$ and $\check{\mathsf{T}}_r$ $(r = \hat{\imath}, 3)$ are defined by

$$\check{\mathsf{T}}_0 := -\frac{1}{2} \mathsf{W}_i^A \frac{\partial}{\partial \mathsf{W}_i^A} \,, \quad \check{\mathsf{T}}_r := -\frac{1}{2} \sigma_{rj}{}^k \mathsf{W}_k^A \frac{\partial}{\partial \mathsf{W}_j^A} \,, \tag{4.2.14}$$

and $\check{\varphi}$ is defined by

$$\check{\varphi} := -\frac{i}{2}\left[\ln\left(\epsilon^{ij}\varpi_{i\dot{\alpha}}\varpi_j^{\dot{\alpha}}\right) - \ln\left(\epsilon_{ij}\epsilon^{\alpha\beta}\frac{\partial}{\partial\rho_i^\alpha}\frac{\partial}{\partial\rho_j^\beta}\right)\right]. \tag{4.2.15}$$

Equations (4.2.13a)–(4.2.13d) imply that F is actually independent of a, b^3, e, and h. Hence it follows that F is a function of the twistors W_i^A only. The holomorphic functions of W_i^A, such as F, are often referred to as the twistor functions. As can be seen immediately, Eqs. (4.2.13h) and (4.2.13i) are respectively equivalent to

$$\varpi_{i\dot{\alpha}}\varpi_j^{\dot{\alpha}}F = \frac{m}{\sqrt{2}}\epsilon_{ij}e^{i\check{\varphi}}F\,, \tag{4.2.16a}$$

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\rho_i^\alpha}\frac{\partial}{\partial\rho_j^\beta}F = \frac{m}{\sqrt{2}}\epsilon^{ij}e^{-i\check{\varphi}}F\,. \tag{4.2.16b}$$

Combining Eqs. (4.2.13e) and (4.2.13f), we have

$$\mathsf{W}_1^A\frac{\partial}{\partial\mathsf{W}_1^A}F = -2(s_1+1)F\,, \tag{4.2.17a}$$

$$\mathsf{W}_2^A\frac{\partial}{\partial\mathsf{W}_2^A}F = -2(s_2+1)F\,, \tag{4.2.17b}$$

where

$$s_1 := \frac{1}{2}(s+t)\,, \quad s_2 := \frac{1}{2}(s-t)\,. \tag{4.2.18}$$

The pair of Eqs. (4.2.13e) and (4.2.13f) is equivalent to the pair of Eqs. (4.2.17a) and (4.2.17b). Obviously, Eqs. (4.2.17a) and (4.2.17b) are simultaneously satisfied by a homogeneous twistor function of degree $-2s_1-2$ w.r.t. W_1^A and degree $-2s_2-2$ w.r.t. W_2^A. These degrees must be integers so that F can be a single-valued function of W_i^A. In this way, the allowed values of s_1 and s_2 are restricted to arbitrary integer or half-integer values, and accordingly s and t are also restricted to arbitrary integer or half-integer values. We thus see that the Chern-Simons coefficients $2s$ and $2t$, which are coefficients of the 1-dimensional Chern-Simons terms S_a and $S_{\mathsf{b}3}$, respectively, are quantized to be arbitrary integer values.

The operators $\check{\mathsf{T}}_r$ fulfill the $SU(2)$ commutation relation

$$\left[\check{\mathsf{T}}_r, \check{\mathsf{T}}_s\right] = i\epsilon_{rst}\check{\mathsf{T}}_t\,. \tag{4.2.19}$$

Following the general method for solving the eigenvalue problem in the $SU(2)$ Lie algebra [39], we can simultaneously solve the eigenvalue equation for the Casimir operator $\check{\mathsf{T}}_r\check{\mathsf{T}}_r = \check{\mathsf{T}}_{\hat{\imath}}\check{\mathsf{T}}_{\hat{\imath}} + \check{\mathsf{T}}_3\check{\mathsf{T}}_3$, i.e.,

$$\check{\mathsf{T}}_r\check{\mathsf{T}}_r F = \Lambda F\,, \tag{4.2.20}$$

and Eq. (4.2.13f) to obtain

$$\Lambda = I(I+1)\,, \quad I = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \tag{4.2.21a}$$

$$t = -I, -I+1, \dots, I-1, I\,. \tag{4.2.21b}$$

In deriving Eqs. (4.2.21a) and (4.2.21b), we assume the existence of a positive-definite inner product in the function space consisting of twistor functions. (As for the twistor formulation of a massless system, a twistor-function space with a positive-definite inner product has been established [40].) Since t takes integer or half-integer values as explained above, I also takes integer or half-integer values accordingly. From Eqs. (4.2.13f), (4.2.13g), (4.2.20), and (4.2.21a), the allowed values of the positive constant k are determined to be

$$k = 2\sqrt{I(I+1) - t^2}\,. \tag{4.2.22}$$

In this way, the coefficient of $S_{\mathrm{b}12}$ is also quantized in addition to the Chern-Simons coefficients. It is now clear that the twistor function F is characterized by the set of three quantum numbers (s, I, t), or equivalently, by (I, s_1, s_2); for this reason, it is convenient to label F as $F_{s,I,t}$ or F_{I,s_1,s_2}.

4.3 Penrose transform and the generalized DFP equation

In this section, we define a spinor field of arbitrary rank by the Penrose transform of F_{I,s_1,s_2}. We also demonstrate that this spinor field satisfies generalized DFP equations with $SU(2)$ indices.

Let us consider the Penrose transform of F_{I,s_1,s_2} specified by

$$\begin{aligned}&\Psi^{i_1\dots i_p}_{\alpha_1\dots\alpha_p;j_1\dots j_q,\dot{\alpha}_1\dots\dot{\alpha}_q}(z)\\ &= \frac{1}{(2\pi i)^4}\oint_{\Sigma} e^{ip\check{\varphi}}\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\frac{\partial}{\partial\rho^{\alpha_1}_{i_1}}\cdots\frac{\partial}{\partial\rho^{\alpha_p}_{i_p}}F_{I,s_1,s_2}(\mathsf{W})\,d^4\varpi\end{aligned} \tag{4.3.1}$$

with

$$d^4\varpi := d\varpi_{1\dot{0}} \wedge d\varpi_{1\dot{1}} \wedge d\varpi_{2\dot{0}} \wedge d\varpi_{2\dot{1}} \tag{4.3.2}$$

to define the rank-$(p+q)$ spinor field $\mathit{\Psi}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}$ (occasionally abbreviated as $\mathit{\Psi}$) on complexified Minkowski space $\mathbb{C}\mathbf{M}$. Here, $\mathit{\Sigma}$ denotes a suitable four-dimensional contour. Equation (4.3.1) is identified as a non-projective form of the Penrose transform in the massive case [3].[1] It should be noted that $\mathit{\Psi}$ has the upper and lower $SU(2)$ indices in addition to the dotted and undotted spinor indices. Because of the structure of Eq. (4.3.1), the number of upper (lower) $SU(2)$ indices is equal to the number of undotted (dotted) spinor indices. It is obvious that $\mathit{\Psi}$ has the symmetric properties

$$\mathit{\Psi}^{i_1 \dots i_m \dots i_n \dots i_p}_{\alpha_1 \dots \alpha_m \dots \alpha_n \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q} = \mathit{\Psi}^{i_1 \dots i_n \dots i_m \dots i_p}_{\alpha_1 \dots \alpha_n \dots \alpha_m \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q} \,, \tag{4.3.3a}$$

$$\mathit{\Psi}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p; j_1 \dots j_a \dots j_b \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_a \dots \dot{\alpha}_b \dots \dot{\alpha}_q} = \mathit{\Psi}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p; j_1 \dots j_b \dots j_a \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_b \dots \dot{\alpha}_a \dots \dot{\alpha}_q} \,. \tag{4.3.3b}$$

Suppose that among $i_1, \dots, i_p$, the number of 1's is p_1 and the number of 2's is $p_2 (= p - p_1)$. Similarly, suppose that among $j_1, \dots, j_q$, the number of 1's is q_1 and the number of 2's is $q_2 (= q - q_1)$. The integral in Eq. (4.3.1) can remain non-vanishing if

$$s_1 = \frac{1}{2}(q_1 - p_1) \,, \quad s_2 = \frac{1}{2}(q_2 - p_2) \,. \tag{4.3.4}$$

Combining Eqs. (4.2.18) and (4.3.4), we have

$$s = \frac{1}{2}(q_1 - p_1 + q_2 - p_2) \,, \tag{4.3.5a}$$

$$t = \frac{1}{2}(q_1 - p_1 - q_2 + p_2) \,. \tag{4.3.5b}$$

[1] The two-dimensional projective form of the Penrose transform (4.3.1) is given by

$$\begin{aligned} &\mathit{\Psi}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p; j_1 \dots j_q, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z) \\ &= \frac{1}{(2\pi i)^2} \oint_{\mathit{\Gamma}} e^{ip\check{\varphi}} \varpi_{j_1 \dot{\alpha}_1} \cdots \varpi_{j_q \dot{\alpha}_q} \\ &\quad \times \frac{\partial}{\partial \rho^{\alpha_1}_{i_1}} \cdots \frac{\partial}{\partial \rho^{\alpha_p}_{i_p}} F_{I, s_1, s_2}(\mathsf{W}) \varpi_{1\dot{\beta}} d\varpi^{\dot{\beta}}_1 \wedge \varpi_{2\dot{\gamma}} d\varpi^{\dot{\gamma}}_2 \,, \end{aligned}$$

where $\mathit{\Gamma}$ denotes a suitable two-dimensional contour [4]. We can also find the three-dimensional projective form of the Penrose transform (4.3.1) [11].

Now we can show that

$$\frac{\partial}{\partial z_{\beta\dot\beta}}F(\mathsf{W})=\frac{\partial\rho_k^\gamma}{\partial z_{\beta\dot\beta}}\frac{\partial}{\partial\rho_k^\gamma}F(\mathsf{W})=\frac{\partial\varrho_k^\gamma}{\partial z_{\beta\dot\beta}}\frac{\partial}{\partial\rho_k^\gamma}F(\mathsf{W})$$
$$=\frac{\partial\big(iz^{\gamma\dot\gamma}\varpi_{k\dot\gamma}\big)}{\partial z_{\beta\dot\beta}}\frac{\partial}{\partial\rho_k^\gamma}F(\mathsf{W})=i\varpi_k^{\dot\beta}\epsilon^{\beta\gamma}\frac{\partial}{\partial\rho_k^\gamma}F(\mathsf{W})\,. \tag{4.3.6}$$

Here the weak equality $\rho_j^\gamma\approx\varrho_j^\gamma$, Eq. (3.3.11), and the formula $\partial/\partial z_{\beta\dot\beta}=\epsilon^{\beta\alpha}\epsilon^{\dot\beta\dot\alpha}\partial/\partial z^{\alpha\dot\alpha}$ have been used. The derivative of $\varPsi$ w.r.t. $z_{\beta\dot\beta}$ can be calculated by using Eq. (4.3.6) as follows:

$$\frac{\partial}{\partial z_{\beta\dot\beta}}\varPsi^{i_1\ldots i_p}_{\alpha_1\ldots\alpha_p;j_1\ldots j_q,\dot\alpha_1\ldots\dot\alpha_q}(z)$$
$$=\frac{1}{(2\pi i)^4}\oint_\Sigma e^{ip\check\varphi}\varpi_{j_1\dot\alpha_1}\cdots\varpi_{j_q\dot\alpha_q}\frac{\partial}{\partial\rho_{i_1}^{\alpha_1}}\cdots\frac{\partial}{\partial\rho_{i_p}^{\alpha_p}}\frac{\partial}{\partial z_{\beta\dot\beta}}F_{I,s_1,s_2}(\mathsf{W})\,d^4\varpi$$
$$=\frac{i}{(2\pi i)^4}\oint_\Sigma e^{ip\check\varphi}\varpi_{j_1\dot\alpha_1}\varpi_k^{\dot\beta}\varpi_{j_2\dot\alpha_2}\cdots\varpi_{j_q\dot\alpha_q}\frac{\partial}{\partial\rho_{i_2}^{\alpha_2}}\cdots\frac{\partial}{\partial\rho_{i_p}^{\alpha_p}}\epsilon^{\beta\gamma}\frac{\partial}{\partial\rho_{i_1}^{\alpha_1}}\frac{\partial}{\partial\rho_k^\gamma}F_{I,s_1,s_2}(\mathsf{W})\,d^4\varpi\,. \tag{4.3.7}$$

Contracting over the indices $\dot\beta$ and $\dot\alpha_1$ in Eq. (4.3.7) and using Eq. (4.2.16a), we obtain

$$\frac{\partial}{\partial z_{\beta\dot\beta}}\varPsi^{i_1\ldots i_p}_{\alpha_1\ldots\alpha_p;j_1\ldots j_q,\dot\beta\dot\alpha_2\ldots\dot\alpha_q}(z)$$
$$=\frac{m}{\sqrt2}\epsilon^{\beta\gamma}\epsilon_{j_1k}\frac{i}{(2\pi i)^4}\oint_\Sigma e^{i(p+1)\check\varphi}\varpi_{j_2\dot\alpha_2}\cdots\varpi_{j_q\dot\alpha_q}\frac{\partial}{\partial\rho_k^\gamma}\frac{\partial}{\partial\rho_{i_1}^{\alpha_1}}\cdots\frac{\partial}{\partial\rho_{i_p}^{\alpha_p}}F_{I,s_1,s_2}(\mathsf{W})\,d^4\varpi$$
$$=\frac{im}{\sqrt2}\epsilon^{\beta\gamma}\epsilon_{j_1k}\varPsi^{ki_1\ldots i_p}_{\gamma\alpha_1\ldots\alpha_p;j_2\ldots j_q,\dot\alpha_2\ldots\dot\alpha_q}(z)\,. \tag{4.3.8}$$

Similarly, contracting over the indices β and α_1 in Eq. (4.3.7) and using Eq. (4.2.16b), we obtain

$$\frac{\partial}{\partial z_{\beta\dot\beta}}\varPsi^{i_1\ldots i_p}_{\beta\alpha_2\ldots\alpha_p;j_1\ldots j_q,\dot\alpha_1\ldots\dot\alpha_q}(z)$$
$$=\frac{m}{\sqrt2}\epsilon^{\dot\beta\dot\gamma}\epsilon^{i_1k}\frac{i}{(2\pi i)^4}\oint_\Sigma e^{i(p-1)\check\varphi}\varpi_{k\dot\gamma}\varpi_{j_1\dot\alpha_1}\cdots\varpi_{j_q\dot\alpha_q}\frac{\partial}{\partial\rho_{i_2}^{\alpha_2}}\cdots\frac{\partial}{\partial\rho_{i_p}^{\alpha_p}}F_{I,s_1,s_2}(\mathsf{W})\,d^4\varpi$$
$$=\frac{im}{\sqrt2}\epsilon^{\dot\beta\dot\gamma}\epsilon^{i_1k}\varPsi^{i_2\ldots i_p}_{\alpha_2\ldots\alpha_p;kj_1\ldots j_q,\dot\gamma\dot\alpha_1\ldots\dot\alpha_q}(z)\,. \tag{4.3.9}$$

In this way, it has been shown that the spinor field Ψ satisfies the generalized DFP equations with $SU(2)$ indices

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Psi^{i_1\ldots i_p}_{\alpha_1\ldots\alpha_p;j_1\ldots j_q,\dot{\beta}\dot{\alpha}_2\ldots\dot{\alpha}_q}+m\epsilon^{\beta\gamma}\epsilon_{j_1k}\Psi^{ki_1\ldots i_p}_{\gamma\alpha_1\ldots\alpha_p;j_2\ldots j_q,\dot{\alpha}_2\ldots\dot{\alpha}_q}=0\,, \tag{4.3.10a}$$

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Psi^{i_1\ldots i_p}_{\beta\alpha_2\ldots\alpha_p;j_1\ldots j_q,\dot{\alpha}_1\ldots\dot{\alpha}_q}+m\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon^{i_1k}\Psi^{i_2\ldots i_p}_{\alpha_2\ldots\alpha_p;kj_1\ldots j_q,\dot{\gamma}\dot{\alpha}_1\ldots\dot{\alpha}_q}=0\,. \tag{4.3.10b}$$

Using Eqs. (4.3.10a) and (4.3.10b) and noting

$$\frac{\partial}{\partial z^{\alpha\dot{\beta}}}\frac{\partial}{\partial z_{\beta\dot{\beta}}}=\frac{1}{2}\delta^{\beta}_{\alpha}\frac{\partial}{\partial z^{\gamma\dot{\gamma}}}\frac{\partial}{\partial z_{\gamma\dot{\gamma}}}\,, \tag{4.3.11}$$

we can derive the Klein-Gordon equation

$$\left(\frac{\partial}{\partial z^{\beta\dot{\beta}}}\frac{\partial}{\partial z_{\beta\dot{\beta}}}+m^2\right)\Psi^{i_1\ldots i_p}_{\alpha_1\ldots\alpha_p;j_1\ldots j_q,\dot{\alpha}_1\ldots\dot{\alpha}_q}=0\,. \tag{4.3.12}$$

This makes it clear that Ψ is a field of mass m. Thus, we obtain a spinor field of arbitrary rank with mass m by means of the Penrose transform (4.3.1).

4.4 Rank-one spinor fields and physical meanings of the gauge symmetries

In this section, we investigate the rank-one spinor fields in detail to clarify the physical meanings of the $U(1)_a$, $U(1)_{\mathsf{b}}$, and $SU(2)$ symmetries as well as those of the constants s and t.

Now we particularly consider Eq. (4.3.10a) in the case $(p,q)=(0,1)$ and Eq. (4.3.10b) in the case $(p,q)=(1,0)$, which respectively read

$$i\sqrt{2}\frac{\partial}{\partial z^{\alpha\dot{\beta}}}\Psi_i^{\dot{\beta}}(z)-m\epsilon_{ij}\Psi^j_{\alpha}(z)=0\,, \tag{4.4.1a}$$

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\alpha}}}\Psi^i_{\beta}(z)+m\epsilon^{ij}\Psi_j^{\dot{\alpha}}(z)=0\,, \tag{4.4.1b}$$

with $\Psi_i^{\dot{\beta}}:=\epsilon^{\dot{\beta}\dot{\gamma}}\Psi_{i\dot{\gamma}}$. Equation (4.4.1a) with $i=1$ and Eq. (4.4.1b) with $i=2$ can be combined in the form of the ordinary Dirac equation

$$D\psi_1(z)=0\,,\quad \psi_1(z):=\begin{pmatrix}\Psi^2_{\beta}(z)\\ \Psi_1^{\dot{\beta}}(z)\end{pmatrix}, \tag{4.4.2}$$

while Eq. (4.4.1a) with $i=2$ and Eq. (4.4.1b) with $i=1$ can be combined, after replacing $z^{\alpha\dot{\alpha}}$ by $-z^{\alpha\dot{\alpha}}$, as

$$D\psi_2(z)=0\,,\quad \psi_2(z):=\begin{pmatrix} \Psi^1_\beta(-z)\\ \Psi_2^{\dot{\beta}}(-z)\end{pmatrix}. \tag{4.4.3}$$

In Eqs. (4.4.2) and (4.4.3), D denotes the Dirac operator

$$D:=\begin{pmatrix} -m\delta_\alpha^\beta & i\sqrt{2}\dfrac{\partial}{\partial z^{\alpha\dot{\beta}}}\\ i\sqrt{2}\dfrac{\partial}{\partial z_{\beta\dot{\alpha}}} & -m\delta^{\dot{\alpha}}_{\dot{\beta}}\end{pmatrix}. \tag{4.4.4}$$

The charge conjugate of $\psi_1(z)$ is found to be

$$\begin{aligned}\psi_1^{\rm c}(z)&:=\begin{pmatrix} 0 & -\epsilon_{\beta\gamma}\\ \epsilon^{\dot{\beta}\dot{\gamma}} & 0\end{pmatrix}\overline{\psi_1(\bar{z})}\\ &=\begin{pmatrix} 0 & -\epsilon_{\beta\gamma}\\ \epsilon^{\dot{\beta}\dot{\gamma}} & 0\end{pmatrix}\begin{pmatrix} \bar{\Psi}_{2\dot{\gamma}}(z)\\ \bar{\Psi}^{1\gamma}(z)\end{pmatrix}=\begin{pmatrix} \bar{\Psi}^1_\beta(z)\\ \bar{\Psi}_2^{\dot{\beta}}(z)\end{pmatrix},\end{aligned} \tag{4.4.5}$$

where the arguments of ψ_1, namely $z^{\alpha\dot{\alpha}}$, have been replaced by their complex conjugates $\bar{z}^{\alpha\dot{\alpha}}:=\overline{z^{\alpha\dot{\alpha}}}$ so that $\psi_1^{\rm c}$ can be a holomorphic function of $z^{\alpha\dot{\alpha}}$. Using the complex conjugates of Eqs. (4.4.1a) and (4.4.1b), we can show that $D\psi_1^{\rm c}(z)=0$. Since ψ_2 and $\psi_1^{\rm c}$ satisfy the same Dirac equation and have the same spinor and $SU(2)$ indices, they can be identified with each other up to an overall constant.[2] (This identification may be confirmed by the CPT symmetry.) If $\psi_1(z)$ is a spinor field of a particle with four-momentum $(E,\boldsymbol{p})$, then $\psi_1^{\rm c}(z)\big(\simeq\psi_2(z)\big)$ is regarded as

[2]The plane wave solution of Eq. (4.4.1) given by

$$\begin{aligned}\Psi^i_\alpha(z)&=-Ce^{i\varphi/2}\bar{\varpi}^i_\alpha\exp\!\big(-iz^{\gamma\dot{\gamma}}\bar{\varpi}^k_\gamma\varpi_{k\dot{\gamma}}\big)\,,\\ \Psi_i^{\dot{\alpha}}(z)&=Ce^{-i\varphi/2}\varpi_i^{\dot{\alpha}}\exp\!\big(-iz^{\gamma\dot{\gamma}}\bar{\varpi}^k_\gamma\varpi_{k\dot{\gamma}}\big)\end{aligned}$$

fulfills the conditions $\Psi^i_\alpha(-z)=-\big(C/\bar{C}\big)\bar{\Psi}^i_\alpha(z)$ and $\Psi_i^{\dot{\alpha}}(-z)=-\big(C/\bar{C}\big)\bar{\Psi}_i^{\dot{\alpha}}(z)$. Here, C is a complex constant and φ is given in Eq. (4.1.26e). These conditions lead to $\psi_2(z)=-\big(C/\bar{C}\big)\psi_1^{\rm c}(z)$, and hence, in this case, ψ_2 and $\psi_1^{\rm c}$ can indeed be identified with each other. For verifying that the plane wave solution satisfies Eq. (4.4.1), it is convenient to use the classical counterparts of Eqs. (4.2.16a) and (4.2.16b):

$$\varpi_{i\dot{\alpha}}\varpi_j^{\dot{\alpha}}=\frac{m}{\sqrt{2}}\epsilon_{ij}e^{i\varphi},\quad \bar{\varpi}^i_\alpha\bar{\varpi}^{j\alpha}=\frac{m}{\sqrt{2}}\epsilon^{ij}e^{-i\varphi}.$$

	particle	antiparticle
left-handed	Ψ_{α}^{2}	Ψ_{α}^{1}
right-handed	$\Psi_{1}^{\dot{\alpha}}$	$\Psi_{2}^{\dot{\alpha}}$

Table 4.1: A classification of the rank-one spinor fields.

	s	t		s	t
Ψ_{α}^{2}	$-\dfrac{1}{2}$	$\dfrac{1}{2}$	Ψ_{α}^{1}	$-\dfrac{1}{2}$	$-\dfrac{1}{2}$
$\Psi_{1}^{\dot{\alpha}}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\Psi_{1}^{\dot{\alpha}}$	$\dfrac{1}{2}$	$-\dfrac{1}{2}$

Table 4.2: The values of s and t of the rank-one spinor fields.

a spinor field of a corresponding antiparticle with four-momentum $(-E, -\boldsymbol{p})$. Accordingly, $\psi_2(-z) = \big(\Psi_{\alpha}^{1}(z), \Psi_{2}^{\dot{\alpha}}(z)\big)^{\mathrm{T}}$ is considered a spinor field of the antiparticle with four-momentum $(E, \boldsymbol{p})$. In the light of this fact, it is clear that $\Psi_{\alpha}^{2}(z)$ and $\Psi_{\alpha}^{1}(z)$ represent a left-handed particle and a corresponding left-handed antiparticle, respectively, while $\Psi_{1}^{\dot{\alpha}}(z)$ and $\Psi_{2}^{\dot{\alpha}}(z)$ represent a right-handed particle and a corresponding right-handed antiparticle, respectively, as summarized in Table 4.1. We thus see that the index i of $\Psi_{\alpha}^{i}(z)$ and $\Psi_{i}^{\dot{\alpha}}(z)$ distinguishes between a particle and its antiparticle.

Using Eq. (4.3.5), we can obtain the possible values of s and t for each of the rank-one spinor fields as in Table 4.2. We observe that the left-handed spinor fields $\Psi_{\alpha}^{i}(z)$ $(i = 1, 2)$ have $s = -1/2$, while the right-handed spinor fields $\Psi_{i}^{\dot{\alpha}}(z)$ $(i = 1, 2)$ have $s = 1/2$. Hence, s turns out to be a quantum number specifying the chirality of a spinor field. Since s is an eigenvalue of $\check{\mathsf{T}}_0$ up to the additive constant 2, as can be seen from (4.2.13e), $\check{\mathsf{T}}_0$ can be interpreted as the operator of chirality. Accordingly, $U(1)_a$ can be identified as the gauge group of chirality, and the $U(1)_a$ symmetry is physically understood as a gauge symmetry leading to chirality conservation. We also observe that the particle spinor fields $\Psi_{\alpha}^{2}(z)$ and $\Psi_{1}^{\dot{\alpha}}(z)$ have $t = 1/2$, while the antiparticle spinor fields $\Psi_{\alpha}^{1}(z)$ and $\Psi_{2}^{\dot{\alpha}}(z)$ have $t = -1/2$. Hence, t turns out to be a quantum number distinguishing between a particle and its antiparticle. Then it follows that t is proportional to the electric charge of the particle/antiparticle. Since t is an eigenvalue of $\check{\mathsf{T}}_3$ as can be seen from (4.2.13f), $\check{\mathsf{T}}_3$ can be interpreted as the operator of electric charge up to a constant of proportionality. Accordingly, $U(1)_{\mathsf{b}}$ can be identified with the gauge group of electric charge, and the $U(1)_{\mathsf{b}}$ symmetry is physically understood as a gauge symmetry leading to electric charge conservation.

Now we recall that our study has been performed in the unitary gauge in which the GGS action takes the form of Eq. (3.3.9) or Eq. (3.3.13). In the unitary gauge, the local $SU(2)$ symmetry is hidden and the $U(1)_{\mathsf{b}}$ symmetry is linearly realized in accordance with Eq. (2.12). The manifestly $SU(2)$ covariant formulation can be developed on the basis of the action (3.3.12). The rank-one spinor fields found in this formulation, denoted by $\varOmega_i^{\dot{\alpha}}$ and $\varOmega_\alpha^i$, are related to $\varPsi_i^{\dot{\alpha}}$ and $\varPsi_\alpha^i$ by[3]

$$\varOmega_i^{\dot{\alpha}}(z) = V_i{}^j \varPsi_j^{\dot{\alpha}}(z)\,, \qquad \varOmega_\alpha^i(z) = \varPsi_\alpha^j(z) V^\dagger{}_j{}^i\,. \tag{4.4.6}$$

Because V is independent of $z^{\alpha\dot{\alpha}}$, we can readily verify by using Eqs. (4.4.1a) and (4.4.1b) that

$$i\sqrt{2}\frac{\partial}{\partial z^{\alpha\dot{\beta}}}\varOmega_i^{\dot{\beta}}(z) - m\epsilon_{ij}\varOmega_\alpha^j(z) = 0\,, \tag{4.4.7a}$$

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\alpha}}}\varOmega_\beta^i(z) + m\epsilon^{ij}\varOmega_j^{\dot{\alpha}}(z) = 0\,. \tag{4.4.7b}$$

Following the above consideration for $\varPsi_i^{\dot{\alpha}}(z)$ and $\varPsi_\alpha^i(z)$, we see that $\varOmega_\alpha^2(z)$ and $\varOmega_\alpha^1(z)$ constitute a doublet of left-handed particle and antiparticle spinor fields, while $\varOmega_1^{\dot{\alpha}}(z)$ and $\varOmega_2^{\dot{\alpha}}(z)$ constitute a doublet of right-handed particle and antiparticle spinor fields. Under the $SU(2)$ transformation, $\varOmega_i^{\dot{\alpha}}$ and $\varOmega_\alpha^i$ transform linearly as

$$\varOmega_i^{\dot{\alpha}} \rightarrow \varOmega_i'^{\dot{\alpha}} = U_i{}^j\varOmega_j^{\dot{\alpha}}\,, \qquad \varOmega_\alpha^i \rightarrow \varOmega_\alpha'^i = \varOmega_\alpha^j U^\dagger{}_j{}^i\,, \tag{4.4.8}$$

[3]The rank-$(p+q)$ spinor field in the manifestly $SU(2)$ covariant formulation is given by

$$\begin{aligned}&\varOmega^{i_1\ldots i_p}_{\alpha_1\ldots\alpha_p;j_1\ldots j_q,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)\\ &= \frac{1}{(2\pi i)^4}\oint_\Sigma e^{ip\check{\varphi}}\pi_{j_1\dot{\alpha}_1}\cdots\pi_{j_q\dot{\alpha}_q}\\ &\quad\times\frac{\partial}{\partial\mu^{\alpha_1}_{i_1}}\cdots\frac{\partial}{\partial\mu^{\alpha_p}_{i_p}}F_{I,s_1,s_2}(\mu,\pi)d^4\pi\,,\end{aligned}$$

where μ_i^α is a spinor related to ω_i^α by the weak equality $\mu_i^\alpha \approx \omega_i^\alpha$.

whereas $\Psi_i^{\dot{\alpha}}$ and Ψ_α^i transform according to the $U(1)_{\mathsf{b}}$ transformation

$$\Psi_i^{\dot{\alpha}} \to \Psi_i'^{\dot{\alpha}} = \Theta_i{}^j \Psi_j^{\dot{\alpha}}\,, \qquad \Psi_\alpha^i \to \Psi_\alpha'^i = \Psi_\alpha^j \Theta^\dagger{}_j{}^i\,. \tag{4.4.9}$$

As seen from Eq. (4.4.8), the $SU(2)$ transformation causes a continuous transformation between the particle spinor field Ω_α^2 $\left(\Omega_1^{\dot{\alpha}}\right)$ and the antiparticle spinor field Ω_α^1 $\left(\Omega_2^{\dot{\alpha}}\right)$. The $SU(2)$ symmetry therefore turns out to be a gauge symmetry realized in the particle-antiparticle doublets $(\Omega_\alpha^2, \Omega_\alpha^1)$ and $\left(\Omega_1^{\dot{\alpha}}, \Omega_2^{\dot{\alpha}}\right)$. Such a symmetry, however, is not observed in nature; hence, it should be considered that the $SU(2)$ symmetry is hidden or broken. The formulation in the unitary gauge is appropriate for this situation, because, in the unitary gauge, the $SU(2)$ symmetry is hidden and the $U(1)_{\mathsf{b}}$ symmetry is manifestly exhibited instead.

Chapter 5

Spinor formulation and canonical quantization of a massive spinning particle

In this chapter, we consider a spinor formulation of a massive spinning particle and the subsequent canonical quantization. In this formulation, we adopt the space-time and spinor variables as fundamental dynamical variables, after decomposing the twistor variables in the GGS action. Here, the mass-shell condition with a real mass parameter is incorporated into the action, instead of the mass-shell condition with the complexified mass parameter. We can expect that this approach clarifies relations between the twistor and ordinary space-time formulations of a massive spinning particle and makes it possible to consider coupling to external fields. We study the canonical Hamiltonian formalism based on the GGS action in accordance with Dirac's recipe for constrained Hamiltonian systems. Subsequently, we perform the canonical quantization of this system. As a result, simultaneous differential equations for a wave function of the space-time and momentum-spinor variables are derived. These equations are solved, yielding plane-wave solutions. We define positive and negative frequency spinor wave functions as linear combinations of the plane wave solutions. It is shown that the spinor wave functions satisfy the generalized DFP equations with $SU(2)$ indices. It is also demonstrated that the spinor wave functions can be expressed in the form of Penrose transforms. In addition, we construct the exponential generating function for the spinor wave functions. Finally, physical meanings of the $U(1)$ and $SU(2)$ symmetries are clarified.

5.1 The GGS action in spinor formulation

In this section, we express the GGS action in terms of space-time variable and spinor variables.

The mass-shell condition incorporated in the GGS action (3.3.13) is a pair of Eqs. (3.1.3a) and (3.1.3b) in the unitary gauge, while Eq. (3.1.3) equivalent to them is now rather convenient for spinor formulation. Therefore we consider the GGS action in which Eq. (3.1.3) is adopted instead of the pair of Eqs. (3.1.3a) and (3.1.3b) as the mass-shell condition:

$$S = \int_{\tau_0}^{\tau_1} d\tau \Bigg[\frac{i}{2}\Big(\bar{\mathsf{Z}}^i_A \dot{\mathsf{Z}}^A_i - \mathsf{Z}^A_i \dot{\bar{\mathsf{Z}}}^i_A\Big) + a\big(\bar{\mathsf{Z}}^i_A \mathsf{Z}^A_i - 2s\big) + \mathsf{b}^3\big(\bar{\mathsf{Z}}^j_A \sigma_{3j}{}^k \mathsf{Z}^A_k - 2t\big) \\ + \mathsf{b}^{\hat{\imath}} \bar{\mathsf{Z}}^j_A \sigma_{\hat{\imath}j}{}^k \mathsf{Z}^A_k - \frac{1}{2\mathsf{e}} \mathsf{b}^{\hat{\imath}} \mathsf{b}^{\hat{\imath}} - \frac{k^2}{2}\mathsf{e} + \frac{f}{2}(\bar{\varpi}^{i\alpha} \varpi^{\dot{\alpha}}_i \bar{\varpi}^k_\alpha \varpi_{k\dot{\alpha}} - m^2)\Bigg] , \tag{5.1.1}$$

where $f = f(\tau)$ is treated as a real scalar-density field of weight 1 on $\mathcal{T}$. It is assumed that f does not change under the $U(1)_a$ and $U(1)_{\mathsf{b}}$ transformations,

$$f \to f' = f. \tag{5.1.2}$$

It is obvious that the variation with respect to f yields the mass-shell condition

$$\bar{\varpi}^{i\alpha} \varpi^{\dot{\alpha}}_i \bar{\varpi}^k_\alpha \varpi_{k\dot{\alpha}} = m^2 . \tag{5.1.3}$$

Eq. (5.1.1) can be written in terms of spinor components ϱ^α_i, $\varpi_{i\dot{\alpha}}$ and their complex conjugates of the twistors in unitary gauge (see right below Eq. (3.3.10)) as

$$S = \int_{\tau_0}^{\tau_1} d\tau \Bigg[\frac{i}{2}\big(\bar{\varpi}^i_\alpha \dot{\varrho}^\alpha_i + \bar{\varrho}^{i\dot{\alpha}} \dot{\varpi}_{i\dot{\alpha}} - \varrho^\alpha_i \dot{\bar{\varpi}}^i_\alpha - \varpi_{i\dot{\alpha}} \dot{\bar{\varrho}}^{i\dot{\alpha}}\big) + a\big(\bar{\varpi}^i_\alpha \varrho^\alpha_i + \bar{\varrho}^{i\dot{\alpha}} \varpi_{i\dot{\alpha}} - 2s\big) \\ + \mathsf{b}^3\big(\bar{\varpi}^j_\alpha \sigma_{3j}{}^k \varrho^\alpha_k + \bar{\varrho}^{j\dot{\alpha}} \sigma_{3j}{}^k \varpi_{k\dot{\alpha}} - 2t\big) + \mathsf{b}^{\hat{\imath}}\big(\bar{\varpi}^j_\alpha \sigma_{\hat{\imath}j}{}^k \varrho^\alpha_k + \bar{\varrho}^{j\dot{\alpha}} \sigma_{\hat{\imath}j}{}^k \varpi_{k\dot{\alpha}}\big) \\ - \frac{1}{2\mathsf{e}} \mathsf{b}^{\hat{\imath}} \mathsf{b}^{\hat{\imath}} - \frac{k^2}{2}\mathsf{e} + \frac{f}{2}(\bar{\varpi}^{i\alpha} \varpi^{\dot{\alpha}}_i \bar{\varpi}^k_\alpha \varpi_{k\dot{\alpha}} - m^2)\Bigg] . \tag{5.1.4}$$

As seen in Eq. (3.3.11), the spinor variable ϱ^α_i is related with another spinor variable $\varpi_{i\dot{\alpha}}$ by $\varrho^\alpha_i = iz^{\alpha\dot{\alpha}} \varpi_{i\dot{\alpha}}$. The coordinates $z^{\alpha\dot{\alpha}}$ can be decomposed as $z^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} - iy^{\alpha\dot{\alpha}}$, where $x^{\alpha\dot{\alpha}}$ and $y^{\alpha\dot{\alpha}}$ are elements of Hermitian matrices, satisfying the Hermitian conditions $\overline{x^{\beta\dot{\alpha}}} = x^{\alpha\dot{\beta}}$ and $\overline{y^{\beta\dot{\alpha}}} = y^{\alpha\dot{\beta}}$. The matrix elements of $x^{\alpha\dot{\alpha}}$ are identified with coordinates of a point in Minkowski space $\mathbf{M}$. Because the coordinates $z^{\alpha\dot{\alpha}}$ are treated as scalar fields on $\mathcal{T}$, $x^{\alpha\dot{\alpha}}$ and $y^{\alpha\dot{\alpha}}$ behave as scalar fields on $\mathcal{T}$. From Eq. (3.3.11) with the decomposition of $z^{\alpha\dot{\alpha}}$ and spinor variables defined as

$$\psi_i^\alpha := y^{\alpha\dot{\alpha}}\varpi_{i\dot{\alpha}} , \qquad \bar{\psi}^{i\dot{\alpha}} := y^{\alpha\dot{\alpha}}\bar{\varpi}_\alpha^i , \tag{5.1.5}$$

the spinor variables ϱ_i^α and its complex conjugate $\bar{\varrho}^{i\dot{\alpha}}$ can be written as

$$\varrho_i^\alpha = ix^{\alpha\dot{\alpha}}\varpi_{i\dot{\alpha}} + \psi_i^\alpha , \qquad \bar{\varrho}^{i\dot{\alpha}} = -ix^{\alpha\dot{\alpha}}\bar{\varpi}_\alpha^i + \bar{\psi}^{i\dot{\alpha}} . \tag{5.1.6}$$

Clearly, the spinor variables ψ_i^α and $\bar{\psi}^{i\dot{\alpha}}$ behave as scalar fields on $\mathcal{T}$. Substituting Eq. (5.1.6) into Eq. (5.1.4) , we obtain

$$\begin{aligned} S_s = \int d\tau \Big[& - \dot{x}^{\alpha\dot{\alpha}}\bar{\varpi}_\alpha^i\varpi_{i\dot{\alpha}} - i(\psi_i^\alpha\dot{\bar{\varpi}}_\alpha^i - \bar{\psi}^{i\dot{\alpha}}\dot{\varpi}_{i\dot{\alpha}}) + a(\bar{\varpi}_\alpha^i\psi_i^\alpha + \bar{\psi}^{i\dot{\alpha}}\varpi_{i\dot{\alpha}} - 2s) \\ & + \mathsf{b}^{\hat{\imath}}\left(\bar{\varpi}_\alpha^k\sigma_{\hat{\imath}k}{}^i\psi_i^\alpha + \bar{\psi}^{k\dot{\alpha}}\sigma_{\hat{\imath}k}{}^i\varpi_{i\dot{\alpha}}\right) + \mathsf{b}^3\left(\bar{\varpi}_\alpha^k\sigma_{3k}{}^i\psi_i^\alpha + \bar{\psi}^{k\dot{\alpha}}\sigma_{3k}{}^i\varpi_{i\dot{\alpha}} - 2t\right) \\ & - \frac{1}{2\mathsf{e}}\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}} - \frac{k^2}{2}\mathsf{e} + \frac{f}{2}(\bar{\varpi}^{i\alpha}\varpi_i^{\dot{\alpha}}\bar{\varpi}_\alpha^k\varpi_{k\dot{\alpha}} - m^2)\Big] . \end{aligned} \tag{5.1.7}$$

This is the GGS action written in terms of space-time and spinor variables. We see that S_s is reparametrization invariant. With Eq. (5.1.6), we find that $x^{\alpha\dot{\alpha}}$, $\varpi_{i\dot{\alpha}}$, $\bar{\varpi}_\alpha^i$, ψ_i^α and $\bar{\psi}^{i\dot{\alpha}}$ transform under the $U(1)_a$ transformation (3.3.3) as

$$x^{\alpha\dot{\alpha}} \to x'^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} , \tag{5.1.8a}$$
$$\varpi_{i\dot{\alpha}} \to \varpi'_{i\dot{\alpha}} = e^{i\theta(\tau)}\varpi_{i\dot{\alpha}} , \tag{5.1.8b}$$
$$\bar{\varpi}_\alpha^i \to \bar{\varpi}'^i_\alpha = e^{-i\theta(\tau))}\bar{\varpi}_\alpha^i , \tag{5.1.8c}$$
$$\psi_i^\alpha \to \psi'^\alpha_i = e^{i\theta(\tau)}\psi_i^\alpha , \tag{5.1.8d}$$
$$\bar{\psi}^{i\dot{\alpha}} \to \bar{\psi}'^{i\dot{\alpha}} = e^{-i\theta(\tau)}\bar{\psi}^{i\dot{\alpha}} . \tag{5.1.8e}$$

On the other hand, we find that $x^{\alpha\dot{\alpha}}$, $\varpi_{i\dot{\alpha}}$, $\bar{\varpi}_\alpha^i$, ψ_i^α and $\bar{\psi}^{i\dot{\alpha}}$ transform under the $U(1)_{\mathsf{b}}$ transformation (3.3.4) as

$$x^{\alpha\dot{\alpha}} \to x'^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} , \tag{5.1.9a}$$
$$\varpi_{i\dot{\alpha}} \to \varpi'_{i\dot{\alpha}} = \Theta_i{}^j(\tau)\varpi_{j\dot{\alpha}} , \tag{5.1.9b}$$
$$\bar{\varpi}_\alpha^i \to \bar{\varpi}'^i_\alpha = \bar{\varpi}_\alpha^j\Theta^\dagger{}_j{}^i(\tau) , \tag{5.1.9c}$$
$$\psi_i^\alpha \to \psi'^\alpha_i = \Theta_i{}^j(\tau)\psi_j^\alpha , \tag{5.1.9d}$$
$$\bar{\psi}^{i\dot{\alpha}} \to \bar{\psi}'^{i\dot{\alpha}} = \bar{\psi}^{j\dot{\alpha}}\Theta^{\dagger i}_j(\tau) . \tag{5.1.9e}$$

It can easily be verified that S_s remains invariant under $U(1)_a$ transformation given by Eqs. (3.3.3f), (3.3.3g), (5.1.2) and (5.1.8) and under $U(1)_{\mathsf{b}}$ transformation given by Eqs. (3.3.4f), (3.3.4g), (5.1.2) and (5.1.9). By applying the Noether's theorem, the conserved quantity corresponding to the translation $x^{\alpha\dot{\alpha}} \to x^{\alpha\dot{\alpha}} + k^{\alpha\dot{\alpha}}$ is obtained as $p_{\alpha\dot{\alpha}} := \bar{\varpi}^i_\alpha \varpi_{i\dot{\alpha}}$, where $k^{\alpha\dot{\alpha}}$ are elements of a constant Hermitian matrix, satisfying the Hermitian condition $\overline{k^{\beta\dot{\alpha}}} = k^{\alpha\dot{\beta}}$. Due to the mass-shell condition, it follows that $p_{\alpha\dot{\alpha}} p^{\alpha\dot{\alpha}} = m^2$. This shows that the action S_s describes a massive particle.

5.2 Canonical formalism

In this section, we study the canonical Hamiltonian formalism of the model governed by the action S_s.

Let L_s the Lagrangian defined in Eq. (5.1.7) as the integrand of S_s. We treat the variables $(x^{\alpha\dot{\alpha}}, \bar{\varpi}^i_\alpha, \varpi_{i\dot{\alpha}}, \psi^\alpha_i, \bar{\psi}^{i\dot{\alpha}}$, a, $\mathsf{b}^{\hat{\imath}}$, b^3, e, $f)$ contained in the Lagrangian L as canonical coordinates. Their corresponding conjugate momenta are defined by

$$P^{(x)}_{\alpha\dot{\alpha}} := \frac{\partial L}{\partial \dot{x}^{\alpha\dot{\alpha}}} = -\bar{\varpi}^i_\alpha \varpi_{i\dot{\alpha}} \ , \tag{5.2.1a}$$

$$P_{(\bar{\varpi})}{}^\alpha_i := \frac{\partial L}{\partial \dot{\bar{\varpi}}^i_\alpha} = -i\psi^\alpha_i \ , \tag{5.2.1b}$$

$$P^{i\dot{\alpha}}_{(\varpi)} := \frac{\partial L}{\partial \dot{\varpi}_{i\dot{\alpha}}} = i\bar{\psi}^{i\dot{\alpha}} \ , \tag{5.2.1c}$$

$$P_{(\psi)}{}^i_\alpha := \frac{\partial L}{\partial \dot{\psi}^\alpha_i} = 0 \ , \tag{5.2.1d}$$

$$P_{(\bar{\psi})i\dot{\alpha}} := \frac{\partial L}{\partial \dot{\bar{\psi}}^{i\dot{\alpha}}} = 0 \ , \tag{5.2.1e}$$

$$P^{(a)} := \frac{\partial L}{\partial \dot{a}} = 0 \ , \tag{5.2.1f}$$

$$P^{(\mathsf{b})}_r := \frac{\partial L}{\partial \dot{\mathsf{b}}^r} = 0, \tag{5.2.1g}$$

$$P^{(\mathsf{e})} := \frac{\partial L}{\partial \dot{\mathsf{e}}} = 0 \ , \tag{5.2.1h}$$

$$P^{(f)} := \frac{\partial L}{\partial \dot{f}} = 0 \ . \tag{5.2.1i}$$

The canonical Hamiltonian is found from Eqs. (5.1.7) and (5.2.1) to be

$$\begin{aligned} H_C := & - a(\bar{\varpi}^i_\alpha \psi^\alpha_i + \bar{\psi}^{i\dot{\alpha}} \varpi_{i\dot{\alpha}} - 2s) - \mathsf{b}^{\hat{\imath}} \left(\bar{\varpi}^k_\alpha \sigma_{\hat{\imath}k}{}^i \psi^\alpha_i + \bar{\psi}^{k\dot{\alpha}} \sigma_{\hat{\imath}k}{}^i \varpi_{i\dot{\alpha}} \right) \\ & - \mathsf{b}^3 \left(\bar{\varpi}^k_\alpha \sigma_{3k}{}^i \psi^\alpha_i + \bar{\psi}^{k\dot{\alpha}} \sigma_{3k}{}^i \varpi_{i\dot{\alpha}} - 2t \right) + \frac{1}{2\mathsf{e}} \mathsf{b}^{\hat{\imath}} \mathsf{b}^{\hat{\imath}} + \frac{k^2}{2} \mathsf{e} \\ & - \frac{f}{2} (\bar{\varpi}^{i\alpha} \varpi^{\dot{\alpha}}_i \bar{\varpi}^k_\alpha \varpi_{k\dot{\alpha}} - m^2) \end{aligned} \tag{5.2.2}$$

The non-vanishing Poisson brackets between the canonical variables are given by

$$\begin{gathered} \left\{ x^{\alpha\dot{\alpha}}, P^{(x)}_{\beta\dot{\beta}} \right\} = \delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}} \ , \qquad \left\{ \bar{\varpi}^i_\alpha, P_{(\bar{\varpi})}{}^\beta_j \right\} = \delta^i_j \delta^\beta_\alpha \ , \qquad \left\{ \varpi_{i\dot{\alpha}}, P^{j\dot{\beta}}_{(\varpi)} \right\} = \delta^j_i \delta^{\dot{\beta}}_{\dot{\alpha}} \ , \\ \left\{ \psi^\alpha_i, P_{(\psi)}{}^j_\beta \right\} = \delta^j_i \delta^\alpha_\beta \ , \qquad \left\{ \bar{\psi}^{i\dot{\alpha}}, P_{(\bar{\psi})j\dot{\beta}} \right\} = \delta^i_j \delta^{\dot{\alpha}}_{\dot{\beta}} \ , \qquad \left\{ a, P_{(a)} \right\} = 1 \\ \left\{ \mathsf{b}^r, P^{(\mathsf{b})}_t \right\} = \delta^r_t \ , \qquad \left\{ \mathsf{e}, P_{(\mathsf{e})} \right\} = 1 \ , \qquad \left\{ f, P_{(f)} \right\} = 1. \end{gathered} \tag{5.2.3}$$

The Poisson bracket between two arbitrary analytic functions of the canonical variables can be calculated using the fundamental Poisson brackets in Eq. (5.2.3).

Equations (5.2.1a)–(5.2.1i) are read, respectively, as the primary constraints

$$\phi^{(x)}_{\alpha\dot{\alpha}} := P^{(x)}_{\alpha\dot{\alpha}} + \bar{\varpi}^i_\alpha \varpi_{i\dot{\alpha}} \approx 0 \ , \tag{5.2.4a}$$

$$\phi_{(\bar{\varpi})}{}^\alpha_i := P_{(\bar{\varpi})}{}^\alpha_i + i\psi^\alpha_i \approx 0 \ , \tag{5.2.4b}$$

$$\phi_{(\varpi)}{}^{i\dot{\alpha}} := P_{(\varpi)}{}^{i\dot{\alpha}} - i\bar{\psi}^{i\dot{\alpha}} \approx 0 \ , \tag{5.2.4c}$$

$$\phi_{(\psi)}{}^i_\alpha := P_{(\psi)}{}^i_\alpha \approx 0 \ , \tag{5.2.4d}$$

$$\phi_{(\bar{\psi})i\dot{\alpha}} := P_{(\bar{\psi})i\dot{\alpha}} \approx 0 \ , \tag{5.2.4e}$$

$$\phi^{(a)} := P^{(a)} \approx 0 \ , \tag{5.2.4f}$$

$$\phi^{(\mathsf{b})}_r := P^{(\mathsf{b})}_r \approx 0 \ , \tag{5.2.4g}$$

$$\phi^{(\mathsf{e})} := P^{(\mathsf{e})} \approx 0 \ , \tag{5.2.4h}$$

$$\phi^{(f)} := P^{(f)} \approx 0 \ . \tag{5.2.4i}$$

where the symbol "$\approx$" denotes the weak equality. Now, we apply the Dirac formulation for constrained Hamiltonian systems [36]-[38] to the present model. To this end, we first calculate the Poisson brackets between the constraint functions $\phi^{(x)}_{\alpha\dot{\alpha}}$, $\phi_{(\bar{\varpi})}{}^\alpha_i$, $\phi^{i\dot{\alpha}}_{(\varpi)}$, $\phi_{(\psi)}{}^i_\alpha$, $\phi_{(\bar{\psi})i\dot{\alpha}}$, $\phi^{(a)}$, $\phi^{(\mathsf{b})}_r$, $\phi^{(\mathsf{e})}$ and $\phi^{(f)}$ obtaining the following non-vanishing Poisson brackets:

$$\begin{aligned} \left\{ \phi^{(x)}_{\alpha\dot{\alpha}}, \phi_{(\bar{\varpi})}{}^\beta_i \right\} &= \delta^\beta_\alpha \varpi_{i\dot{\alpha}} \ , & \left\{ \phi^{(x)}_{\alpha\dot{\alpha}}, \phi^{i\dot{\beta}}_{(\varpi)} \right\} &= \bar{\varpi}^i_\alpha \delta^{\dot{\beta}}_{\dot{\alpha}} \ , \\ \left\{ \phi_{(\bar{\varpi})}{}^\alpha_i, \phi_{(\psi)}{}^j_\beta \right\} &= i\delta^j_i \delta^\alpha_\beta \ , & \left\{ \phi^{i\dot{\alpha}}_{(\varpi)}, \phi_{(\bar{\psi})j\dot{\beta}} \right\} &= -i\delta^i_j \delta^{\dot{\alpha}}_{\dot{\beta}} \ . \end{aligned} \tag{5.2.5}$$

We can also obtain

$$\{\phi^{(x)}_{\alpha\dot\alpha}, H_C\} = 0 \ , \tag{5.2.6a}$$

$$\{\phi_{(\bar\varpi)}{}^{\alpha}_{i}, H_C\} = a\psi^\alpha_i + \mathsf{b}^3\sigma_{3i}{}^j\psi^\alpha_j + \mathsf{b}^{\hat\imath}\sigma_{\hat\imath i}{}^j\psi^\alpha_j + f\bar\varpi^{j\alpha}\varpi^{\dot\beta}_j\varpi_{i\dot\beta} \ , \tag{5.2.6b}$$

$$\{\phi_{(\varpi)}{}^{i\dot\alpha}, H_C\} = a\bar\psi^{i\dot\alpha} + \mathsf{b}^3\sigma_{3j}{}^i\bar\psi^{j\dot\alpha} + \mathsf{b}^{\hat\imath}\sigma_{\hat\imath j}{}^i\bar\psi^{j\dot\alpha} + f\varpi^{\dot\alpha}_j\bar\varpi^{j\beta}\bar\varpi^i_\beta \ , \tag{5.2.6c}$$

$$\{\phi_{(\psi)}{}^{i}_{\alpha}, H_C\} = a\bar\varpi^i_\alpha + \mathsf{b}^3\bar\varpi^j_\alpha\sigma_{3j}{}^i + \mathsf{b}^{\hat\imath}\bar\varpi^j_\alpha\sigma_{\hat\imath j}{}^i, \tag{5.2.6d}$$

$$\{\phi_{(\bar\psi)i\dot\alpha}, H_C\} = a\varpi_{i\dot\alpha} + \mathsf{b}^3\sigma_{3i}{}^j\varpi_{j\dot\alpha} + \mathsf{b}^{\hat\imath}\sigma_{\hat\imath i}{}^j\varpi_{j\dot\alpha} \ , \tag{5.2.6e}$$

$$\{\phi^{(a)}, H_C\} = \psi^\alpha_i\bar\varpi^i_\alpha + \bar\psi^{i\dot\alpha}\varpi_{i\dot\alpha} - 2s \ , \tag{5.2.6f}$$

$$\left\{\phi^{(\mathsf{b})}_{\hat\imath}, H_C\right\} = \left(\bar\varpi^k_\alpha\sigma_{\hat\imath k}{}^i\psi^\alpha_i + \bar\psi^{k\dot\alpha}\sigma_{\hat\imath k}{}^i\varpi_{i\dot\alpha}\right) - \frac{\mathsf{b}^{\hat\imath}}{\mathsf{e}} \ , \tag{5.2.6g}$$

$$\{\phi^{(\mathsf{b})}_3, H_C\} = \bar\varpi^k_\alpha\sigma_{3k}{}^i\psi^\alpha_i + \bar\psi^{k\dot\alpha}\sigma_{3k}{}^i\varpi_{i\dot\alpha} - 2t \quad , \tag{5.2.6h}$$

$$\{\phi^{(\mathsf{e})}, H_C\} = \frac{1}{2\mathsf{e}^2}\left(\mathsf{b}^{\hat\imath}\mathsf{b}^{\hat\imath} - k^2e^2\right) \ , \tag{5.2.6i}$$

$$\{\phi^{(f)}, H_C\} = \frac{1}{2}(\bar\varpi^{i\alpha}\varpi^{\dot\alpha}_i\bar\varpi^k_\alpha\varpi_{k\dot\alpha} - m^2) \ . \tag{5.2.6j}$$

Introducing the Lagrange multipliers $u^{\alpha\dot\alpha}_{(x)}$, $u_{(\bar\varpi)}{}^{i}_{\alpha}$, $u_{(\varpi)i\dot\alpha}$, $u_{(\psi)}{}^{\alpha}_{i}$, $u^{i\dot\alpha}_{(\bar\psi)}$, $u_{(a)}$, $u^r_{(\mathsf{b})}$, $u_{(\mathsf{e})}$ and $u_{(f)}$, we define the total Hamiltonian

$$\begin{aligned} H_{\mathrm{T}} := H_{\mathrm{C}} &+ u^{\alpha\dot\alpha}_{(x)}\phi^{(x)}_{\alpha\dot\alpha} + u_{(\bar\varpi)}{}^{i}_{\alpha}\phi_{(\bar\varpi)}{}^{\alpha}_{i} + u_{(\varpi)i\dot\alpha}\phi^{i\dot\alpha}_{(\varpi)} + u_{(\psi)}{}^{\alpha}_{i}\phi_{(\psi)}{}^{i}_{\alpha} + u^{i\dot\alpha}_{(\bar\psi)}\phi_{(\bar\psi)i\dot\alpha} \\ &+ u_{(a)}\phi^{(a)} + u^r_{(\mathsf{b})}\phi^{(\mathsf{b})}_r + u_{(\mathsf{e})}\phi^{(\mathsf{e})} + u_{(f)}\phi^{(f)} \end{aligned} \tag{5.2.7}$$

With this Hamiltonian, the canonical equation for a function F of the canonical variables is given by

$$\dot F = \{F, H_{\mathrm{T}}\} \, . \tag{5.2.8}$$

The primary constraints (5.2.4a)–(5.2.4i) must be preserved in time, because they are valid at any time. The time evolutions of the constraints functions can be evaluated using Eqs. (5.2.6a)–(5.2.6i), and as a result, we have the consistency conditions

$$\dot{\phi}^{(x)}_{\alpha\dot{\alpha}} = \left\{\phi^{(x)}_{\alpha\dot{\alpha}},\, H_T\right\} \approx u_{(\bar{\varpi})}{}^{i}_{\alpha}\varpi_{i\dot{\alpha}} + u_{(\varpi)i\dot{\alpha}}\bar{\varpi}^{i}_{\alpha} \approx 0 \;, \tag{5.2.9a}$$

$$\begin{aligned}\dot{\phi}_{(\bar{\varpi})}{}^{\alpha}_{i} &= \left\{\phi_{(\bar{\varpi})}{}^{\alpha}_{i},\, H_T\right\} \\ &\approx a\psi^{\alpha}_{i} + \mathsf{b}^3\sigma_{3i}{}^{j}\psi^{\alpha}_{j} + \mathsf{b}^{\hat{\imath}}\sigma_{\hat{\imath}i}{}^{j}\psi^{\alpha}_{j} + f\bar{\varpi}^{j\alpha}\varpi^{\dot{\beta}}_{j}\varpi_{i\dot{\beta}} - u^{\alpha\dot{\alpha}}_{(x)}\varpi_{i\dot{\alpha}} + iu_{(\psi)}{}^{\alpha}_{i} \\ &\approx 0\,,\end{aligned} \tag{5.2.9b}$$

$$\begin{aligned}\dot{\phi}^{i\dot{\alpha}}_{(\varpi)} &= \left\{\phi^{i\dot{\alpha}}_{(\varpi)},\, H_T\right\} \\ &\approx a\bar{\psi}^{i\dot{\alpha}} + \mathsf{b}^3\sigma_{3j}{}^{i}\bar{\psi}^{j\dot{\alpha}} + \mathsf{b}^{\hat{\imath}}\sigma_{\hat{\imath}j}{}^{i}\bar{\psi}^{j\dot{\alpha}} + f\varpi^{\dot{\alpha}}_{j}\bar{\varpi}^{j\beta}\bar{\varpi}^{i}_{\beta} - u^{\alpha\dot{\alpha}}_{(x)}\bar{\varpi}^{i}_{\alpha} - iu^{i\dot{\alpha}}_{(\bar{\psi})} \\ &\approx 0 \;,\end{aligned} \tag{5.2.9c}$$

$$\dot{\phi}_{(\psi)}{}^{i}_{\alpha} = \left\{\phi_{(\psi)}{}^{i}_{\alpha},\, H_T\right\} \approx a\bar{\varpi}^{i}_{\alpha} + \mathsf{b}^3\bar{\varpi}^{j}_{\alpha}\sigma_{3j}{}^{i} + \mathsf{b}^{\hat{\imath}}\bar{\varpi}^{j}_{\alpha}\sigma_{\hat{\imath}j}{}^{i} - iu_{(\bar{\varpi})}{}^{i}_{\alpha} \approx 0 \;, \tag{5.2.9d}$$

$$\dot{\phi}_{(\bar{\psi})i\dot{\alpha}} = \left\{\phi_{(\bar{\psi})i\dot{\alpha}},\, H_T\right\} \approx a\varpi_{i\dot{\alpha}} + \mathsf{b}^3\sigma_{3i}{}^{j}\varpi_{j\dot{\alpha}} + \mathsf{b}^{\hat{\imath}}\sigma_{\hat{\imath}i}{}^{j}\varpi_{j\dot{\alpha}} + iu_{(\varpi)i\dot{\alpha}} \approx 0 \;, \tag{5.2.9e}$$

$$\dot{\phi}_{(a)} = \left\{\phi_{(a)},\, H_T\right\} \approx \bar{\varpi}^{i}_{\alpha}\psi^{\alpha}_{i} + \bar{\psi}^{i\dot{\alpha}}\varpi_{i\dot{\alpha}} - 2s \approx 0 \;, \tag{5.2.9f}$$

$$\dot{\phi}^{(\mathsf{b})}_{\hat{\imath}} = \left\{\phi^{(\mathsf{b})}_{\hat{\imath}}, H_T\right\} = \left(\bar{\varpi}^{k}_{\alpha}\sigma_{\hat{\imath}k}{}^{i}\psi^{\alpha}_{i} + \bar{\psi}^{k\dot{\alpha}}\sigma_{\hat{\imath}k}{}^{i}\varpi_{i\dot{\alpha}}\right) - \frac{\mathsf{b}_{\hat{\imath}}}{e} \approx 0 \;, \tag{5.2.9g}$$

$$\dot{\phi}^{(\mathsf{b})}_{3} = \left\{\phi^{(\mathsf{b})}_{3},\, H_T\right\} \approx \bar{\varpi}^{k}_{\alpha}\sigma_{3k}{}^{i}\psi^{\alpha}_{i} + \bar{\psi}^{k\dot{\alpha}}\sigma_{3k}{}^{i}\varpi_{i\dot{\alpha}} - 2t \approx 0 \;, \tag{5.2.9h}$$

$$\dot{\phi}_{(\mathsf{e})} = \left\{\phi_{(\mathsf{e})}, H_T\right\} = \frac{1}{2e^2}\left(\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}} - k^2\mathsf{e}^2\right) \approx 0 \;, \tag{5.2.9i}$$

$$\dot{\phi}_{(f)} = \left\{\phi_{(f)},\, H_T\right\} \approx \frac{1}{2}(\bar{\varpi}^{i\alpha}\varpi^{\dot{\alpha}}_{i}\bar{\varpi}^{k}_{\alpha}\varpi_{k\dot{\alpha}} - m^2) \approx 0 \;. \tag{5.2.9j}$$

Equations (5.2.9d) and (5.2.9e) determine $u_{(\bar{\varpi})}{}^{i}_{\alpha}$ and $u_{(\varpi)i\dot{\alpha}}$, respectively, as follows:

$$u_{(\bar{\varpi})}{}^{i}_{\alpha} = -i\left(a\bar{\varpi}^{i}_{\alpha} + \mathsf{b}^3\bar{\varpi}^{j}_{\alpha}\sigma_{3j}{}^{i} + \mathsf{b}^{\hat{\imath}}\bar{\varpi}^{j}_{\alpha}\sigma_{\hat{\imath}j}{}^{i}\right) \;, \tag{5.2.10a}$$

$$u_{(\varpi)i\dot{\alpha}} = i\left(a\varpi_{i\dot{\alpha}} + \mathsf{b}^3\sigma_{3i}{}^{j}\varpi_{j\dot{\alpha}} + \mathsf{b}^{\hat{\imath}}\sigma_{\hat{\imath}i}{}^{j}\varpi_{j\dot{\alpha}}\right) \;. \tag{5.2.10b}$$

Substituting these into Eq.(5.2.9a), we see that $\dot{\phi}^{(x)}_{\alpha\dot{\alpha}} \approx 0$ is identically satisfied; hence, Eq. (5.2.9a) gives no new constraints. If $u^{\alpha\dot{\alpha}}_{(x)}$ is fixed to a specific function of the canonical variables, $u_{(\psi)}{}^{\alpha}_{i}$ and $u^{i\dot{\alpha}}_{(\bar{\psi})}$ are determined from Eqs. (5.2.9b) and (5.2.9c), respectively, as follows:

$$u_{(\psi)}{}^{\alpha}_{i} = i\left(a\psi^{\alpha}_{i} + \mathsf{b}^3\sigma_{3i}{}^{j}\psi^{\alpha}_{j} + \mathsf{b}^{\hat{\imath}}\sigma_{\hat{\imath}i}{}^{j}\psi^{\alpha}_{j} + f\bar{\varpi}^{j\alpha}\varpi^{\dot{\beta}}_{j}\varpi_{i\dot{\beta}} - u^{\alpha\dot{\alpha}}_{(x)}\varpi_{i\dot{\alpha}}\right) \;, \tag{5.2.11a}$$

$$u^{i\dot{\alpha}}_{(\bar{\psi})} = -i\left(a\bar{\psi}^{i\dot{\alpha}} + \mathsf{b}^3\bar{\psi}^{j\dot{\alpha}}\sigma_{3j}{}^{i} + \mathsf{b}^{\hat{\imath}}\bar{\psi}^{j\dot{\alpha}}\sigma_{\hat{\imath}j}{}^{i} + f\varpi^{\dot{\alpha}}_{j}\bar{\varpi}^{j\beta}\bar{\varpi}^{i}_{\beta} - u^{\alpha\dot{\alpha}}_{(x)}\bar{\varpi}^{i}_{\alpha}\right) \;. \tag{5.2.11b}$$

In contrast, equations (5.2.9f)–(5.2.9j) give rise to secondary constraints

$$\chi_{(a)} := \bar{\varpi}^i_\alpha \psi^\alpha_i + \bar{\psi}^{i\dot{\alpha}} \varpi_{i\dot{\alpha}} - 2s \approx 0 \ , \tag{5.2.12a}$$

$$\chi^{(\mathsf{b})}_{\hat{\imath}} := \bar{\varpi}^k_\alpha \sigma_{\hat{\imath}k}{}^i \psi^\alpha_i + \bar{\psi}^{k\dot{\alpha}} \sigma_{\hat{\imath}k}{}^i \varpi_{i\dot{\alpha}} - \frac{b_{\hat{\imath}}}{\mathsf{e}} \approx 0 \ , \tag{5.2.12b}$$

$$\chi^{(\mathsf{b})}_3 := \bar{\varpi}^k_\alpha \sigma_{3k}{}^i \psi^\alpha_i + \bar{\psi}^{k\dot{\alpha}} \sigma_{3k}{}^i \varpi_{i\dot{\alpha}} - 2t \approx 0 \ . \tag{5.2.12c}$$

$$\chi_{(\mathsf{e})} := \frac{1}{2} \left(\mathsf{b}^{\hat{\imath}} \mathsf{b}^{\hat{\imath}} - k^2 \mathsf{e}^2 \right) \approx 0 \ , \tag{5.2.12d}$$

$$\chi_{(f)} := \frac{1}{2} (\bar{\varpi}^{i\alpha} \varpi^{\dot{\alpha}}_i \bar{\varpi}^k_\alpha \varpi_{k\dot{\alpha}} - m^2) \approx 0 \ . \tag{5.2.12e}$$

The non-vanishing Poisson brackets between $\chi_{(a)}$, $\chi^{(\mathsf{b})}_3$, $\chi^{(\mathsf{b})}_{\hat{\imath}}$, $\chi_{(f)}$, $\chi_{(\mathsf{e})}$ and the primary constraint functions are found to be

$$\begin{aligned}
&\{\chi_{(a)}, \phi_{(\bar{\varpi})}{}^\alpha_i\} = \psi^\alpha_i \ , && \{\chi_{(a)}, \phi_{(\varpi)}{}^{i\dot{\alpha}}\} = \bar{\psi}^{i\dot{\alpha}} \ , \\
&\{\chi_{(a)}, \phi_{(\psi)}{}^i_\alpha\} = \bar{\varpi}^i_\alpha \ , && \{\chi_{(a)}, \phi_{(\bar{\psi})i\dot{\alpha}}\} = \varpi_{i\dot{\alpha}} \ , \\
&\{\chi^{(\mathsf{b})}_3, \phi_{(\bar{\varpi})}{}^\alpha_i\} = \sigma_{3i}{}^j \psi^\alpha_j \ , && \{\chi^{(\mathsf{b})}_3, \phi_{(\varpi)}{}^{i\dot{\alpha}}\} = \bar{\psi}^{j\dot{\alpha}} \sigma_{3j}{}^i \ , \\
&\{\chi^{(\mathsf{b})}_3, \phi_{(\psi)}{}^i_\alpha\} = \bar{\varpi}^j_\alpha \sigma_{3j}{}^i \ , && \{\chi^{(\mathsf{b})}_3, \phi_{(\bar{\psi})i\dot{\alpha}}\} = \sigma_{3i}{}^j \varpi_{j\dot{\alpha}} \ , \\
&\{\chi^{(\mathsf{b})}_{\hat{\imath}}, \phi_{(\bar{\varpi})}{}^\alpha_i\} = \sigma_{\hat{\imath}i}{}^j \psi^\alpha_j \ , && \{\chi^{(\mathsf{b})}_{\hat{\imath}}, \phi_{(\varpi)}{}^{i\dot{\alpha}}\} = \bar{\psi}^{j\dot{\alpha}} \sigma_{\hat{\imath}j}{}^i \ , \\
&\{\chi^{(\mathsf{b})}_{\hat{\imath}}, \phi_{(\psi)}{}^i_\alpha\} = \bar{\varpi}^j_\alpha \sigma_{\hat{\imath}j}{}^i \ , && \{\chi^{(\mathsf{b})}_{\hat{\imath}}, \phi_{(\bar{\psi})i\dot{\alpha}}\} = \sigma_{\hat{\imath}i}{}^j \varpi_{j\dot{\alpha}} \ , \\
&\left\{\chi^{(\mathsf{b})}_{\hat{\imath}}, \phi^{(\mathsf{b})}_{\hat{\jmath}}\right\} = -\frac{\delta_{\hat{\imath}\hat{\jmath}}}{\mathsf{e}} && \left\{\chi^{(\mathsf{b})}_{\hat{\imath}}, \phi_{(\mathsf{e})}\right\} = \frac{\mathsf{b}_{\hat{\imath}}}{\mathsf{e}^2} \ , \\
&\left\{\chi_{(\mathsf{e})}, \phi^{(\mathsf{b})}_{\hat{\imath}}\right\} = \mathsf{b}_{\hat{\imath}} \ , && \left\{\chi_{(\mathsf{e})}, \phi_{(\mathsf{e})}\right\} = -k^2 \mathsf{e}, \\
&\{\chi_{(f)}, \phi_{(\bar{\varpi})}{}^\alpha_i\} = \bar{\varpi}^{k\alpha} \varpi^{\dot{\beta}}_k \varpi_{i\dot{\beta}} \ , && \{\chi_{(f)}, \phi_{(\varpi)}{}^{i\dot{\alpha}}\} = \varpi^{\dot{\alpha}}_k \bar{\varpi}^{k\beta} \bar{\varpi}^i_\beta \, .
\end{aligned} \tag{5.2.13}$$

Next, we investigate the time evolution of the secondary constraint functions using Eqs. (5.2.5) and (5.2.13). The time evolution of $\chi^{(a)}$ is evaluated as

$$\dot{\chi}^{(a)} = u_{(\bar{\varpi})}{}^i_\alpha \psi^\alpha_i + u_{(\psi)}{}^\alpha_i \bar{\varpi}^i_\alpha + u_{(\varpi)i\dot{\alpha}} \bar{\psi}^{i\dot{\alpha}} + u^{i\dot{\alpha}}_{(\bar{\psi})} \varpi_{i\dot{\alpha}} \, . \tag{5.2.14}$$

The condition $\dot{\chi}^{(a)} \approx 0$ is identically fulfilled with the aid of Eqs. (5.2.10) and (5.2.11), and hence no new constraints are obtained from $\dot{\chi}^{(a)} \approx 0$. The time evolution of $\chi^{(\mathsf{b})}_{\hat{\imath}}$ and $\chi^{(\mathsf{b})}_{3}$ are evaluated as

$$
\begin{aligned}
\dot{\chi}^{(\mathsf{b})}_{\hat{\imath}} &= \left\{\chi^{(\mathsf{b})}_{\hat{\imath}},\, H_T\right\} \\
&= \left(u_{(\bar{\varpi})}{}^{k}_{\alpha}\psi^{\alpha}_{i} + \bar{\varpi}^{k}_{\alpha}u_{(\psi)}{}^{\alpha}_{i} + \bar{\psi}^{k\dot{\alpha}}u_{(\varpi)i\dot{\alpha}} + u_{(\bar{\psi})}{}^{k\dot{\alpha}}\varpi_{i\dot{\alpha}}\right)\sigma_{\hat{\imath}k}{}^{i} - \frac{1}{\mathsf{e}}\left(u^{\hat{\imath}}_{(\mathsf{b})} - u_{(\mathsf{e})}\frac{\mathsf{b}^{\hat{\imath}}}{\mathsf{e}}\right) \\
&= 2\epsilon^{\hat{\imath}\hat{\jmath}}\left\{\mathsf{b}^{3}\left(\bar{\varpi}^{j}_{\alpha}\sigma_{\hat{\jmath}j}{}^{i}\psi_{i}{}^{\alpha} + \bar{\psi}^{k\dot{\alpha}}\sigma_{\hat{\jmath}k}{}^{j}\varpi_{j\dot{\alpha}}\right) - \mathsf{b}^{\hat{\jmath}}\left(\bar{\varpi}^{j}_{\alpha}\sigma_{3j}{}^{i}\psi^{\alpha}_{i} + \bar{\psi}^{k\dot{\alpha}}\sigma_{3k}{}^{j}\varpi_{j\dot{\alpha}}\right)\right\} \\
&\quad - \frac{1}{\mathsf{e}}\left(u^{\hat{\imath}}_{(\mathsf{b})} - u_{(\mathsf{e})}\frac{\mathsf{b}^{\hat{\imath}}}{\mathsf{e}}\right) \\
&\approx 2\epsilon^{\hat{\imath}\hat{\jmath}}\mathsf{b}^{\hat{\jmath}}\left(\frac{\mathsf{b}^{3}}{\mathsf{e}} - 2t\right) - \frac{1}{\mathsf{e}}\left(u^{\hat{\imath}}_{(\mathsf{b})} - u_{(\mathsf{e})}\frac{\mathsf{b}^{\hat{\imath}}}{\mathsf{e}}\right),
\end{aligned} \tag{5.2.15}
$$

$$
\begin{aligned}
\dot{\chi}^{(\mathsf{b})}_{3} &= \left\{\chi^{(\mathsf{b})}_{3},\, H_T\right\} \\
&= u_{(\bar{\varpi})}{}^{i}_{\alpha}\sigma_{3i}{}^{j}\psi^{\alpha}_{j} + \bar{\varpi}^{j}_{\alpha}\sigma_{3j}{}^{i}u_{(\psi)}{}^{\beta}_{i} + \bar{\psi}^{j\dot{\alpha}}\sigma_{3j}{}^{i}u_{(\varpi)i\dot{\alpha}} + u_{(\bar{\psi})}{}^{i\dot{\alpha}}\sigma_{3i}{}^{j}\varpi_{j\dot{\alpha}} \approx -\frac{2}{\mathsf{e}}\epsilon^{\hat{\imath}\hat{\jmath}}\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\jmath}} = 0
\end{aligned} \tag{5.2.16}
$$

by using Eqs. (5.2.10), (5.2.11), (5.2.12c) and (5.2.12b). Then we see that the condition $\dot{\chi}^{(\mathsf{b})}_{\hat{\imath}} \approx 0$ determines $u^{\hat{\imath}}_{(\mathsf{b})}$ as follows:

$$
u^{\hat{\imath}}_{(\mathsf{b})} = u_{(\mathsf{e})}\frac{\mathsf{b}^{\hat{\imath}}}{\mathsf{e}} + 2\epsilon^{\hat{\imath}\hat{\jmath}}\mathsf{b}^{\hat{\jmath}}\left(\mathsf{b}^{3} - 2t\mathsf{e}\right)\,, \tag{5.2.17}
$$

while $\dot{\chi}^{(\mathsf{b})}_{3}$ is identically satisfied. The time evolution of $\chi^{(\mathsf{e})}$ is calculated as

$$
\dot{\chi}^{(\mathsf{e})} = \left\{\chi^{(\mathsf{e})},\, H_T\right\} = \frac{u^{(\mathsf{e})}}{\mathsf{e}}\left(\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}} - k^{2}\mathsf{e}^{2}\right) \approx 0 \tag{5.2.18}
$$

by using Eqs. (5.2.17) and (5.2.12d). Hence the condition $\dot{\chi} \approx 0$ is identically satisfied. The time evolution of $\chi^{(f)}$ is evaluated as

$$
\dot{\chi}^{(f)} = (u_{(\bar{\varpi})}{}^{i}_{\alpha}\varpi_{i\dot{\alpha}} + \bar{\varpi}^{i}_{\alpha}u_{(\varpi)i\dot{\alpha}})\bar{\varpi}^{j\alpha}\varpi^{\dot{\alpha}}_{j} \approx 0 \tag{5.2.19}
$$

by using Eq. (5.2.10). Hence the condition $\dot{\chi}^{(f)} \approx 0$ is identically fulfilled. From the above analysis, we see that no further constraints can be derived; thus, the procedure for deriving constraints is now completed. We also see that $u_{(\varpi)}{}^{i}_{\alpha}$, $u_{(\bar{\varpi})i\dot{\alpha}}$, $u_{(\psi)}{}^{\alpha}_{i}$, $u^{i\dot{\alpha}}_{(\bar{\psi})}$ and $u^{\hat{\imath}}_{(\mathsf{b})}$ are determined to be what are written in terms of other variables such as the canonical coordinates, while $u^{\alpha\dot{\alpha}}_{(x)}$, $u_{(a)}$, $u^{3}_{(\mathsf{b})}$, $u_{(\mathsf{e})}$ and $u_{(f)}$ still remain as arbitrary functions of τ.

We have obtained all the non-vanishing Poisson brackets between the constraint functions, as in Eqs. (5.2.5) and (5.2.13). However, it is difficult to classify the constraints into first and second classes on the basis of Eqs. (5.2.4) and (5.2.12) together with the vanishing Poisson brackets between the constraint functions. To find simpler forms of the relevant Poisson brackets, now we define

$$\tilde{\phi}^{(x)}_{\alpha\dot{\alpha}} := \phi^{(x)}_{\alpha\dot{\alpha}} - i\phi_{(\psi)}{}^{i}_{\alpha}\varpi_{i\dot{\alpha}} + i\bar{\varpi}^{i}_{\alpha}\phi_{(\bar{\psi})i\dot{\alpha}} \,, \tag{5.2.20a}$$

$$\phi^{(\mathsf{b})}_{+} := \mathsf{b}^{\hat{\imath}}\phi^{(\mathsf{b})}_{\hat{\imath}} \,, \tag{5.2.20b}$$

$$\phi^{(\mathsf{b})}_{-} := \mathsf{b}^{\hat{\imath}}\epsilon^{\hat{\imath}\hat{\jmath}}\phi^{(\mathsf{b})}_{\hat{\jmath}} \,, \tag{5.2.20c}$$

$$\tilde{\phi}^{(\mathsf{e})} := \phi^{(\mathsf{e})} + \sigma_{\hat{\imath}j}{}^{k}\left(\bar{\varpi}^{j}_{\alpha}\psi^{\alpha}_{k} + \bar{\psi}^{j\dot{\alpha}}\varpi_{k\dot{\alpha}}\right)\phi^{(\mathsf{b})}_{\hat{\imath}} \,, \tag{5.2.20d}$$

$$\tilde{\chi}^{(a)} := \chi^{(a)} - i\phi_{(\psi)}{}^{i}_{\alpha}\psi^{\alpha}_{i} + i\bar{\psi}^{i\dot{\alpha}}\phi_{(\bar{\psi})i\dot{\alpha}} + i\bar{\varpi}^{i}_{\alpha}\phi_{(\bar{\varpi})}{}^{\alpha}_{i} - i\phi^{i\dot{\alpha}}_{(\varpi)}\varpi_{i\dot{\alpha}} \,, \tag{5.2.20e}$$

$$\tilde{\chi}^{(\mathsf{b})}_{3} := \chi^{(\mathsf{b})}_{3} + \Lambda_{3} + 2\mathsf{b}^{\hat{\imath}}\epsilon^{\hat{\imath}\hat{\jmath}}\phi^{(\mathsf{b})}_{\hat{\jmath}} \,, \tag{5.2.20f}$$

$$\tilde{\chi}^{(\mathsf{b})}_{+} := \mathsf{b}^{\hat{\imath}}\left(\chi^{(\mathsf{b})}_{\hat{\imath}} + \Lambda_{\hat{\imath}}\right) \,, \tag{5.2.20g}$$

$$\tilde{\chi}^{(\mathsf{b})}_{-} := \mathsf{b}^{\hat{\imath}}\epsilon^{\hat{\imath}\hat{\jmath}}\left(\chi^{(\mathsf{b})}_{\hat{\jmath}} + \Lambda_{\hat{\jmath}}\right) + 4t\mathsf{e}\mathsf{b}^{\hat{\imath}}\phi^{(\mathsf{b})}_{\hat{\imath}} \,, \tag{5.2.20h}$$

$$\tilde{\chi}^{(f)} := \chi^{(f)} + i\bar{\varpi}^{j\alpha}\phi_{(\psi)}{}^{i}_{\alpha}\varpi^{\dot{\alpha}}_{i}\varpi_{j\dot{\alpha}} - i\varpi^{\dot{\alpha}}_{j}\phi_{(\bar{\psi})i\dot{\alpha}}\bar{\varpi}^{i\alpha}\bar{\varpi}^{j}_{\alpha} \,, \tag{5.2.20i}$$

$$\tilde{\chi}^{(\mathsf{e})} := \chi^{(\mathsf{e})} + \mathsf{e}\mathsf{b}^{\hat{\imath}}\chi^{(\mathsf{b})}_{\hat{\imath}} \,, \tag{5.2.20j}$$

where $\Lambda_r\,(r=\hat{\imath},3)$ are defined as

$$\Lambda_r := i\sigma_{ri}{}^{j}\left(\bar{\varpi}^{i}_{\alpha}\phi_{(\bar{\varpi})}{}^{\alpha}_{j} - \phi^{i\dot{\alpha}}_{(\varpi)}\varpi_{j\dot{\alpha}} + \bar{\psi}^{i\dot{\alpha}}\phi_{(\bar{\psi})j\dot{\alpha}} - \phi_{(\psi)}{}^{i}_{\alpha}\psi^{\alpha}_{j}\right) . \tag{5.2.21}$$

It is immediately seen that the set of all constraints $\Big(\phi^{(x)}_{\alpha\dot{\alpha}},\ \phi_{(\bar{\varpi})}{}^{\alpha}_{i},\ \phi^{i\dot{\alpha}}_{(\varpi)},\ \phi_{(\psi)}{}^{i}_{\alpha},\ \phi_{(\bar{\psi})i\dot{\alpha}},$ $\phi^{(a)},\ \phi^{(\mathsf{b})}_{\hat{\imath}},\ \phi^{(\mathsf{b})}_{3},\ \phi^{(\mathsf{e})},\ \phi^{(f)},\ \chi^{(a)},\ \chi^{(\mathsf{b})}_{\hat{\imath}},\ \chi^{(\mathsf{b})}_{3},\ \chi^{(\mathsf{e})},\ \chi^{(f)}\Big) \approx 0$ is equivalent to the new set of constraints $\Big(\tilde{\phi}^{(x)}_{\alpha\dot{\alpha}},\ \phi_{(\bar{\varpi})}{}^{\alpha}_{i},\ \phi^{i\dot{\alpha}}_{(\varpi)},\ \phi_{(\psi)}{}^{i}_{\alpha},\ \phi_{(\bar{\psi})i\dot{\alpha}},\ \phi^{(a)},\ \phi^{(\mathsf{b})}_{+},\ \phi^{(\mathsf{b})}_{-},\ \phi^{(\mathsf{b})}_{3},\ \tilde{\phi}^{(\mathsf{e})},\ \phi^{(f)},\ \tilde{\chi}^{(a)},$ $\tilde{\chi}^{(\mathsf{b})}_{+},\ \tilde{\chi}^{(\mathsf{b})}_{-},\ \chi^{(\mathsf{b})}_{3},\ \tilde{\chi}^{(\mathsf{e})},\ \tilde{\chi}^{(f)}\Big) \approx 0$. We can show that except for

$$\begin{aligned} \left\{\phi_{(\bar{\varpi})}{}^{\alpha}_{i}, \phi_{(\psi)}{}^{j}_{\beta}\right\} &= i\delta^{j}_{i}\delta^{\alpha}_{\beta} \,, & \left\{\phi^{i\dot{\alpha}}_{(\varpi)}, \phi_{(\bar{\psi})j\dot{\beta}}\right\} &= -i\delta^{i}_{j}\delta^{\dot{\alpha}}_{\dot{\beta}} \,, \\ \left\{\phi^{(\mathsf{b})}_{+}, \tilde{\chi}^{(\mathsf{b})}_{+}\right\} &= \frac{\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}}}{e} \,, & \left\{\phi^{(\mathsf{b})}_{-}, \tilde{\chi}^{(\mathsf{b})}_{-}\right\} &= \frac{\mathsf{b}^{\hat{\imath}}\mathsf{b}^{\hat{\imath}}}{e} \,, \end{aligned} \tag{5.2.22}$$

all the other Poisson brackets between the constraint functions in the new set vanish. In this way, the relevant Poisson brackets are simplified in terms of $\tilde{\phi}^{(x)}_{\alpha\dot{\alpha}}$, $\phi^{(\mathsf{b})}_{+}$, $\phi^{(\mathsf{b})}_{-}$, $\tilde{\phi}^{(\mathsf{e})}$, $\tilde{\chi}^{(a)}$, $\tilde{\chi}^{(\mathsf{b})}_{3}$, $\tilde{\chi}^{(\mathsf{b})}_{+}$, $\tilde{\chi}^{(\mathsf{b})}_{-}$, $\tilde{\chi}^{(f)}$ and $\tilde{\chi}^{(\mathsf{e})}$, and the matrix consisting of those Poisson brackets has a maximal invertible submatrix as

$$
\begin{array}{c|cccccccc|}
 & \phi_{(\bar\varpi)}{}_j^{\beta} & \phi^{j\dot\beta}_{(\varpi)} & \phi_{(\psi)}{}^j_{\beta} & \phi_{(\bar\psi)j\dot\beta} & \phi^{(\mathsf{b})}_{+} & \phi^{(\mathsf{b})}_{-} & \tilde\chi^{(\mathsf{b})}_{+} & \tilde\chi^{(\mathsf{b})}_{-} \\
\phi_{(\bar\varpi)}{}_i^{\alpha} & 0 & 0 & i\delta_i^j\delta^\alpha_\beta & 0 & 0 & 0 & 0 & 0 \\
\phi^{i\dot\alpha}_{(\varpi)} & 0 & 0 & 0 & -i\delta^i_j\delta^{\dot\alpha}_{\dot\beta} & 0 & 0 & 0 & 0 \\
\phi_{(\psi)}{}^i_{\alpha} & -i\delta^i_j\delta^\beta_\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{(\bar\psi)i\dot\alpha} & 0 & i\delta^j_i\delta^{\dot\beta}_{\dot\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi^{(\mathsf{b})}_{+} & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{b}^{\hat\imath}\mathsf{b}^{\hat\imath}/\mathsf{e} & 0 \\
\phi^{(\mathsf{b})}_{-} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{b}^{\hat\imath}\mathsf{b}^{\hat\imath}/\mathsf{e} \\
\tilde\chi^{(\mathsf{b})}_{+} & 0 & 0 & 0 & 0 & -\mathsf{b}^{\hat\imath}\mathsf{b}^{\hat\imath}/\mathsf{e} & 0 & 0 & 0 \\
\tilde\chi^{(\mathsf{b})}_{-} & 0 & 0 & 0 & 0 & 0 & -\mathsf{b}^{\hat\imath}\mathsf{b}^{\hat\imath}/\mathsf{e} & 0 & 0
\end{array} . \tag{5.2.23}
$$

We can see from this matrix that $\tilde\phi^{(x)}_{\alpha\dot\alpha}$, $\phi^{(a)}$, $\phi^{(\mathsf{b})}_3$, $\tilde\phi^{(\mathsf{e})}$, $\phi^{(f)}$, $\tilde\chi^{(a)}$, $\tilde\chi^{(\mathsf{b})}_3$, $\tilde\chi^{(\mathsf{e})}$ and $\tilde\chi^{(f)}$ are first-class constraints, while $\phi_{(\bar\varpi)}{}_i^{\alpha}$, $\phi^{i\dot\alpha}_{(\varpi)}$ $\phi_{(\psi)}{}^i_{\alpha}$, $\phi_{(\bar\psi)i\dot\alpha}$, $\phi^{(\mathsf{b})}_{+}$, $\phi^{(\mathsf{b})}_{-}$, $\tilde\chi^{(\mathsf{b})}_{+}$and $\tilde\chi^{(\mathsf{b})}_{-}$ are second-class constraints.

Following Dirac's approach to second-class constraints, we define the Dirac bracket with the aid of the largest invertible submatrix of the matrix (5.2.23). For arbitrary functions f and g of the canonical variables, the Dirac bracket is defined by

$$
\begin{aligned}
&\{f,g\}_{\mathrm{D}} \\
&:= \{f,g\} - i\{f,\phi_{(\bar\varpi)}{}_i^{\alpha}\}\{\phi_{(\psi)}{}^i_{\alpha},g\} + i\,\{f,\phi^{i\dot\alpha}_{(\varpi)}\}\{\phi_{(\bar\psi)i\dot\alpha},g\} + i\,\{f,\phi_{(\psi)}{}^i_{\alpha}\}\{\phi_{(\bar\varpi)}{}_i^{\alpha},g\} \\
&\quad - i\,\{f,\phi_{(\bar\psi)i\dot\alpha}\}\{\phi^{i\dot\alpha}_{(\varpi)},g\} + \frac{\mathsf{e}}{\mathsf{b}^{\hat\imath}\mathsf{b}^{\hat\imath}}\Big(\Big\{f,\phi^{(\mathsf{b})}_{+}\}\{\tilde\chi^{(\mathsf{b})}_{+},g\Big\} - \Big\{f,\tilde\chi^{(\mathsf{b})}_{+}\Big\}\Big\{\phi^{(\mathsf{b})}_{+},g\Big\} \\
&\quad + \Big\{f,\phi^{(\mathsf{b})}_{-}\Big\}\Big\{\tilde\chi^{(\mathsf{b})}_{-},g\Big\} - \Big\{f,\tilde\chi^{(\mathsf{b})}_{-}\Big\}\Big\{\phi^{(\mathsf{b})}_{-},g\Big\}\Big)\ .
\end{aligned} \tag{5.2.24}
$$

Because the Dirac bracket between f and each of the functions $\phi_{(\bar\varpi)}{}_i^{\alpha}$, $\phi^{i\dot\alpha}_{(\varpi)}$, $\phi_{(\psi)}{}^i_{\alpha}$, $\phi_{(\bar\psi)i\dot\alpha}$, $\phi^{(\mathsf{b})}_{+}$, $\phi^{(\mathsf{b})}_{-}$, $\tilde\chi^{(\mathsf{b})}_{+}$ and $\tilde\chi^{(\mathsf{b})}_{-}$ vanishes identically, the second-class constraint can be set strongly equal to zero and may be expressed as $\phi_{(\bar\varpi)}{}_i^{\alpha}=0$, $\phi^{i\dot\alpha}_{(\varpi)}=0$, $\phi_{(\psi)}{}^i_{\alpha}=0$, $\phi_{(\bar\psi)i\dot\alpha}=0$, $\phi^{(\mathsf{b})}_{+}=0$, $\tilde\phi^{(\mathsf{b})}_{-}=0$, $\tilde\chi^{(\mathsf{b})}_{+}=0$ and $\tilde\chi^{(\mathsf{b})}_{-}=0$, as long as the Dirac bracket $\{f,g\}_{\mathrm{D}}$ is adopted. We see that the second-class constraints lead to

$$
P_{(\bar\varpi)}{}_i^{\alpha} = -i\psi^\alpha_i, \qquad P^{i\dot\alpha}_{(\varpi)} = i\bar\psi^{i\dot\alpha}, \tag{5.2.25a}
$$

$$
P_{(\psi)}{}^i_{\alpha} = 0, \qquad P^{i\dot\alpha}_{(\bar\psi)} = 0, \tag{5.2.25b}
$$

$$
\mathsf{b}^{\hat\imath} = \mathsf{e}\sigma_{\hat\imath j}{}^{k}\left(\bar\varpi^j_\alpha\psi^\alpha_k + \bar\psi^{j\dot\alpha}\varpi_{k\dot\alpha}\right), \qquad P^{(\mathsf{b})}_{\hat\imath} = 0. \tag{5.2.25c}
$$

Accordingly, ψ_i^α and $\bar{\psi}^{i\dot{\alpha}}$ can be identified with the conjugate momenta of $\bar{\varpi}_\alpha^i$ and $\varpi_{i\dot{\alpha}}$, respectively (up to multiplicative constants), and $\mathsf{b}^{\hat{\imath}}$ become functions of $\bar{\varpi}_\alpha^i$, $\varpi_{i\dot{\alpha}}$, ψ_i^α, $\bar{\psi}^{i\dot{\alpha}}$ and e. Hereafter, with the Dirac bracket $\{f, g\}_{\mathrm{D}}$, we treat $\left(x^{\alpha\dot{\alpha}},\, \bar{\varpi}_\alpha^i,\, \varpi_{i\dot{\alpha}},\, a,\, \mathsf{b}^3,\, \mathsf{e},\, f\right)$ as canonical coordinates and treat $\Big(P_{\alpha\dot{\alpha}}^{(x)},\, \psi_i^\alpha,\, \bar{\psi}^{i\dot{\alpha}},\, P^{(a)},$ $P_3^{(\mathsf{b})},\, P^{(\mathsf{e})}, P^{(f)}\Big)$ as their corresponding conjugate momenta. The nonvanishing Dirac brackets between these canonical variables are found from Eqs. (5.2.24) and (5.2.3) to be

$$\left\{x^{\alpha\dot{\alpha}},\, P_{\beta\dot{\beta}}^{(x)}\right\}_{\mathrm{D}} = \delta_\beta^\alpha \delta_{\dot{\beta}}^{\dot{\alpha}}\,, \qquad \left\{f,\, P_{(f)}\right\}_{\mathrm{D}} = 1\,, \tag{5.2.26a}$$

$$\left\{\bar{\varpi}_\alpha^i,\, \psi_j^\beta\right\}_{\mathrm{D}} = i\delta_j^i \delta_\alpha^\beta\,, \qquad \left\{\varpi_{i\dot{\alpha}},\, \bar{\psi}^{j\dot{\beta}}\right\}_{\mathrm{D}} = -i\delta_i^j \delta_{\dot{\alpha}}^{\dot{\beta}}\,, \tag{5.2.26b}$$

$$\left\{a,\, P_{(a)}\right\}_{\mathrm{D}} = 1\,, \qquad \left\{\mathsf{b}^3,\, P_3^{(\mathsf{b})}\right\}_{\mathrm{D}} = 1\,, \tag{5.2.26c}$$

$$\left\{\mathsf{e},\, P_{(\mathsf{e})}\right\}_{\mathrm{D}} = 1\,. \tag{5.2.26d}$$

Because the second-class constraints are now strong equations, Eqs. (5.2.20a), (5.2.20d), (5.2.20e), (5.2.20f), (5.2.20i) and (5.2.20j) reduce to $\tilde{\phi}_{\alpha\dot{\alpha}}^{(x)} = \phi_{\alpha\dot{\alpha}}^{(x)}$, $\tilde{\phi}^{(\mathsf{e})} = \phi^{(\mathsf{e})}$, $\tilde{\chi}^{(a)} = \chi^{(a)}$, $\tilde{\chi}_3^{(\mathsf{b})} = \chi_3^{(\mathsf{b})}$, $\tilde{\chi}^{(\mathsf{e})} = \chi^{(\mathsf{e})}$ and $\tilde{\chi}^{(f)} = \chi^{(f)}$. Substituting the first equation in Eq. (5.2.25c) into Eq. (5.2.12d), we see that the first-class constraint $\chi^{(\mathsf{e})} \approx 0$ can be expressed as

$$\check{\chi}^{(\mathsf{e})} := 4T_{\hat{\imath}}T_{\hat{\imath}} - k^2 \approx 0\,, \tag{5.2.27}$$

where $T_{\hat{\imath}}$ are defined in

$$T_0 := \frac{1}{2}\left(\bar{\varpi}_\alpha^i \psi_i^\alpha + \bar{\psi}^{i\dot{\alpha}}\varpi_{i\dot{\alpha}}\right)\,, \qquad T_r := \frac{1}{2}\sigma_{rj}{}^k\left(\bar{\varpi}_\alpha^j \psi_k^\alpha + \bar{\psi}^{j\dot{\alpha}}\varpi_{k\dot{\alpha}}\right)\,. \tag{5.2.28}$$

Using Eq. (5.2.26), we see that T_0 and T_r constitute a bases of the $U(1)_a \times SU(2)$ Lie algebra in the following sense:

$$\{T_0,\, T_r\}_{\mathrm{D}} = 0\,, \quad \{T_r,\, T_s\}_{\mathrm{D}} = \epsilon_{rst}T_t\,. \tag{5.2.29}$$

From the above analysis of the constrained Hamiltonian system, it follows that the set of all the first-class constraints that we should take into account is eventually

$$\left(\phi_{\alpha\dot{\alpha}}^{(x)},\, \phi^{(a)},\, \phi_3^{(\mathsf{b})},\, \phi^{(\mathsf{e})},\, \phi^{(f)},\, \chi^{(a)},\, \chi_3^{(\mathsf{b})},\, \check{\chi}^{(\mathsf{e})},\, \chi^{(f)}\right) \approx 0\,. \tag{5.2.30}$$

5.3 Canonical quantization

In this section, we carry out canonical quantization of the Hamiltonian system analyzed in Sec. 5.2. To this end, in accordance with Dirac's procedure of quantization, we introduce the operators $\hat{f}$ and $\hat{g}$ corresponding to the functions f and g, respectively, and set the commutation relation

$$\left[\hat{f}, \hat{g}\right] = i\, \widehat{\{f, g\}}_{\mathrm{D}} \tag{5.3.1}$$

in units such that $\hbar = 1$. Here, $\widehat{\{f, g\}}_{\mathrm{D}}$ denotes the operator corresponding to the Dirac bracket $\{f, g\}_{\mathrm{D}}$. From Eqs. (5.2.26) and (5.3.1), we have the canonical commutation relations

$$\left[\hat{x}^{\alpha\dot{\alpha}},\, \hat{P}^{(x)}_{\beta\dot{\beta}}\right] = i\delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}}\,, \qquad \left[\hat{f},\, \hat{P}_{(f)}\right\} = i\,, \tag{5.3.2a}$$

$$\left[\hat{\bar{\varpi}}{}^{i}_{\alpha},\, \hat{\psi}^{\beta}_{j}\right] = -\delta^{i}_{j}\delta^{\beta}_{\alpha}\,, \qquad \left[\hat{\varpi}_{i\dot{\alpha}},\, \hat{\bar{\psi}}{}^{j\dot{\beta}}\right] = \delta^{j}_{i}\delta^{\dot{\beta}}_{\dot{\alpha}}\,, \tag{5.3.2b}$$

$$\left[\hat{a},\, \hat{P}^{(a)}\right] = i\,, \qquad \left[\hat{\mathsf{b}}^{3},\, \hat{P}^{(\mathsf{b})}_{3}\right] = i\,, \tag{5.3.2c}$$

$$\left[\hat{\mathsf{e}},\, \hat{P}^{(\mathsf{e})}\right] = i\,. \tag{5.3.2d}$$

The other canonical commutation relations vanish.

In the quantization procedure, the first-class constraints in Eq. (5.2.30) lead to the physical state conditions

$$\hat{\phi}^{(x)}_{\alpha\dot{\alpha}}|\mathit{\Phi}\rangle = \left(\hat{P}^{(x)}_{\alpha\dot{\alpha}} + \hat{\bar{\varpi}}{}^{i}_{\alpha}\hat{\varpi}_{i\dot{\alpha}}\right)|\mathit{\Phi}\rangle = 0, \tag{5.3.3a}$$

$$\hat{\phi}^{(a)}|\mathit{\Phi}\rangle = \hat{P}^{(a)}|\mathit{\Phi}\rangle = 0, \tag{5.3.3b}$$

$$\hat{\phi}^{(\mathsf{b})}_{3}|\mathit{\Phi}\rangle = \hat{P}^{(\mathsf{b})}_{3}|\mathit{\Phi}\rangle = 0, \tag{5.3.3c}$$

$$\hat{\phi}^{(\mathsf{e})}|\mathit{\Phi}\rangle = \hat{P}^{(\mathsf{e})}|\mathit{\Phi}\rangle = 0, \tag{5.3.3d}$$

$$\hat{\phi}^{(f)}|\mathit{\Phi}\rangle = \hat{P}^{(f)}|\mathit{\Phi}\rangle = 0, \tag{5.3.3e}$$

$$\hat{\chi}^{(a)}|\mathit{\Phi}\rangle = \left(\hat{\bar{\varpi}}{}^{i}_{\alpha}\hat{\psi}^{\alpha}_{i} + \hat{\bar{\psi}}{}^{i\dot{\alpha}}\hat{\varpi}_{i\dot{\alpha}} - 2s\right)|\mathit{\Phi}\rangle = 2\left(\hat{T}_{0} - s\right)|\mathit{\Phi}\rangle = 0, \tag{5.3.3f}$$

$$\hat{\chi}^{(\mathsf{b})}_{3}|\mathit{\Phi}\rangle = \left[\sigma_{3j}{}^{k}\left(\hat{\bar{\varpi}}{}^{j}_{\alpha}\hat{\psi}^{\alpha}_{k} + \hat{\bar{\psi}}{}^{j\dot{\alpha}}\hat{\varpi}_{k\dot{\alpha}}\right) - 2t\right]|\mathit{\Phi}\rangle = 2\left(\hat{T}_{3} - t\right)|\mathit{\Phi}\rangle = 0, \tag{5.3.3g}$$

$$\hat{\chi}^{(\mathsf{e})}|\mathit{\Phi}\rangle = \left(\hat{T}_{\hat{\imath}}\hat{T}_{\hat{\imath}} - \frac{k^{2}}{4}\right)|\mathit{\Phi}\rangle = 0, \tag{5.3.3h}$$

$$\hat{\chi}^{(f)}|\mathit{\Phi}\rangle = \frac{1}{2}\left(\hat{\bar{\varpi}}{}^{i\alpha}\hat{\varpi}^{\dot{\alpha}}_{i}\hat{\bar{\varpi}}{}^{j}_{\alpha}\hat{\varpi}_{j\dot{\alpha}} - m^{2}\right)|\mathit{\Phi}\rangle = 0, \tag{5.3.3i}$$

where $|\Phi\rangle$ denotes a physical state, $\hat{T}_0$ and $\hat{T}_r$ ($r = \hat{\imath}, 3$) are defined by

$$\hat{T}_0 := \frac{1}{2}\left(\hat{\bar{\varpi}}^i_\alpha \hat{\psi}^\alpha_i + \hat{\varpi}_{i\dot{\alpha}} \hat{\bar{\psi}}^{i\dot{\alpha}}\right), \qquad \hat{T}_r := \frac{1}{2}\sigma_{rj}{}^k \left(\hat{\bar{\varpi}}^j_\alpha \hat{\psi}^\alpha_k + \hat{\varpi}_{k\dot{\alpha}} \hat{\bar{\psi}}^{j\dot{\alpha}}\right). \tag{5.3.4}$$

In defining the operators $\hat{\phi}^{(x)}_{\alpha\dot{\alpha}}$, $\hat{\chi}^{(a)}$, $\hat{\chi}^{(\mathsf{b})}_3$ and $\hat{\bar{\chi}}^{(\mathsf{e})}$, we have obeyed the Weyl ordering rule. Then we have used the relevant canonical commutation relations to simplify the Weyl ordered operators. Using Eq. (5.3.2b), we can easily show that

$$\left[\hat{T}_0, \hat{T}_r\right] = 0\,, \qquad \left[\hat{T}_r, \hat{T}_s\right] = i\epsilon_{rst}\hat{T}_t\,, \tag{5.3.5}$$

which is precisely the quantum mechanical counterpart of Eq. (5.2.29). It is evident that $\hat{T}_0$ is the generator of $U(1)_a$ and $\hat{T}_r$ ($r = 1, 2, 3$) are the generators of $SU(2)$. In particular, $\hat{T}_3$ is the generator of $U(1)_{\mathsf{b}}$.

Now, we introduce the bra-vector

$$\begin{aligned}&\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \\ &:= \langle 0| \exp\left(ix^{\alpha\dot{\alpha}}\hat{P}^{(x)}_{\alpha\dot{\alpha}} + if\hat{P}^{(f)} + \bar{\varpi}^i_\alpha \hat{\psi}^\alpha_i - \varpi_{i\dot{\alpha}}\hat{\bar{\psi}}^{i\dot{\alpha}} + ia\hat{P}^{(a)} + i\mathsf{b}^3\hat{P}^{(\mathsf{b})}_3 + i\mathsf{e}\hat{P}^{(\mathsf{e})}\right)\end{aligned} \tag{5.3.6}$$

with the reference bra-vector $\langle 0|$ specified by

$$\langle 0|\hat{x}^{\alpha\dot{\alpha}} = \langle 0|\hat{f} = \langle 0|\hat{\bar{\varpi}}^i_\alpha = \langle 0|\hat{\varpi}_{i\dot{\alpha}} = \langle 0|\hat{a} = \langle 0|\hat{\mathsf{b}}^3 = \langle 0|\hat{\mathsf{e}} = 0\,. \tag{5.3.7}$$

Using the commutation relations in Eq. (5.3.2), we can show that

$$\begin{aligned}
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\hat{x}^{\alpha\dot{\alpha}} &= x^{\alpha\dot{\alpha}}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\,,\\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\hat{f} &= f\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\,,\\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\hat{\bar{\varpi}}^i_\alpha &= \bar{\varpi}^i_\alpha\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\,,\\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\hat{\varpi}_{i\dot{\alpha}} &= \varpi_{i\dot{\alpha}}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\,,\\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\hat{a} &= a\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\,,\\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\hat{\mathsf{b}}^3 &= \mathsf{b}^3\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\,,\\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\hat{\mathsf{e}} &= \mathsf{e}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}|\,.
\end{aligned} \tag{5.3.8}$$

Also, it is easy to see that

$$
\begin{aligned}
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}, \mathsf{e}| \hat{P}^{(x)}_{\alpha\dot{\alpha}} &= -i\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \,, \\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \hat{P}^{(f)} &= -i\frac{\partial}{\partial f}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \,, \\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \hat{\psi}^{\alpha}_{i} &= \frac{\partial}{\partial \bar{\varpi}^{i}_{\alpha}}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \,, \\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \hat{\bar{\psi}}^{i\dot{\alpha}} &= -\frac{\partial}{\partial \varpi_{i\dot{\alpha}}}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \,, \\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \hat{P}^{(a)} &= -i\frac{\partial}{\partial a}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \,, \\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \hat{P}^{(\mathsf{b})}_{3} &= -i\frac{\partial}{\partial \mathsf{b}^3}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \,, \\
\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \hat{P}^{(\mathsf{e})} &= -i\frac{\partial}{\partial \mathsf{e}}\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}| \,. \qquad (5.3.9)
\end{aligned}
$$

Multiplying each of Eqs. (5.3.3a)–(5.3.3i) by $\langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}, \mathsf{e}|$ on the left and using Eqs. (5.3.8) and (5.3.9), we have

$$
\begin{aligned}
\left(-i\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + \bar{\varpi}^{i}_{\alpha}\varpi_{i\dot{\alpha}}\right) \mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10a) \\
-i\frac{\partial}{\partial a}\mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10b) \\
-i\frac{\partial}{\partial \mathsf{b}^3}\mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10c) \\
-i\frac{\partial}{\partial \mathsf{e}}\mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10d) \\
-i\frac{\partial}{\partial f}\mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10e) \\
(\mathcal{T}_0 - s)\, \mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10f) \\
(\mathcal{T}_3 - t)\, \mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10g) \\
\left(\mathcal{T}_{\hat{\imath}}\mathcal{T}_{\hat{\imath}} - \frac{k^2}{4}\right) \mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0, && (5.3.10h) \\
\left(\bar{\varpi}^{i\alpha}\varpi^{\dot{\alpha}}_{i}\bar{\varpi}^{j}_{\alpha}\varpi_{j\dot{\alpha}} - m^2\right) \mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) &= 0\,, && (5.3.10i)
\end{aligned}
$$

with $\mathit{\Phi}(x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}) := \langle x, f, \bar{\varpi}, \varpi, a, \mathsf{b}^3, \mathsf{e}\,|\mathit{\Phi}\rangle$. Here, $\mathcal{T}_0$ and $\mathcal{T}_r$ $(r = \hat{\imath}, 3)$ are defined by

$$
\mathcal{T}_0 := \frac{1}{2}\left(\bar{\varpi}^{i}_{\alpha}\frac{\partial}{\partial \bar{\varpi}^{i}_{\alpha}} - \varpi_{i\dot{\alpha}}\frac{\partial}{\partial \varpi_{i\dot{\alpha}}}\right), \qquad \mathcal{T}_r := \frac{1}{2}\sigma_{rj}{}^{k}\left(\bar{\varpi}^{j}_{\alpha}\frac{\partial}{\partial \bar{\varpi}^{k}_{\alpha}} - \varpi_{k\dot{\alpha}}\frac{\partial}{\partial \varpi_{j\dot{\alpha}}}\right). \qquad (5.3.11)
$$

Equations (5.3.10b)–(5.3.10e) imply that Φ is independent of a, b^3, e and f. Hence it follows that Φ is a function of $x^{\alpha\dot{\alpha}}$, $\bar{\varpi}^i_\alpha$, and $\varpi_{i\dot{\alpha}}$. Equations (5.3.10a), (5.3.10f) and (5.3.10g) can be simultaneously solved for any arbitrary real constants s and t. However, if the solution is required to be a Lorentz spinor consisting only of $\bar{\varpi}^i_\alpha$, $\varpi_{i\dot{\alpha}}$ and $x^{\alpha\dot{\alpha}}$, it is restricted to

$$\tilde{\Phi}^{i_1\cdots i_p}_{\alpha_1\cdots\alpha_p;\, j_1\cdots j_q,\, \dot{\alpha}_1\cdots\dot{\alpha}_q}(x, \bar{\varpi}, \varpi) = \bar{\varpi}^{i_1}_{\alpha_1}\cdots\bar{\varpi}^{i_p}_{\alpha_p}\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\exp\left(-ix^{\alpha\dot{\alpha}}\bar{\varpi}^i_\alpha\varpi_{i\dot{\alpha}}\right), \tag{5.3.12}$$

and accordingly s and t are determined to be

$$s = \frac{p_1 + p_2 - q_1 - q_2}{2}, \quad t = \frac{p_1 - p_2 - q_1 + q_2}{2}, \qquad p_1,\, p_2,\, q_1,\, q_2 = 0, 1, 2, \dots\ . \tag{5.3.13}$$

Here, p_1 is the number of $\bar{\varpi}^1_\alpha$ in Eq. (5.3.12) and $p_2(= p - p_1)$ is the number of $\bar{\varpi}^2_\alpha$. Similarly, q_1 is the number of $\varpi_{1\dot{\alpha}}$ in Eq. (5.3.12) and $q_2(= q - q_1)$ is the number of $\varpi_{2\dot{\alpha}}$. In this way, the allowed values of constants s and t turn out to be either integer or half-integer values. It is obvious from Eq. (5.3.12) that $\tilde{\Phi}$ has the symmetric properties:

$$\tilde{\Phi}^{i_1\dots i_m\dots i_n\dots i_p}_{\alpha_1\dots\alpha_m\dots\alpha_n\dots\alpha_p;j_1\dots j_q,\dot{\alpha}_1\dots\dot{\alpha}_q} = \tilde{\Phi}^{i_1\dots i_n\dots i_m\dots i_p}_{\alpha_1\dots\alpha_n\dots\alpha_m\dots\alpha_p;j_1\dots j_q,\dot{\alpha}_1\dots\dot{\alpha}_q}, \tag{5.3.14a}$$

$$\tilde{\Phi}^{i_1\dots i_p}_{\alpha_1\dots\alpha_p;j_1\dots j_a\dots j_b\dots j_q,\dot{\alpha}_1\dots\dot{\alpha}_a\dots\dot{\alpha}_b\dots\dot{\alpha}_q} = \tilde{\Phi}^{i_1\dots i_p}_{\alpha_1\dots\alpha_p;j_1\dots j_b\dots j_a\dots j_q,\dot{\alpha}_1\dots\dot{\alpha}_b\dots\dot{\alpha}_a\dots\dot{\alpha}_q}. \tag{5.3.14b}$$

The operators $\mathcal{T}_r$ fulfill the $SU(2)$ commutation relation

$$[\mathcal{T}_r,\, \mathcal{T}_s] = i\epsilon_{rst}\mathcal{T}_t\,. \tag{5.3.15}$$

Following the general method for solving the eigenvalue problem in the $SU(2)$ Lie algebra, we can simultaneously solve the eigenvalue equation for the Casimir operator $\mathcal{T}_r\mathcal{T}_r = \mathcal{T}_{\hat{\imath}}\mathcal{T}_{\hat{\imath}} + \mathcal{T}_3\mathcal{T}_3$, i.e.,

$$\mathcal{T}_r\mathcal{T}_r\Phi = \Lambda\Phi \tag{5.3.16}$$

and Eq. (5.3.10g) to obtain

$$\Lambda = I(I+1), \quad I = 0,\, \frac{1}{2},\, 1,\, \frac{3}{2},\, \dots, \tag{5.3.17}$$

$$t = -I,\, -I+1,\, \dots, I-1,\, I. \tag{5.3.18}$$

For convenience, we introduce

$$\zeta^{i_1\ldots i_p i_{p+1}\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q} := \epsilon^{i_{p+1}k_1}\cdots\epsilon^{i_{p+q}k_q}\tilde{\Phi}^{i_1\cdots i_p}_{\alpha_1\cdots\alpha_p;\,k_1\cdots k_q,\,\dot{\alpha}_1\cdots\dot{\alpha}_q} \,. \tag{5.3.19}$$

Using this, we can find that the eigenfunction Φ, being the solution of (5.3.10a), (5.3.10f), (5.3.10g) and (5.3.10h), is given by symmetrizing the indices $i_1, \ldots, i_{p+q}$ in Eq. (5.3.19) as

$$\begin{aligned} &\Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\dot{\alpha}_1\ldots\dot{\alpha}_q}(x,\bar{\varpi},\varpi) \\ &:= \zeta^{(i_1\ldots i_{p+q})}_{\alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(x,\bar{\varpi},\varpi) \equiv \sum_{\text{perm.}} \zeta^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(x,\bar{\varpi},\varpi)\,, \end{aligned} \tag{5.3.20}$$

where $\sum_{\text{perm.}}$ denotes the sum over all permutations of $\{i_1, \cdots, i_{p+q}\}$. In addition, I is determined to be

$$I = \frac{p+q}{2} = \frac{p_1 + p_2 + q_1 + q_2}{2}\,. \tag{5.3.21}$$

It is follows from Eqs. (5.3.14a), (5.3.14b) and (5.3.20) that Φ has the symmetric properties

$$\Phi^{i_1\ldots i_{p+q}}_{\ldots\alpha_a\ldots\alpha_b\ldots;\dot{\alpha}_1\ldots\dot{\alpha}_q} = \Phi^{i_1\ldots i_{p+q}}_{\ldots\alpha_b\ldots\alpha_a\ldots;\dot{\alpha}_1\ldots\dot{\alpha}_q}\,, \tag{5.3.22a}$$

$$\Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\ldots\dot{\alpha}_a\ldots\dot{\alpha}_b\ldots} = \Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\ldots\dot{\alpha}_a\ldots\dot{\alpha}_b\ldots}\,. \tag{5.3.22b}$$

From Eqs. (5.3.10g), (5.3.10h), (5.3.16) and (5.3.17), the allowed values of the positive constant k are determined to be

$$k = 2\sqrt{I(I+1) - t^2}\,. \tag{5.3.23}$$

It is now clear that Φ is characterized by the set of three quantum numbers $(s,\, I,\, t)$ or, equivalently, by $(p,\, q,\, t)$. In addition, Φ is also characterized by an another set $(s_1,\, s_2,\, I)$, where $s_1 := s + t\,,\ s_2 := s - t\,.$

Since the coordinate time is given by $x^0 = (x^{0\dot{0}} + x^{1\dot{1}})/\sqrt{2}$, we see that Eq. (5.3.20) describes a plane wave of the positive-frequency $(|\varpi_{1\dot{0}}|^2 + |\varpi_{2\dot{0}}|^2 + |\varpi_{1\dot{1}}|^2 + |\varpi_{2\dot{1}}|^2)/\sqrt{2}$. A negative-frequency plane-wave function can be obtained by taking the complex conjugate of Eq. (5.3.20).

5.4 Positive and negative frequency fields and the generalized DFP equations

In this section, we construct well-defined positive-frequency and negative-frequency spinor wave functions from the plane-wave solutions, considering a regularization method to have well-defined spinor wave functions. We also find Penrose transforms via appropriate Fourier-Laplace transforms.

5.4.1 Positive-frequency wave function

We consider the positive-frequency spinor wave function defined by

$$\begin{aligned}
&\Psi^{+i_1\dots i_{p+q}}_{\alpha_1\dots\alpha_p;\dot\alpha_1\dots\dot\alpha_q}(x) \\
&:= \frac{(-1)^p}{(2\pi i)^8}\int_{\mathbb{C}^4}\tilde f^+(\bar\varpi,\varpi)\Phi^{i_1\dots i_{p+q}}_{\alpha_1\dots\alpha_p;\dot\alpha_1\dots\dot\alpha_q}(x,\bar\varpi,\varpi)d^2\bar\varpi^1\wedge d^2\bar\varpi^2\wedge d^2\varpi_1\wedge d^2\varpi_2 \\
&= \frac{(-1)^p}{(2\pi i)^8}\int_{\mathbb{C}^4}\sum_{\text{perm.}}\bar\varpi^{i_1}_{\alpha_1}\cdots\bar\varpi^{i_p}_{\alpha_p}\epsilon^{i_{p+1}j_1}\cdots\epsilon^{i_{p+q}j_q}\varpi_{j_1\dot\alpha_1}\cdots\varpi_{j_q\dot\alpha_q}\tilde f^+(\bar\varpi,\varpi) \\
&\quad\times\exp\left(-ix^{\beta\dot\beta}\bar\varpi^k_\beta\varpi_{k\dot\beta}\right)d^2\bar\varpi^1\wedge d^2\bar\varpi^2\wedge d^2\varpi_1\wedge d^2\varpi_2\ ,
\end{aligned}\tag{5.4.1}$$

where $d^2\bar\varpi^i := d\bar\varpi^i_0\wedge d\bar\varpi^i_1$, $d^2\varpi_i := d\varpi_{i\dot 0}\wedge d\varpi_{i\dot 1}$ $(i = 1, 2)$. This function is just a linear combination of $\Phi^{i_1\dots i_{p+q}}_{\alpha_1\dots\alpha_p;\dot\alpha_1\dots\dot\alpha_q}$ with the coefficient function $\tilde f^+$. When the absolute value of the integrand increase or sufficiently slowly decreases in the asymptotic region specified by $(|\varpi_{1\dot 0}|^2+|\varpi_{2\dot 0}|^2+|\varpi_{1\dot 1}|^2+|\varpi_{2\dot 1}|^2)\to\infty$, the integral in Eq. (5.4.1) is not well-defined. To make this integral well-defined, we replace $x^{\alpha\dot\alpha}$ by $z^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - iy^{\alpha\dot\alpha}$ so that the integrand can include the multiplicative exponential factor $\exp\left(-y^{\beta\dot\beta}\bar\varpi^j_\beta\varpi_{j\dot\beta}\right)$. The exponent $y^{\beta\dot\beta}\bar\varpi^j_\beta\varpi_{j\dot\beta}$ can be written as

$$\begin{aligned}
&y^{\beta\dot\beta}\bar\varpi^k_\beta\varpi_{k\dot\beta} \\
&= \frac{1}{\sqrt 2}\left(y^0+|\boldsymbol y|\right)\left(|\lambda_{1\dot 0}|^2+|\lambda_{2\dot 0}|^2\right)+\frac{1}{\sqrt 2}\left((y^0-|\boldsymbol y|\right)\left(|\lambda_{1\dot 1}|^2+|\lambda_{2\dot 1}|^2\right)
\end{aligned}\tag{5.4.2}$$

in terms of the real variables y^μ $(\mu = 0, 1, 2, 3)$ and the spinor $\lambda_{i\dot\alpha} := U_{\dot\alpha}{}^{\dot\beta}(y)\varpi_{i\dot\beta}$. Here, $|\boldsymbol y| := \sqrt{(y^1)^2+(y^2)^2+(y^3)^2}$ and

$$U(y) := \frac{1}{2|\boldsymbol y|(y^3+|\boldsymbol y|)}\begin{pmatrix} y^3+|\boldsymbol y| & y^1+iy^2 \\ y^1-iy^2 & -y^3-|\boldsymbol y|\end{pmatrix}\ . \tag{5.4.3}$$

This matrix is both unitary and Hermitian. From Eq. (5.4.2), we see that $y^{\beta\dot\beta}\bar\varpi^k_\beta\varpi_{k\dot\beta}$ is positive definite if and only if $y_\mu y^\mu \equiv (y^0)^2 - |\boldsymbol y|^2 > 0$ and $y^0 > 0$. These two conditions for y^μ together define a region called the forward (or future) tube:

$$\mathbb{C}\mathbf{M}^+ := \{(z^\mu)\in\mathbb{C}\mathbf{M}^\sharp \mid z^\mu = x^\mu - iy^\mu,\ y_\mu y^\mu > 0,\ y^0 > 0\}\ . \tag{5.4.4}$$

Here, $\mathbb{C}\mathbf{M}^\sharp$ denotes the conformal compactification of complexified Minkowski space $\mathbb{C}\mathbf{M}$. Since $y^{\beta\dot{\beta}}\bar{\varpi}^k_\beta\varpi_{k\dot{\beta}} > 0$ is valid in $\mathbb{C}\mathbf{M}^+$, the integral in

$$\begin{aligned}
&\Psi^{+\,i_1\ldots i_{p+q}}_{\;\alpha_1\ldots\alpha_p;\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)\\
&= \frac{(-1)^p}{(2\pi i)^8}\int_{\mathbb{C}^4}\sum_{\text{perm.}}\bar{\varpi}^{i_1}_{\alpha_1}\cdots\bar{\varpi}^{i_p}_{\alpha_p}\epsilon^{i_{p+1}j_1}\cdots\epsilon^{i_{p+q}j_q}\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\tilde{f}^+(\bar{\varpi},\varpi)\\
&\qquad\times\exp\left(-iz^{\beta\dot{\beta}}\bar{\varpi}^k_\beta\varpi_{k\dot{\beta}}\right)d^2\bar{\varpi}^1\wedge d^2\bar{\varpi}^2\wedge d^2\varpi_1\wedge d^2\varpi_2\,,
\end{aligned}\tag{5.4.5}$$

is well-defined for $z^\mu \in \mathbb{C}\mathbf{M}^+$. Therefore, the positive-frequency spinor wave functions is suitably defined on $\mathbb{C}\mathbf{M}^+$. In this function, $\exp\left(-y^{\beta\dot{\beta}}\bar{\varpi}^k_\beta\varpi_{k\dot{\beta}}\right)$ plays the role of a damping factor. The corresponding spinor wave function on $\mathbf{M}$ is given by

$$\Psi^{+\,i_1\ldots i_{p+q}}_{\;\alpha_1\ldots\alpha_p;\dot{\alpha}_1\ldots\dot{\alpha}_q}(x) := \lim_{y^0\downarrow 0}\Psi^{+\,i_1\ldots i_{p+q}}_{\;\alpha_1\ldots\alpha_p;\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)\,. \tag{5.4.6}$$

We can find that Eq.(5.3.10i) is equivalent to formulas

$$\bar{\varpi}^i_\alpha\bar{\varpi}^{j\alpha} = \epsilon^{ij}\frac{m}{\sqrt{2}}\mathrm{e}^{i\tilde{\varphi}}\,,\quad \varpi_{i\dot{\alpha}}\varpi^{\dot{\alpha}}_j = \epsilon_{ij}\frac{m}{\sqrt{2}}\mathrm{e}^{-i\tilde{\varphi}} \tag{5.4.7}$$

where $\tilde{\varphi}$ is an arbitrary real constant. Hereafter, we choose $\tilde{\varphi}$, in such a way that $\mathrm{e}^{i\tilde{\varphi}} = 1$. By using these formulas, it is easily seen that $\Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z,\bar{\varpi},\varpi)$ satisfies

$$\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot{\beta}\dot{\alpha}_2\ldots\dot{\alpha}_q} = i\frac{m}{\sqrt{2}}\epsilon^{\beta\gamma}\Phi^{i_1\ldots i_{p+q}}_{\gamma\alpha_1\ldots\alpha_p;\,\dot{\alpha}_2\ldots\dot{\alpha}_q}\,, \tag{5.4.8a}$$

$$\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Phi^{i_1\ldots i_{p+q}}_{\beta\alpha_2\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q} = -i\frac{m}{\sqrt{2}}\epsilon^{\beta\gamma}\Phi^{i_1\ldots i_{p+q}}_{\gamma\alpha_1\ldots\alpha_p;\,\dot{\alpha}_2\ldots\dot{\alpha}_q}\,. \tag{5.4.8b}$$

From Eqs. (5.4.5) and (5.4.8), we can prove that $\Psi^{+\,i_1\ldots i_{p+q}}_{\;\alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)$ satisfies the Dirac-Fierz-Pauli equations with $SU(2)$ indices

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Psi^{+\,i_1\ldots i_{p+q}}_{\;\alpha_1\ldots\alpha_p;\dot{\beta}\dot{\alpha}_2\ldots\dot{\alpha}_q}(z) - m\epsilon^{\beta\gamma}\Psi^{+\,i_1\ldots i_{p+q}}_{\;\gamma\alpha_1\ldots\alpha_p;\,\dot{\alpha}_2\ldots\dot{\alpha}_q}(z) = 0\,, \tag{5.4.9a}$$

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Psi^{+\,i_1\ldots i_{p+q}}_{\;\beta\alpha_2\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z) + m\epsilon^{\dot{\beta}\dot{\gamma}}\Psi^{+\,i_1\ldots i_{p+q}}_{\;\alpha_2\ldots\alpha_p;\,\dot{\gamma}\dot{\alpha}_1\ldots\dot{\alpha}_q}(z) = 0\,. \tag{5.4.9b}$$

Therefore the function $\Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z,\bar{\varpi},\varpi)$ is a particular solution of these equations. Using Eqs. (5.4.9a) and (5.4.9b) and noting

$$\frac{\partial}{\partial z^{\alpha\dot{\beta}}}\frac{\partial}{\partial z_{\beta\dot{\beta}}} = \frac{1}{2}\delta^\beta_\alpha\frac{\partial}{\partial z^{\gamma\dot{\gamma}}}\frac{\partial}{\partial z_{\gamma\dot{\gamma}}}\,, \tag{5.4.10}$$

we can derive the Klein-Gordon equation

$$\left(\frac{\partial}{\partial z^{\gamma\dot{\gamma}}}\frac{\partial}{\partial z_{\gamma\dot{\gamma}}}+m^2\right)\Psi^{+i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)=0\,. \tag{5.4.11}$$

This makes it clear that Ψ^+ is a field of mass m. Thus we obtain a spinor field of rank $p+q$ with mass m.

Now, let us consider the Fourier-Laplace transform of $\tilde{f}^+(\bar{\varpi},\varpi)$ with respect to $\bar{\varpi}^i_\alpha$

$$f^+(\varrho,\varpi):=\frac{1}{(2\pi i)^4}\oint_{\Pi^+}\tilde{f}^+(\bar{\varpi},\varpi)\exp\left(-\bar{\varpi}^i_\alpha\varrho^\alpha_i\right)d^2\bar{\varpi}^1\wedge d^2\bar{\varpi}^2 \tag{5.4.12}$$

Here, ϱ^α_i is defined by Eq. (3.3.11), and the integral is taken over a suitable four-dimensional contour, Π^+, chosen in such a manner that f^+ becomes a holomorphic function of ϱ^α_i and $\varpi_{i\dot{\alpha}}$. (The Fourier-Laplace transform (5.4.12) is consistent with the representation $\hat{\bar{\varpi}}^i_\alpha=-\partial/\partial\varrho^\alpha_i$.) Since the pair of ϱ^α_i and $\varpi_{i\dot{\alpha}}$ is precisely the twistor $\mathsf{Z}^A_i=(\varrho^\alpha_i,\,\varpi_{i\dot{\alpha}})$, the function f^+ is regarded as a holomorphic function on (nonprojective) twistor space $\mathbf{T}\times\mathbf{T}$, the direct product of two four-dimensional complex spaces coordinatized by $(\varrho^\alpha_1,\,\varpi_{1\dot{\alpha}})$ and $(\varrho^\alpha_2,\,\varpi_{2\dot{\alpha}})$ respectively, and can be expressed as $f^+(\mathsf{Z}_i)$. From the first equality of

$$y^{\beta\dot{\beta}}\bar{\varpi}^i_\beta\varpi_{i\dot{\beta}}=\mathrm{Re}(\bar{\varpi}^i_\alpha\varrho^\alpha_i)=\frac{1}{2}\left(\bar{\varpi}^i_\alpha\varrho^\alpha_i+\bar{\varrho}^{i\dot{\alpha}}\varpi_{i\dot{\alpha}}\right)\,, \tag{5.4.13}$$

it is clear that f^+ is well-defined on the condition $y^{\beta\dot{\beta}}\bar{\varpi}^i_\beta\varpi_{i\dot{\beta}}>0$. In other words, f^+ is actually well-defined on a region

$$(\mathbf{T}\times\mathbf{T})^+:=\left\{(\varrho^\alpha_i,\,\varpi_{i\dot{\alpha}})\in\mathbf{T}\times\mathbf{T}\,|\bar{\varpi}^i_\alpha\varrho^\alpha_i+\bar{\varrho}^{i\dot{\alpha}}\varpi_{i\dot{\alpha}}>0\right\}\,. \tag{5.4.14}$$

Noting that $\partial/\partial\varrho^\alpha_i\exp\left(-\bar{\varpi}^j_\beta\varrho^\beta_j\right)=-\bar{\varpi}^i_\alpha\exp\left(-\bar{\varpi}^j_\beta\varrho^\beta_j\right)$, we can write Eq. (5.4.5) in terms of $f^+(\varrho,\varpi)$ as

$$\begin{aligned}&\Psi^{+i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)\\&=\frac{1}{(2\pi i)^4}\oint_{\Sigma^+}\sum_{\text{perm.}}\epsilon^{i_{p+1}j_1}\cdots\epsilon^{i_{p+q}j_q}\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\frac{\partial}{\partial\varrho^{\alpha_1}_{i_1}}\cdots\frac{\partial}{\partial\varrho^{\alpha_p}_{i_p}}f^+(\varrho,\varpi)\\&\quad\times d^2\varpi_1\wedge d^2\varpi_2\,,\end{aligned} \tag{5.4.15}$$

where $(z^\mu)\in\mathbb{C}\mathbf{M}^+$, and Σ^+ is another four-dimensional contour. Equation (5.4.15) is identified as a nonprojective form of the Penrose transform in the massive case.

The exterior derivative of the integrand including $d^2\varpi_1 \wedge d^2\varpi_2$ vanishes with z^μ held constant

$$d\left(\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\frac{\partial}{\partial\varrho^{\alpha_1}_{i_1}}\cdots\frac{\partial}{\partial\varrho^{\alpha_p}_{i_p}}f^+(\varrho,\varpi)d^2\varpi_1\wedge d^2\varpi_2\right)=0\,. \tag{5.4.16}$$

Therefore, it can be proven by using Poincaré's lemma and Stokes' theorem that the integral itself remains invariant under the deformations of $\varSigma^+$ that are carried out continuously in the domain of the integrand. Suppose now that f^+ is homogeneous of degree $\tilde{s}_1$ with respect to $\varpi_{1\dot{\alpha}}$ and $\tilde{s}_2$ with respect to $\varpi_{2\dot{\alpha}}$, that is, $f^+(c\varrho_1, c\varpi_1, c\varrho_2, c\varpi_2) = c^{\tilde{s}_1+\tilde{s}_2}f^+(\varrho,\varpi)$ $(c \in \mathbb{C})$. Then under the replacement of $\varpi_{i\dot{\alpha}}$ by $c_i\varpi_{i\dot{\alpha}}$(no sum with respect to i), the integral changes into the multiplied by $c^{q_1+q_2-p_1-p_2+4+\tilde{s}_1+\tilde{s}_2}$ by virtue of the deformation invariance of the integral. However, this replacement cannot change the integral actually, because the $\varpi_{i\dot{\alpha}}$ are merely variables of integration. Hence, it follows that the integral vanishes if $p_1+p_2-q_1-q_2-4 \neq \tilde{s}_1+\tilde{s}_2$; only in the case of

$$p_1+p_2-q_1-q_2-4=\tilde{s}_1+\tilde{s}_2 \tag{5.4.17}$$

the integral may remain nonvanishing. In this case, the integrand including $d^2\varpi_1 \wedge d^2\varpi_2$ can be expressed as the exterior product of $d\varpi_{1\dot{0}}/\varpi_{1\dot{0}}$ and a 3-form consisting of $\zeta := \varpi_{1\dot{1}}/\varpi_{1\dot{0}}$ and $\xi_{\dot{\alpha}} := \varpi_{2\dot{\alpha}}/\varpi_{1\dot{0}}$ $(\dot{\alpha}=\dot{0},\dot{1})$. (Here, $\varpi_{1\dot{0}}$, ζ and $\xi_{\dot{\alpha}}$ are treated as independent variables.) After carrying out the contour integration over $\varpi_{1\dot{0}}$ along a topological circle surrounds $\varpi_{1\dot{0}}=0$, Eq. (5.4.15) reduces to

$$\begin{aligned}&\varPsi^{+\,i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_q;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)\\&=\frac{1}{(2\pi i)^3}\oint_{\varGamma^+}\sum_{\text{perm.}}\epsilon^{i_{p+1}j_1}\cdots\epsilon^{i_{p+q}j_q}\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\frac{\partial}{\partial\varrho^{\alpha_1}_{i_1}}\cdots\frac{\partial}{\partial\varrho^{\alpha_p}_{i_p}}f^+(\varrho,\varpi)\\&\qquad\times\frac{1}{3}\epsilon^{ij}\epsilon^{kl}\pi_{i\dot{\alpha}}d\varpi^{\dot{\alpha}}_k\wedge d\varpi_{j\dot{\beta}}\wedge d\varpi^{\dot{\beta}}_l\,,\end{aligned} \tag{5.4.18}$$

where $\varGamma^+$ denotes a three-dimensional closed contour on the $\mathbb{C}\mathbf{P}^3$ coordinatized by $(\zeta,\,\xi_{\dot{0}},\,\xi_{\dot{1}})$. Equation (5.4.18) is identified as a three-dimensional projective form of the Penrose transform [11]. It is easy to show that f^+ satisfies

$$\left(-\varrho^\alpha_i\frac{\partial}{\partial\varrho^\alpha_i}-\varpi_{i\dot{\alpha}}\frac{\partial}{\partial\varpi_{i\dot{\alpha}}}+2s-4\right)f^+(\varrho,\varpi)=0\,. \tag{5.4.19}$$

This looks like the eigenvalue equation (4.2.13e) for the generator $\check{\mathsf{T}}_0$ of the $U(1)_a$ transformation, which the twistor function $F(\mathsf{W}^A_i)$ obey in twistor formulation, but the sign of s is opposite.

5.4.2 Negative-frequency wave function

A (well-defined) negative-frequency spinor wave function can be obtained immediately by taking the complex conjugate of $\Psi^{+\,i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot\alpha_1\ldots\dot\alpha_q}(z)$. The wave function obtained in this manner is, however, a function of $\overline{z^\mu}$ and hence is anti-holomorphic. In the following, we construct a *holomorphic* negative-frequency spinor wave function.

Let $\tilde f^-(\bar\varpi,\varpi)$ be a complex function similar to $\tilde f^+(\bar\varpi,\varpi)$. The negative-frequency counterpart of $\Psi^{+\,i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot\alpha_1\ldots\dot\alpha_q}(x)$ in Eq. (5.4.1) is defined by

$$\begin{aligned}
&\Psi^{-\,i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot\alpha_1\ldots\dot\alpha_q}(x)\\
&:=\frac{1}{(2\pi i)^8}\int_{\mathbb{C}^4}\tilde f^-(\bar\varpi,\varpi)\,\Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot\alpha_1\ldots\dot\alpha_q}(-x,\bar\varpi,\varpi)d^2\bar\varpi^1\wedge d^2\bar\varpi^2\wedge d^2\varpi_1\wedge d^2\varpi_2\\
&=\frac{1}{(2\pi i)^8}\int_{\mathbb{C}^4}\sum_{\text{perm.}}\bar\varpi^{i_1}_{\alpha_1}\cdots\bar\varpi^{i_p}_{\alpha_p}\epsilon^{i_{p+1}j_1}\cdots\epsilon^{i_{p+q}j_q}\varpi_{j_1\dot\alpha_1}\cdots\varpi_{j_q\dot\alpha_q}\tilde f^-(\bar\varpi,\varpi)\\
&\quad\times\exp\left(ix^{\beta\dot\beta}\bar\varpi^k_\beta\varpi_{k\dot\beta}\right)d^2\bar\varpi^1\wedge d^2\bar\varpi^2\wedge d^2\varpi_1\wedge d^2\varpi_2\,,
\end{aligned}\tag{5.4.20}$$

where $\Phi^{i_1\ldots i_{p+q}}_{\alpha_1\ldots\alpha_p;\,\dot\alpha_1\ldots\dot\alpha_q}(-x,\bar\varpi,\varpi)$ obeys the complex conjugates of Eqs. (5.3.10a), (5.3.10f)–(5.3.10h), and its corresponding values of s, t, and I are determined to be

$$\begin{aligned}
&s=-\frac{p_1+p_2-q_1-q_2}{2}\,, &&t=-\frac{p_1-p_2-q_1+q_2}{2}\,,\\
&I=\frac{p_1+p_2+q_1+q_2}{2}=\frac{p+q}{2}\,, &&p_1,\,p_2,\,q_1,\,q_2=0,1,2,\ldots.
\end{aligned}\tag{5.4.21}$$

Note that, s and t are different from Eq. (5.3.13) only in the sign. The integral in Eq. (5.4.20) itself is not well-defined in general, and we therefore replace $x^{\alpha\dot\alpha}$ with $z^{\alpha\dot\alpha}=x^{\alpha\dot\alpha}-iy^{\alpha\dot\alpha}$ by following the case of the positive-frequency spinor wave function. Owing the replacement, the integrand is modified so as to include the damping factor $\exp\left(y^{\beta\dot\beta}\bar\varpi^i_\alpha\varpi_{i\dot\alpha}\right)$ valid on the simultaneous conditions $y_\mu y^\mu>0$ and $y^0<0$. These conditions together defined a region called the backward (or past) tube:

$$\mathbb{C}\mathbf{M}^-:=\left\{(z^\mu)\in\mathbb{C}\mathbf{M}^\sharp\mid z^\mu=x^\mu-iy^\mu,\,y_\mu y^\mu>0,\,y^0<0\right\}\,.\tag{5.4.22}$$

Since $y^{\beta\dot{\beta}}\varpi^i_\alpha\varpi_{i\dot{\alpha}}$ is fulfilled in $\mathbb{C}\mathbf{M}^-$, the integral in

$$\begin{aligned}
&\Psi^{-\,i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)\\
&= \frac{1}{(2\pi i)^8}\int_{\mathbb{C}^4}\sum_{\text{perm.}} \bar{\varpi}^{i_1}_{\alpha_1}\cdots\bar{\varpi}^{i_p}_{\alpha_p}\epsilon^{i_{p+1}j_1}\cdots\epsilon^{i_{p+q}j_q}\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\tilde{f}^-(\bar{\varpi},\varpi)\\
&\quad\times\exp\left(iz^{\beta\dot{\beta}}\bar{\varpi}^k_\beta\varpi_{k\dot{\beta}}\right)d^2\bar{\varpi}^1\wedge d^2\bar{\varpi}^2\wedge d^2\varpi_1\wedge d^2\varpi_2 \qquad (5.4.23)
\end{aligned}$$

is well-defined for $z^\mu \in \mathbb{C}\mathbf{M}^-$. It thus follows that the holomorphic negative-frequency spinor wave function is properly defined on $\mathbb{C}\mathbf{M}^-$. The corresponding spinor wave function on $\mathbf{M}$ is given by

$$\Psi^{-\,i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(x) := \lim_{y^0\uparrow 0}\Psi^{-\,i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)\,. \qquad (5.4.24)$$

Using the formula (5.4.7), we can easily prove that $\Psi^{-\,i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)$ satisfies the Dirac-Fierz-Pauli equations with $SU(2)$ indices (5.4.9a) and (5.4.9b).

Next, we consider the Fourier-Laplace transform of $\tilde{f}^-(\bar{\varpi},\varpi)$ with respect to $\bar{\varpi}^i_\alpha$:

$$f^-(\varrho^\alpha_i,\varpi_{i\dot{\alpha}}) := \frac{1}{(2\pi i)^4}\oint_{\Pi^-}\tilde{f}^-(\bar{\varpi},\varpi)\exp\left(\bar{\varpi}^i_\alpha\varrho^\alpha_i\right)d^2\bar{\varpi}^1\wedge d^2\bar{\varpi}^2\,. \qquad (5.4.25)$$

Here, the integral is taken over a suitable four-dimensional contour, Π^-, chosen in such a manner that f^- becomes a holomorphic function of ϱ^α_i and $\varpi_{i\dot{\alpha}}$. (The Fourier-Laplace transform (5.4.25) is consistent with the *conjugate* representation $\hat{\bar{\varpi}}^i_\alpha = \partial/\partial\varrho^\alpha_i$) It is clear from Eq. (5.4.13) that f^- is well-defined on a region of two-twistor space

$$(\mathbf{T}\times\mathbf{T})^- := \left\{(\varrho^\alpha_i,\,\varpi_{i\dot{\alpha}})\in\mathbf{T}\times\mathbf{T}\,|\,\bar{\varpi}^i_\alpha\varrho^\alpha_i+\bar{\varrho}^{i\dot{\alpha}}\varpi_{i\dot{\alpha}}<0\right\}\,. \qquad (5.4.26)$$

This is the region of $\mathbf{T}\times\mathbf{T}$ corresponding to $\mathbb{C}\mathbf{M}^-$; a correspondence similar to that between $(\mathbf{T}\times\mathbf{T})^+$ and $\mathbb{C}\mathbf{M}^+$ is established between $(\mathbf{T}\times\mathbf{T})^-$ and $\mathbb{C}\mathbf{M}^-$.

We can write Eq. (5.4.23) in terms of $f^-(\varrho,\varpi)$ as

$$\begin{aligned}
\Psi^{-\,i_1\ldots i_{p+q}}_{\ \ \alpha_1\ldots\alpha_p;\,\dot{\alpha}_1\ldots\dot{\alpha}_q}(z) &= \frac{1}{(2\pi i)^4}\oint_{\Sigma^-}\sum_{\text{perm.}}\epsilon^{i_{p+1}j_1}\cdots\epsilon^{i_{p+q}j_q}\varpi_{j_1\dot{\alpha}_1}\cdots\varpi_{j_q\dot{\alpha}_q}\\
&\quad\times\frac{\partial}{\partial\varrho^{\alpha_1}_{i_1}}\cdots\frac{\partial}{\partial\varrho^{\alpha_p}_{i_p}}f^-(\varrho,\varpi)d^2\varpi_1\wedge d^2\varpi_2\,, \qquad (5.4.27)
\end{aligned}$$

where $(z^\mu) \in \mathbb{C}\mathbf{M}^-$ and points $(\varrho_i^\alpha, \varpi_{i\dot\alpha})$ in two-twistor space are located in the region specified by Eq. (5.4.26). The contour Σ^- of integration is another four-dimensional contour. Suppose now that f^- is homogeneous of degree $\tilde{s}'_1$ with respect to $\varpi_{1\dot\alpha}$ and $\tilde{s}'_2$ with respect to $\varpi_{2\dot\alpha}$. Then , if $\tilde{s}'_1 + \tilde{s}'_2 \neq p_1 + p_2 - q_1 - q_2 - 4$, the integral vanishes; if $\tilde{s}'_1 + \tilde{s}'_2 = p_1 + p_2 - q_1 - q_2 - 4$, the integral may remain nonvanishing and can be written as

$$\begin{aligned}
&\Psi^{-i_1 \ldots i_p}{}_{\alpha_1 \ldots \alpha_p;\, j_1 \ldots j_q\, \dot\alpha_1 \ldots \dot\alpha_q}(z) \\
&= \frac{1}{(2\pi i)^3} \oint_{\Gamma^-} \sum_{\text{perm.}} \epsilon^{i_{p+1} j_1} \cdots \epsilon^{i_{p+q} j_q} \varpi_{j_1 \dot\alpha_1} \cdots \varpi_{j_q \dot\alpha_q} \frac{\partial}{\partial \varrho_{i_1}^{\alpha_1}} \cdots \frac{\partial}{\partial \varrho_{i_p}^{\alpha_p}} f^-(\varrho, \varpi) \\
&\quad \times \frac{1}{3} \epsilon^{ij} \epsilon^{kl} \varpi_{i\dot\alpha} d\varpi_k^{\dot\alpha} \wedge d\varpi_{j\dot\beta} \wedge d\varpi_l^{\dot\beta} \ . \qquad (5.4.28)
\end{aligned}$$

where Γ^- denotes a three-dimensional closed contour on $\mathbb{C}\mathbf{P}^3$. In this way, we obtain the negative-frequency wave function $\Psi^{-i_1 \ldots i_{p+q}}{}_{\alpha_1 \ldots \alpha_p;\, \dot\alpha_1 \ldots \dot\alpha_q}(z)$ written in the form of a Penrose transform. We can show that f^- satisfies

$$\left(\varrho_i^\alpha \frac{\partial}{\partial \varrho_i^\alpha} + \varpi_{i\dot\alpha} \frac{\partial}{\partial \varpi_{i\dot\alpha}} + 4 + 2s \right) f^-(\varrho, \pi) = 0 \,, \qquad (5.4.29)$$

where s is given in Eq. (5.4.21).

5.5 Exponential generating function for spinor wave functions

In this section, we treat the spinor wave functions including in the positive and negative frequency wave functions $\Psi^{\pm i_1 \ldots i_{p+q}}{}_{\alpha_1 \ldots \alpha_p;\, \dot\alpha_1 \ldots \dot\alpha_q}$. Then we define the exponential generating function for spinor wave functions. From the exponential generating function, we derive a novel representation of spinor wave functions.

We denote each term in the sum in Eq. (5.4.5) as

$$\begin{aligned}
&\tilde\Psi^{+i_1 \ldots i_p}{}_{\alpha_1 \ldots \alpha_p;\, j_1 \ldots j_q\, \dot\alpha_1 \ldots \dot\alpha_q}(z) \\
&:= \frac{(-1)^p}{(2\pi i)^8} \int_{\mathbb{C}^4} \tilde f^+(\bar\varpi, \varpi) \tilde\Phi^{i_1 \ldots i_p}_{\alpha_1 \ldots \alpha_p;\, j_1 \ldots j_q\, \dot\alpha_1 \ldots \dot\alpha_q}(z, \bar\varpi, \varpi) \\
&\quad \times d^2\bar\varpi^1 \wedge d^2\bar\varpi^2 \wedge d^2\varpi_1 \wedge d^2\varpi_2 \\
&= \frac{(-1)^p}{(2\pi i)^8} \int_{\mathbb{C}^4} \bar\varpi_{\alpha_1}^{i_1} \cdots \bar\varpi_{\alpha_p}^{i_p} \varpi_{j_1 \dot\alpha_1} \cdots \varpi_{j_q \dot\alpha_q} \tilde f^+(\bar\varpi, \varpi) \exp\left(-i z^{\beta\dot\beta} \bar\varpi_\beta^k \varpi_{k\dot\beta} \right) \\
&\quad \times d^2\bar\varpi^1 \wedge d^2\bar\varpi^2 \wedge d^2\varpi_1 \wedge d^2\varpi_2 \ , \qquad (5.5.1)
\end{aligned}$$

where ϵ^{ij}'s have been omitted and $\tilde{\Phi}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z, \bar{\varpi}, \varpi)$ is given by replacing $x^{\alpha\dot{\alpha}}$ with $z^{\alpha\dot{\alpha}} := x^{\alpha\dot{\alpha}} - iy^{\alpha\dot{\alpha}}$ in Eq. (5.3.12). Note that $\tilde{\Phi}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z, \bar{\varpi}, \varpi)$ is a solution of Eqs. (5.3.10f) and (5.3.10g); however, it is not a solution of Eq. (5.3.10h). From Eq. (5.4.18), $\tilde{\Psi}^{+ i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z)$ can be expressed in the form of the Penrose transform as

$$
\begin{aligned}
&\tilde{\Psi}^{+ i_1 \dots i_p}_{\alpha_1 \dots \alpha_q ; j_1 \dots j_p \, \dot{\alpha}_1 \dots \dot{\alpha} q}(z) \\
&= \frac{1}{(2\pi i)^3} \oint_{\Gamma^+} \varpi_{j_1 \dot{\alpha}_1} \cdots \varpi_{j_q \dot{\alpha}_q} \frac{\partial}{\partial \varrho^{\alpha_1}_{i_1}} \cdots \frac{\partial}{\partial \varrho^{\alpha_p}_{i_p}} f^+(\varrho, \varpi) \frac{1}{3} \epsilon^{ij} \epsilon^{kl} \pi_{i\dot{\alpha}} d\varpi^{\dot{\alpha}}_k \wedge d\varpi_{j\dot{\beta}} \wedge d\pi^{\dot{\beta}}_l \; .
\end{aligned}
\tag{5.5.2}
$$

Here we have omitted ϵ^{ij}'s. Similarly, we denote each term in the sum in Eq. (5.4.23) as

$$
\begin{aligned}
&\tilde{\Psi}^{- i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(x) \\
&:= \frac{1}{(2\pi i)^8} \int_{\mathbb{C}^4} \tilde{f}^-(\bar{\varpi}, \varpi) \bar{\tilde{\Phi}}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z, \bar{\varpi}, \varpi) \\
&\quad \times d^2\bar{\varpi}^1 \wedge d^2\bar{\varpi}^2 \wedge d^2\varpi_1 \wedge d^2\varpi_2 \\
&= \frac{1}{(2\pi i)^8} \int_{\mathbb{C}^4} \bar{\varpi}^{i_1}_{\alpha_1} \cdots \bar{\varpi}^{i_p}_{\alpha_p} \varpi_{j_1 \dot{\alpha}_1} \cdots \varpi_{j_q \dot{\alpha}_q} \tilde{f}^-(\bar{\varpi}, \varpi) \exp\left(i z^{\beta\dot{\beta}} \bar{\varpi}^k_\beta \varpi_{k\dot{\beta}} \right) \\
&\quad \times d^2\bar{\varpi}^1 \wedge d^2\bar{\varpi}^2 \wedge d^2\varpi_1 \wedge d^2\varpi_2 \; ,
\end{aligned}
\tag{5.5.3}
$$

where $\bar{\tilde{\Phi}}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z, \bar{\varpi}, \varpi)$ is given by replacing $x^{\alpha\dot{\alpha}}$ with $z^{\alpha\dot{\alpha}} := x^{\alpha\dot{\alpha}} - iy^{\alpha\dot{\alpha}}$ after taking the complex conjugate of Eq. (5.3.12). From Eq. (5.4.28), we find that $\tilde{\Psi}^{- i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z)$ can be written in the form of the Penrose transform as

$$
\begin{aligned}
&\tilde{\Psi}^{- i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \, \dot{\alpha}_1 \dots \dot{\alpha}_q}(z) \\
&= \frac{1}{(2\pi i)^3} \oint_{\Gamma^-} \varpi_{j_1 \dot{\alpha}_1} \cdots \varpi_{j_q \dot{\alpha}_q} \frac{\partial}{\partial \varrho^{\alpha_1}_{i_1}} \cdots \frac{\partial}{\partial \varrho^{\alpha_p}_{i_p}} f^-(\varrho, \varpi) \frac{1}{3} \epsilon^{ij} \epsilon^{kl} \varpi_{i\dot{\alpha}} d\varpi^{\dot{\alpha}}_k \wedge d\varpi_{j\dot{\beta}} \wedge d\varpi^{\dot{\beta}}_l \; .
\end{aligned}
\tag{5.5.4}
$$

From Eqs. (5.5.1) and (5.5.3), it is easily seen that

$$
-i \frac{\partial}{\partial z^{\beta\dot{\beta}}} \tilde{\Psi}^{\pm i_1 \dots i_p}_{\alpha_1 \dots \alpha_p ; j_1 \dots j_q \dot{\alpha}_1 \dots \dot{\alpha}_q}(z) = \tilde{\Psi}^{\pm \, k \, i_1 \dots i_p}_{\beta \alpha_1 \dots \alpha_p ; k \, j_1 \dots j_q \dot{\beta} \dot{\alpha}_1 \dots \dot{\alpha}_q}(z) \; . \tag{5.5.5}
$$

Now we define the exponential generating function, $\mathbf{\Psi}$, for the spinor wave function

$\tilde{\Psi}^{\pm i_1 \dots i_p}_{\alpha_1 \dots \alpha_p; j_1 \dots j_q \dot{\alpha}_1 \dots \dot{\alpha}_q}(z)$:

$$\begin{aligned}\boldsymbol{\Psi}^{\pm}(z, \iota, \kappa) :=& \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \tilde{\Psi}^{\pm i_1 \dots i_p}_{\alpha_1 \dots \alpha_p; j_1 \dots j_n \dot{\alpha}_1 \dots \dot{\alpha}_q}(z) \\ &\times \iota^{\alpha_1}_{i_1} \cdots \iota^{\alpha_p}_{i_p} \kappa^{j_1 \dot{\alpha}_1} \cdots \kappa^{j_q \dot{\alpha}_q} ,\end{aligned} \tag{5.5.6}$$

where ι^{α}_{i} and $\kappa^{i\dot{\alpha}}$ are arbitrary undotted and dotted spinors, respectively. The functions $\tilde{\Psi}^{\pm i_1 \dots i_p}_{\alpha_1 \dots \alpha_p; j_1 \dots j_q \dot{\alpha}_1 \dots \dot{\alpha}_q}(z)$ can be treated as expansion coefficients in the Maclaurin series expansion of $\boldsymbol{\Psi}^{\pm}$ with respect to ι^{α}_{i} and $\kappa^{i\dot{\alpha}}$. Using Eq. (5.5.5), we can show that $\boldsymbol{\Psi}^{\pm}$ satisfies the fundamental equation

$$\left(-i \frac{\partial}{\partial z^{\alpha\dot{\alpha}}} - \frac{\partial^2}{\partial \iota^{\alpha}_{i} \kappa^{i\dot{\alpha}}} \right) \boldsymbol{\Psi}^{\pm}(z, \iota, \kappa) = 0 . \tag{5.5.7}$$

This is precisely the complexification of the so-called *unfolded equations*

$$\left(-i \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} - \frac{\partial^2}{\partial \psi^{\alpha}_{i} \bar{\psi}^{i\dot{\alpha}}} \right) \check{\Phi}(x, \psi, \bar{\psi}) = 0 . \tag{5.5.8}$$

which can be obtained in the present formulation by taking the inner product between Eq. (5.3.3a) and the bra-vector

$$\begin{aligned}&\langle x, f, \psi, \bar{\psi}, a, \mathsf{b}^3, \mathsf{e}| \\ &:= \langle 0| \exp\left(i x^{\alpha\dot{\alpha}} \hat{P}^{(x)}_{\alpha\dot{\alpha}} + i f \hat{P}^{(f)} - \psi^{\alpha}_{i} \hat{\bar{\varpi}}^{i}_{\alpha} + \bar{\psi}^{i\dot{\alpha}} \hat{\varpi}_{i\dot{\alpha}} + i a \hat{P}^{(a)} + i \mathsf{b}^3 \hat{P}^{(\mathsf{b})}_{3} + i \mathsf{e} \hat{P}^{(\mathsf{e})} \right) .\end{aligned} \tag{5.5.9}$$

Here, $\langle \tilde{0}|$ is a reference bra-vector specified by $\langle \tilde{0}| \hat{x}^{\alpha\dot{\alpha}} = \langle \tilde{0}| \hat{f} = \langle \tilde{0}| \hat{\psi}^{i}_{\alpha} = \langle \tilde{0}| \hat{\bar{\psi}}^{i\dot{\alpha}} = \langle \tilde{0}| \hat{a} = \langle \tilde{0}| \hat{\mathsf{b}}^3 = \langle \tilde{0}| \hat{\mathsf{e}} = 0$. The function $\check{\Phi}$ is defined by $\check{\Phi}(x, f, \psi, \bar{\psi}, a, \mathsf{b}^3, \mathsf{e}) := \langle x, f, \psi, \bar{\psi}, a, \mathsf{b}^3, \mathsf{e} | \Phi \rangle$ and is described as $\check{\Phi}(x, \psi, \bar{\psi})$ after taking into account Eqs. (5.3.3b)–(5.3.3e). Substituting Eqs. (5.5.1) and (5.5.3) into (5.5.6), we have

$$\begin{aligned}\boldsymbol{\Psi}^{\pm}(z, \iota, \kappa) =& \frac{1}{(2\pi i)^8} \int_{\mathbb{C}^4} \tilde{f}^{\pm}(\bar{\varpi}, \varpi) \exp\left(\mp i z^{\beta\dot{\beta}} \bar{\varpi}^{k}_{\beta} \varpi_{k\dot{\beta}} \mp \bar{\varpi}^{i}_{\alpha} \iota^{\alpha}_{i} + \varpi_{i\dot{\alpha}} \kappa^{i\dot{\alpha}} \right) \\ &\times d^2\bar{\varpi}^1 \wedge d^2\bar{\varpi}^2 \wedge d^2\varpi_1 \wedge d^2\varpi_2\end{aligned} \tag{5.5.10}$$

With this expression, it is clear that $\boldsymbol{\Psi}^{+}$ and $\boldsymbol{\Psi}^{-}$ are well-defined on $\mathbb{C}\mathbf{M}^{+}$ and $\mathbb{C}\mathbf{M}^{-}$, respectively, owing to the fact that the integrals converge in their corresponding tube domains. Substitution of Eqs. (5.5.2) and (5.5.4) into Eq. (5.5.6) yields

$$\begin{aligned}&\boldsymbol{\Psi}^{\pm}(z,\iota,\kappa)\\&=\frac{1}{(2\pi i)^3}\oint_{\Gamma^{\pm}}\exp\left(\varpi_{i\dot{\alpha}}\kappa^{i\dot{\alpha}}+\iota_i^{\alpha}\frac{\partial}{\partial\varrho_i^{\alpha}}\right)f^{\pm}(\varrho,\varpi)\frac{1}{3}\epsilon^{jk}\epsilon^{lm}\varpi_{j\dot{\beta}}d\varpi_l^{\dot{\beta}}\wedge d\varpi_{k\dot{\gamma}}\wedge d\varpi_m^{\dot{\gamma}}\ ,\end{aligned}\tag{5.5.11}$$

which can be recognized as a collective form of the Penrose transforms (5.5.2) and (5.5.4).

We now note that

$$\begin{aligned}&\tilde{f}^{\pm}\left(\bar{\varpi},\varpi\right)\exp\left(\mp\bar{\varpi}_{\alpha}^{i}\iota_i^{\alpha}+\varpi_{i\dot{\alpha}}\kappa^{i\dot{\alpha}}\right)\\&=\tilde{f}^{\pm}\left(\mp\frac{\partial}{\partial\iota},\frac{\partial}{\partial\kappa}\right)\exp\left(\mp\bar{\varpi}_{\alpha}^{i}\iota_i^{\alpha}+\varpi_{i\dot{\alpha}}\kappa^{i\dot{\alpha}}\right)\ ,\end{aligned}\tag{5.5.12}$$

where $\tilde{f}^{\pm}(\mp\partial/\partial\iota,\partial/\partial\kappa)$ may include the integration operators $(\partial/\partial\iota_i^{\alpha})^{-1}:=\int d\iota_i^{\alpha}$ and $(\partial/\partial\kappa^{i\dot{\alpha}})^{-1}:=\int d\kappa^{i\dot{\alpha}}$, and their higher-order analogs. Applying Eq. (5.5.12) to Eq. (5.5.10), we obtain

$$\begin{aligned}\boldsymbol{\Psi}^{\pm}(z,\iota,\kappa)&=\frac{1}{(2\pi i)^8}\tilde{f}^{\pm}\left(\mp\frac{\partial}{\partial\iota},\frac{\partial}{\partial\kappa}\right)\int_{\mathbb{C}^4}\exp\left(\mp iz^{\beta\dot{\beta}}\bar{\varpi}_{\beta}^{k}\varpi_{\dot{\beta}}\mp\bar{\varpi}_{\alpha}^{i}\iota_i^{\alpha}+\varpi_{i\dot{\alpha}}\kappa^{i\dot{\alpha}}\right)\\&\quad\times d^2\bar{\varpi}^1\wedge d^2\bar{\varpi}^2\wedge d^2\varpi_1\wedge d^2\varpi_2\\&=\frac{1}{(2\pi i)^8}\tilde{f}^{\pm}\left(\mp\frac{\partial}{\partial\iota},\frac{\partial}{\partial\kappa}\right)\exp\left(iz^{-1}_{\dot{\alpha}\alpha}\iota_i^{\alpha}\kappa^{i\dot{\alpha}}\right)\int_{\mathbb{C}^4}\exp\left(\mp iz^{\beta\dot{\beta}}\bar{\varpi}_{\beta}^{j}\varpi_{j\dot{\beta}}\right)\\&\quad\times d^2\bar{\varpi}^1\wedge d^2\bar{\varpi}^2\wedge d^2\varpi_1\wedge d^2\varpi_2\ .\end{aligned}\tag{5.5.13}$$

Here, $z^{-1}_{\alpha\dot{\alpha}}$ denote the matrix elements such that $z^{\alpha\dot{\gamma}}z^{-1}_{\dot{\gamma}\beta}=\delta^{\alpha}{}_{\beta}$ and $z^{-1}_{\dot{\alpha}\gamma}z^{\gamma\dot{\beta}}=\delta_{\dot{\alpha}}{}^{\dot{\beta}}$. Carrying out the integration in Eq. (5.5.13) leads to

$$\boldsymbol{\Psi}^{\pm}(z,\iota,\kappa)=\frac{1}{(2\pi)^4}\det\left(z^{-1}_{\dot{\beta}\beta}\right)\tilde{f}^{\pm}\left(\mp\frac{\partial}{\partial\iota},\frac{\partial}{\partial\kappa}\right)\exp\left(iz^{-1}_{\dot{\alpha}\alpha}\iota_i^{\alpha}\kappa^{i\dot{\alpha}}\right)\ .\tag{5.5.14}$$

We can directly verify that $\boldsymbol{\Psi}^{\pm}$ in Eq. (5.5.14) fulfills Eq. (5.5.7). The spinor wave functions can be derived from Eq. (5.5.14) as the coefficients of the Maclaurin series expansion of $\boldsymbol{\Psi}^{\pm}$ with respect to ι_i^{α} and $\kappa^{i\dot{\alpha}}$

$$\begin{aligned}\tilde{\Psi}^{\pm i_1\ldots i_p}{}_{\alpha_1\ldots\alpha_p;\,j_1\ldots j_q\dot{\alpha}_1\ldots\dot{\alpha}_q}(z)&=\frac{1}{(2\pi)^4}\det\left(z^{-1}_{\dot{\beta}\beta}\right)\frac{\partial^{p+q}}{\partial\iota_{i_1}^{\alpha_1}\cdots\partial\iota_{i_p}^{\alpha_p}\partial\kappa^{j_1\dot{\alpha}}\cdots\kappa^{j_q\dot{\alpha}_q}}\\&\quad\times\tilde{f}^{\pm}\left(\mp\frac{\partial}{\partial\iota},\frac{\partial}{\partial\kappa}\right)\exp\left(iz^{-1}_{\dot{\alpha}\alpha}\iota_i^{\alpha}\kappa^{i\dot{\alpha}}\right)\Big|_{\iota_i^{\alpha}=\kappa^{i\dot{\alpha}}=0}\ .\end{aligned}\tag{5.5.15}$$

In this way, we have obtained a novel representation for each of the spinor wave functions. We now write the contravariant vector corresponding $z^{-1}_{\alpha\dot{\alpha}}$ as $(z^{-1})^{\mu}$. Then it can be shown that $(z^{-1})^{\mu} = 2z^{\mu}/(z_{\nu}z^{\nu})$. The discrete transformation $z^{\mu} \to \frac{1}{2}(z^{-1})^{\mu}$ is known as the conformal inversion transformation. Therefore, it turns out that $\tilde{\Psi}^{\pm i_1 \ldots i_p}{}_{\alpha_1 \ldots \alpha_p; j_1 \ldots j_q \dot{\alpha}_1 \ldots \dot{\alpha}_q}(z)$ in Eq. (5.5.15) is a function of the conformally invereted space-time variables $\frac{1}{2}(z^{-1})^{\mu}$.

5.6 Physical meanings of the internal symmetries

In this section, we investigate the rank-one spinor fields of $I = 1/2$ in detail to clarify physical meanings of the $U(1)_a$, $U(1)_{\mathsf{b}}$ and $SU(2)$ symmetries as well as those of the constants s, t and I. In addition, we demonstrate the rank-two spinor fields of $I = 1$ constitute massive fields obeying the Proca equations.

5.6.1 Case $I = 1/2$

We consider the DFP equations that rank-one spinor fields of $I = 1/2$, namely $\Psi^{\pm i}{}_{\alpha}$ and $\Psi^{\pm i}{}_{\dot{\alpha}}$, obey, which are given by Eq. (5.4.9a) in the case $(p, q) = (0, 1)$ and Eq. (5.4.9b) in the case $(p, q) = (1, 0)$ as

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Psi^{\pm i}{}_{\dot{\beta}}(z) - m\Psi^{\pm i\beta}(z) = 0\,, \tag{5.6.1a}$$

$$i\sqrt{2}\frac{\partial}{\partial z^{\beta\dot{\beta}}}\Psi^{\pm i\beta}(z) - m\Psi^{\pm i}{}_{\dot{\beta}}(z) = 0 \tag{5.6.1b}$$

with $\Psi^{\pm i\beta} := \epsilon^{\beta\gamma}\Psi^{\pm i}{}_{\gamma}$. Equations (5.6.1a) and (5.6.1b) with $i = 1$ can be combined in the form of the ordinary Dirac equation

$$D\psi^{\pm}_1(z) = 0, \qquad \psi^{\pm}_1(z) := \begin{pmatrix} \Psi^{\pm 1\beta}(z) \\ \Psi^{\pm 1}{}_{\dot{\beta}}(z) \end{pmatrix}\,, \tag{5.6.2}$$

while Eqs. (5.6.1a) and (5.6.1b) with $i = 2$ can be combined, after replacing $z^{\alpha\dot{\alpha}}$ by $-z^{\alpha\dot{\alpha}}$, as

$$D\psi^{\pm}_2(z) = 0, \qquad \psi^{\pm}_2(z) := \begin{pmatrix} \Psi^{\pm 2\beta}(-z) \\ \Psi^{\pm 2}{}_{\dot{\beta}}(-z) \end{pmatrix}\,. \tag{5.6.3}$$

	particle	antiparticle
left-handed	$\Psi^{+1\alpha}$	$\Psi^{+2\alpha}$
right-handed	$\Psi^{+1}{}_{\dot{\alpha}}$	$\Psi^{+2}{}_{\dot{\alpha}}$

Table 5.1: A classification of the rank-one spinor fields.

In Eqs. (5.6.2) and (5.6.3), D denotes the Dirac operator

$$D := \begin{pmatrix} -m\delta_{\alpha}^{\beta} & i\sqrt{2}\frac{\partial}{\partial z_{\alpha\dot{\beta}}} \\ i\sqrt{2}\frac{\partial}{\partial z^{\beta\dot{\alpha}}} & -m\delta_{\dot{\beta}}^{\dot{\alpha}} \end{pmatrix} . \tag{5.6.4}$$

The charge conjugate of $\psi_1^{\pm}(z)$ is found to be

$$(\psi_1^{\pm})^{\mathrm{c}}(z) := \begin{pmatrix} 0 & \epsilon^{\beta\gamma} \\ -\epsilon_{\dot{\beta}\dot{\gamma}} & 0 \end{pmatrix} \overline{\psi_1^{\pm}(\bar{z})} = \begin{pmatrix} 0 & \epsilon^{\beta\gamma} \\ -\epsilon_{\dot{\beta}\dot{\gamma}} & 0 \end{pmatrix} \begin{pmatrix} \bar{\Psi}^{\pm\dot{\gamma}}{}_{1}(z) \\ \bar{\Psi}^{\pm}{}_{1\gamma}(z) \end{pmatrix} = \begin{pmatrix} \bar{\Psi}^{\pm\beta}{}_{1}(z) \\ \bar{\Psi}^{\pm}{}_{1\dot{\beta}}(z) \end{pmatrix} , \tag{5.6.5}$$

where the arguments of ψ_1, namely $z^{\alpha\dot{\alpha}}$, have been replaced by their complex conjugates $\bar{z}^{\alpha\dot{\alpha}} := \overline{z^{\alpha\dot{\alpha}}}$ so that $(\psi_1^{\pm})^{\mathrm{c}}$ can be a holomorphic function of $z^{\alpha\dot{\alpha}}$. Using the complex conjugates of Eqs. (5.6.1a) and (5.6.1b), we can see that $D(\psi_1^{\pm})^{\mathrm{c}}(z) = 0$. Since $\psi_2^{\pm}$ and $(\psi_1^{\pm})^{\mathrm{c}}$ satisfy the same Dirac equation and have the same spinor and $SU(2)$ indices, they can be identified with each other up to an overall constant. If ψ_1^{+} represents a spinor field of a particle with four-momentum $(E, \boldsymbol{p})$, then $\psi_2^{+}(z)$ is regarded as a spinor field of a corresponding antiparticle with four-momentum $(-E, -\boldsymbol{p})$ owing to $\psi_2^{+}(z) \simeq (\psi_1^{+})^{\mathrm{c}}(z)$. This means that $\psi_2^{+}(-z) = (\Psi^{+2\alpha}(z), \Psi^{+2}_{\dot{\alpha}}(z))^{\mathrm{T}}$ is considered a spinor field of the antiparticle with four-momentum $(E, \boldsymbol{p})$. In view of this fact, it is clear that $\Psi^{+1\alpha}(z)$ and $\Psi^{+2\alpha}(z)$ represent a left-handed particle and a corresponding left-handed antiparticle, respectively, while $\Psi^{+1}_{\dot{\alpha}}(z)$ and $\Psi^{+2}_{\dot{\alpha}}(z)$ represent a right-handed particle and a corresponding right-handed antiparticle, respectively, as summarized in Table 5.1. We thus find that the index i of $\Psi^{+i\alpha}$ and $\Psi^{+i}_{\dot{\alpha}}$ distinguishes between a particle and its antiparticle. Using Eq. (5.3.13), we can obtain the possible values of s and t for each of the rank-one spinor fields as in Table 5.2. We observe that the left-handed spinor fields $\Psi^{+i\alpha}(z)$ $(i = 1, 2)$ have $s = 1/2$, while the right-handed spinor fields $\Psi^{+i}_{\dot{\alpha}}$ $(i = 1, 2)$ have $s = -1/2$. Hence, s turns out to be a quantum number specifying the chirality of a spinor fields . Since s is an eigenvalues of $\mathcal{T}_0$, as can be seen from (5.3.10f),

	s	t		s	t
$\Psi^{+1\alpha}$	$\frac{1}{2}$	$\frac{1}{2}$	$\Psi^{+2\alpha}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\Psi^{+1}{}_{\dot\alpha}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\Psi^{+2}{}_{\dot\alpha}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 5.2: The values of s and t of the rank-one spinor fields.

$\mathcal{T}_0$ can be interpreted as the operator of chirality. Accordingly, $U(1)_a$ is can be identified as the gauge group of chirality, and the $U(1)_a$ symmetry is physically understood as a gauge symmetry leading to chirality conservation. We also perceive that the particle spinor fields $\Psi^{+1\alpha}(z)$ and $\Psi^{+1}_{\dot\alpha}(z)$ have $t = 1/2$, while the antiparticle spinor fields $\Psi^{+2\alpha}(z)$ and $\Psi^{+2}_{\dot\alpha}(z)$ have $t = -1/2$. Hence, t turns out to be a quantum number distinguishing between a particle and its antiparticle. Then it follows that t is proportional to the electric charge of the particle or antiparticle. Since t is an eigenvalue of $\mathcal{T}_3$ as can be seen from (5.3.10g), $\mathcal{T}_3$ can be interpreted as the operator of electric charge up to a constant of proportionality. Accordingly, $U(1)_{\mathsf{b}}$ can be identified with the gauge group of electric charge, and the $U(1)_{\mathsf{b}}$ symmetry is physically understood as a gauge symmetry leading to electric charge conservation.

5.6.2 Case $I = 1$

We consider the DFP equations satisfied by the rank-two spinor fields of $I = 1$, that is, $\Psi^{\pm ij}_{\alpha\beta}$, $\Psi^{\pm ij}{}_{\alpha\dot\beta}$, and $\Psi^{\pm ij}_{\dot\alpha\dot\beta}$.

The DFP equations satisfied by $\Psi^{\pm ij}_{\alpha\beta}$ and $\Psi^{\pm ij}{}_{\alpha\dot\beta}$ are given by Eq. (5.4.9a) in the case $(p, q) = (1, 1)$ and Eq. (5.4.9b) in the case $(p, q) = (2, 0)$ as

$$i\sqrt{2}\frac{\partial}{\partial z^{\beta\dot\beta}}\Psi^{\pm ij}{}_{\alpha}{}^{\dot\beta} + m\Psi^{\pm ij}_{\alpha\beta} = 0\,, \tag{5.6.6a}$$

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot\beta}}\Psi^{\pm ij}_{\alpha\beta} + m\Psi^{\pm ij}{}_{\alpha}{}^{\dot\beta} = 0\,. \tag{5.6.6b}$$

Similarly, the DFP equations for $\Psi^{\pm ij}{}_{\alpha\dot\beta}$ and $\Psi^{\pm ij}_{\dot\alpha\dot\beta}$ are given by Eq. (5.4.9a) in the case $(p,q)=(0,2)$ and Eq. (5.4.9b) in the case $(p,q)=(1,1)$ as

$$i\sqrt{2}\frac{\partial}{\partial z^{\beta\dot{\beta}}}\Psi^{\pm ij\dot{\alpha}\dot{\beta}}+m\Psi^{\pm ij}{}_{\beta}{}^{\dot{\alpha}}=0\,, \tag{5.6.7a}$$

$$i\sqrt{2}\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Psi^{\pm ij}{}_{\beta}{}^{\dot{\alpha}}+m\Psi^{\pm ij\dot{\alpha}\dot{\beta}}=0\,. \tag{5.6.7b}$$

Using Eqs. (5.6.6a), (5.6.6b), (5.6.7a), and (5.6.7b), we can derive the Klein-Gordon equation for $\Psi^{\pm ij}_{\alpha\beta}$, $\Psi^{\pm ij}_{\alpha\dot{\beta}}$, and $\Psi^{\pm ij}_{\dot{\alpha}\dot{\beta}}$ as

$$\left(\frac{\partial}{\partial z^{\beta\dot{\beta}}}\frac{\partial}{\partial z_{\beta\dot{\beta}}}+m^2\right)\Psi^{\pm ij}_{\alpha\beta}=0, \tag{5.6.8a}$$

$$\left(\frac{\partial}{\partial z^{\beta\dot{\beta}}}\frac{\partial}{\partial z_{\beta\dot{\beta}}}+m^2\right)\Psi^{\pm ij}_{\alpha\dot{\beta}}=0, \tag{5.6.8b}$$

$$\left(\frac{\partial}{\partial z^{\beta\dot{\beta}}}\frac{\partial}{\partial z_{\beta\dot{\beta}}}+m^2\right)\Psi^{\pm ij}_{\dot{\alpha}\dot{\beta}}=0, \tag{5.6.8c}$$

From Eq. (5.6.6a) and Eq. (5.6.7b), we find the symmetric properties

$$\frac{\partial}{\partial z^{\beta\dot{\beta}}}\Psi^{\pm ij\dot{\beta}}_{\ \ \alpha}=\frac{\partial}{\partial z^{\alpha\dot{\beta}}}\Psi^{\pm ij\dot{\beta}}_{\ \ \beta}\,, \tag{5.6.9a}$$

$$\frac{\partial}{\partial z_{\beta\dot{\beta}}}\Psi^{\pm ij\dot{\alpha}}_{\ \ \beta}=\frac{\partial}{\partial z_{\beta\dot{\alpha}}}\Psi^{\pm ij\dot{\beta}}_{\ \ \beta}\,, \tag{5.6.9b}$$

which lead to

$$\frac{\partial}{\partial z_{\alpha\dot{\alpha}}}\Psi^{\pm ij}_{\alpha\dot{\alpha}}=\frac{\partial}{\partial z_{\mu}}\Psi^{\pm ij}_{\mu}=0 \tag{5.6.10}$$

with $\Psi^{\pm ij}_{\mu}:=\sigma^{\alpha\dot{\beta}}_{\mu}\Psi^{\pm ij}_{\alpha\dot{\beta}}$. Multiplying Eq. (5.6.6a) by $\epsilon^{\dot{\alpha}\dot{\beta}}$ and multiplying Eq. (5.6.7b) by $\epsilon^{\alpha\beta}$, we obtain

$$i\sqrt{2}\frac{\partial}{\partial z_{\alpha\dot{\gamma}}}\Psi^{\pm ij\beta}{}_{\dot{\gamma}}\epsilon^{\dot{\alpha}\dot{\beta}}=m\Psi^{\pm ij\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\,, \tag{5.6.11a}$$

$$i\sqrt{2}\frac{\partial}{\partial z_{\gamma\dot{\alpha}}}\Psi^{\pm ij}{}_{\gamma}{}^{\dot{\beta}}\epsilon^{\alpha\beta}=-m\Psi^{\pm ij\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\,. \tag{5.6.11b}$$

By adding Eq. (5.6.11b) to Eq. (5.6.11a), we derive

$$\frac{\partial}{\partial z_{\alpha\dot{\gamma}}}\Psi^{\pm ij\beta}{}_{\dot{\gamma}}\epsilon^{\dot{\alpha}\dot{\beta}}+\frac{\partial}{\partial z_{\gamma\dot{\alpha}}}\Psi^{\pm ij}{}_{\gamma}{}^{\dot{\beta}}\epsilon^{\alpha\beta}=F^{\pm ij\alpha\dot{\alpha}\beta\dot{\beta}}\,, \tag{5.6.12}$$

where

$$F^{\pm ij\alpha\dot{\alpha}\beta\dot{\beta}} := \frac{m}{i\sqrt{2}}\Big(\Psi^{\pm ij\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} - \Psi^{\pm ij\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\Big)\,. \tag{5.6.13}$$

Furthermore, by utilizing the formula [1]

$$\sigma^{\mu\alpha\dot{\gamma}}\sigma^{\nu}_{\alpha\dot{\beta}}\sigma^{\rho\beta\dot{\beta}}\sigma^{\eta}_{\beta\dot{\gamma}} = \frac{1}{2}\Big(g^{\mu\nu}g^{\rho\eta} + g^{\mu\eta}g^{\nu\rho} - g^{\mu\rho}g^{\nu\eta} - i\epsilon^{\mu\nu\rho\eta}\Big)\,, \tag{5.6.14}$$

it follows from Eq. (5.6.12) that

$$\partial^{\mu}\Psi^{\pm ij\nu} - \partial^{\nu}\Psi^{\pm ij\mu} = F^{\pm ij\mu\nu} \tag{5.6.15}$$

with $F^{\pm ij\mu\nu} := \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\alpha\dot{\alpha}}F^{\pm ij\alpha\dot{\alpha}\beta\dot{\beta}}$. Using Eq. (5.6.8b), (5.6.10), and (5.6.15), we can find

$$\partial^{\mu}F^{\pm ij}{}_{\mu\nu} = m^2\Psi^{\pm ij}{}_{\nu}\,. \tag{5.6.16}$$

Equations (5.6.15) and (5.6.16) are precisely the Proca equation for the $SU(2)$ triplets Ψ^{ij}_{μ} and $F^{ij}_{\mu\nu}$. On the other hand, subtracting Eq. (5.6.11b) from Eq. (5.6.11a), we obtain

$$i\left(\frac{\partial}{\partial z_{\alpha\dot{\gamma}}}\Psi^{\pm ij\beta}{}_{\dot{\gamma}}\epsilon^{\dot{\alpha}\dot{\beta}} - \frac{\partial}{\partial z_{\gamma\dot{\alpha}}}\Psi^{\pm ij}{}_{\gamma}{}^{\dot{\beta}}\epsilon^{\alpha\beta}\right) = \tilde{F}^{\pm ij\alpha\dot{\alpha}\beta\dot{\beta}}\,, \tag{5.6.17}$$

where

$$\tilde{F}^{\pm ij\alpha\dot{\alpha}\beta\dot{\beta}} := \frac{m}{\sqrt{2}}\Big(\Psi^{\pm ij\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} + \Psi^{\pm ij\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\Big) \tag{5.6.18}$$

corresponds to the dual tensor $\tilde{F}^{\pm ij\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\eta}F^{\pm ij}_{\rho\eta}$. With the formula (5.6.14), Eq. (5.6.17) reads

$$\frac{1}{2}\epsilon^{\mu\nu\rho\eta}\Big(\partial_{\rho}\Psi^{\pm ij}{}_{\eta} - \partial_{\eta}\Psi^{\pm ij}{}_{\rho}\Big) = \tilde{F}^{\pm ij\mu\nu}\,. \tag{5.6.19}$$

It is evident that this equation, or Eq. (5.6.17), is the dual of Eq. (5.6.15).

[1] The four-dimensional Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ is defined as $\epsilon^{0123} = -\epsilon_{0123} = 1$.

Chapter 6

Summary and discussion

In this thesis, we have presented a gauged twistor model of a free massive spinning particle in four dimensions. This model was formulated in terms of two independent twistors as a non-Abelian extension of the gauged twistor model of a free massless spinning particle in four dimensions, presented in Refs. [27, 28, 29]. The extended model is governed by the GGS action that was elaborated by adding the 1D Chern-Simons terms S_a and $S_{\mathsf{b}3}$ and the novel term S_{be} to the gauged twistorial action $S_{m\mathrm{g}}$ [see Eq. (2.20)]. The GGS action remains invariant under the reparametrization, the local $U(1)_a$ and local $SU(2)$ transformations, although the $SU(2)$ symmetry is nonlinearly realized in the action. In the unitary gauge, the $U(1)_{\mathsf{b}}$ symmetry is manifestly exhibited, while the $SU(2)$ symmetry is hidden.

In Chapter 4, we studied the canonical Hamiltonian formalism of the gauged twistor model and performed its subsequent canonical quantization. The canonical Hamiltonian formalism based on the GGS action was studied in the unitary gauge by following Dirac's recipe for constrained Hamiltonian systems. The classification of the constraints into first and second classes was carried out strictly, and the Dirac brackets between the canonical variables were obtained concretely. It was demonstrated that just sufficient constraints for the twistor variables are consistently derived as the secondary first-class constraints [see Eqs. (4.1.28e)–(4.1.28i)]. The subsequent canonical quantization of the system was performed in terms of the new twistor variables W_i^A and $\bar{\mathsf{W}}_A^i$, because they satisfy the simple Dirac brackets given in Eq. (4.1.33). We have shown that the Chern-Simons coefficients $2s$ and $2t$ are quantized to be arbitrary integer values as a result of the canonical quantization based on the commutation relations (4.2.2a)–(4.2.2e). In general, the quantization of Chern-Simons coefficient is a common consequence in certain theories in which the Chern-Simons terms play crucial roles (see e.g. Refs. [41, 42, 43, 44]). Our gauged twistor model can be regarded as a specific example of such theories. Intriguingly, the coefficient k of $S_{\mathsf{b}12}$ is also quantized via solving the eigenvalue problem of the $SU(2)$ Lie algebra. We found that the twistor functions in our model are eigenfunctions of the relevant differential operators governed by the $U(1)_a \times SU(2)$ Lie algebra [see Eqs. (4.2.13e)–(4.2.13g)]. Each twistor function F is then labeled by a set of three quantum numbers associated with the $U(1)_a \times SU(2)$ Lie algebra. We have carried out the Penrose transform of the twistor function F to obtain a massive spinor field of arbitrary rank defined on complexified Minkowski space [see Eq. (4.3.1)]. As emphasized earlier, this spinor field has the upper and lower $SU(2)$ indices in addition to the dotted and undotted spinor indices. In fact, we observed that the number of upper (lower) $SU(2)$ indices is equal to the number of undotted (dotted) spinor indices. We also demonstrated that the spinor field satisfies the generalized DFP equations with $SU(2)$ indices, given in Eq. (4.3.10). We have investigated the rank-one spinor fields in detail to clarify the physical meanings of the gauge symmetries as well as those of the constants s and t. It turned out that s is a quantum number specifying the chirality of a spinor field and that the $U(1)_a$ symmetry is a gauge symmetry leading to chirality conservation. It also turned out that t is a quantum number proportional to the electric charge of a spinor field and that the $U(1)_{\mathsf{b}}$ symmetry is a gauge symmetry leading to electric charge conservation. The $SU(2)$ symmetry was shown to be a gauge symmetry realized in the particle-antiparticle doublets. Such a symmetry, however, is not observed in nature, so that it should be considered to be hidden or broken. Fortunately our twistor formulation in the unitary gauge is appropriate for describing this situation. Since the $SU(2)$ symmetry is a symmetry realized in the particle-antiparticle doublets, it cannot be identified with the weak isospin symmetry. We thus conclude that the idea proposed by Penrose, Perjés, and Hughston [6, 8, 9, 10, 11] is not valid in our gauged twistor model.

In Chapter 5, we treated the gauged twistor model formulated using the spinor and space-time variables. The GGS action in this spinor formulation is written in terms of the space-time and spinor variables and yields the mass-shell condition in Eq. (5.1.3). The canonical Hamiltonian formalism based on the GGS action (5.1.1) was also studied by taking the space-time and spinor variables as canonical coordinates. The classification of the constraints into first and second classes was accomplished, and the Dirac brackets between the canonical variables were obtained. After the subsequent canonical quantization of the system based on the relevant commutation relations, the physical state conditions defined from the first-class constraints were read as the simultaneous differential equations (5.3.3a)–(5.3.3i). By solving them, we found a plane-wave solution Φ and saw that each of the constants (s, t, k) is quantized as with the result obtained in the gauged twistor formulation. We defined the positive-frequency wave function Ψ^{+} as a linear combination of the plane wave solutions with a coefficient function $\tilde{f}^{+}$ and defined the negative-frequency wave function Ψ^{-} as a linear combination of the plane wave solution with a coefficient function $\tilde{f}^{-}$. It was shown that Ψ^{+} and Ψ^{-} are well-defined on the forward tube $\mathbb{C}\mathbf{M}^{+}$ and on the backward tube $\mathbb{C}\mathbf{M}^{-}$, respectively, and satisfy the DFP equations with $SU(2)$ indices (5.4.9a) and (5.4.9b). Also, it was demonstrated that the spinor wave functions with $SU(2)$ indices can be expressed as the Penrose transforms of the holomorphic functions f^{+} and f^{-} that are defined as the Fourier-Laplace transforms of $\tilde{f}^{+}$ and $\tilde{f}^{-}$, respectively. In this way, we have obtained the Penrose transforms in the case of massive fields via appropriate Fourier-Laplace transforms. Furthermore, we constructed the exponential generating function $\mathbf{\Psi}^{\pm}$ for the spinor wave functions and derived from it a novel representation, Eq. (5.5.15), for each of the spinor wave functions. Then this representation turned out to be a function of the conformally inverted space-time variables $\frac{1}{2}(z^{-1})^{\mu}$. We have also investigated the physical meaning of the $U(1)$ and $SU(2)$ symmetries as well as those of the constants s and t. The results turned out to be identical with those obtained in Chapter 4.

The observation that s is a quantum number specifying the chirality of a spinor field is supported for the following reason: The gauged Shirafuji action for a massless spinning particle enjoys the $U(1)_a$ symmetry and contains its associated constant s [27, 28, 29]. This constant is indeed shown to be the helicity of a massless spinning particle. As is well known, the chirality is an analog of the helicity, while the chirality is a Lorentz invariant quantity valid for massive particles as well as massless particles. (For massless particles, chirality is the same as helicity.) For this reason, in the present twistor model, it is quite natural to identify the Lorentz invariant quantity s as the chirality quantum number.

We have seen that each eigenstate of $\check{\mathsf{T}}_3$ corresponds (via the Penrose transform) to a particle or antiparticle state represented by its own spinor field. Remarkably, we encounter a similar situation in studying the rigid body model [45, 46]. In this model, the rigid body rotation leads to an intrinsic $SU(2)$ symmetry in addition to the spin $SU(2)$ symmetry. Hara *et al.* showed that the eigenstates of the third generator of the intrinsic $SU(2)$ group are assigned to particle and antiparticle spinor fields. They also pointed out that this generator cannot be identified with the third component of the isospin generators. (Accordingly, it turns out that the intrinsic $SU(2)$ symmetry cannot be regarded as the isospin symmetry. This result contradicts the earlier idea concerning isospin proposed in Refs. [47, 48].) We thus see that the gauged twistor model and the rigid body model share common aspects.

Now we recall that the secondary first-class constraints (4.1.28e)–(4.1.28g), or equivalently, Eqs. (4.1.36a), (4.1.36b), and (4.1.38), have been derived systematically on the basis of the $U(1)_a$, $U(1)_{\mathsf{b}}$, and reparametrization symmetries of the GGS action. By contrast, the remaining secondary first-class constraints (4.1.28h) and (4.1.28i) have been derived as a result of incorporating the mass-shell condition (3.1.3) into the GGS action by hand. Considering this fact, we can never say that the present approach for constructing the GGS action is satisfactory from the gauge-theoretical point of view. To make our gauged twistor formulation complete, we need to establish an approach in which the mass-shell condition (3.1.3) is supplied as an inevitable outcome of an extra gauge symmetry.

In this thesis, we have not presented precise definitions of the chirality and charge conjugation for a massive spinor field of arbitrary rank. The chirality may be defined on the basis of the type of spinor indices of the field. For clarifying the definition of charge conjugation and its associated concept of particle-antiparticle, it is necessary to examine coupling of a massive spinor field of arbitrary rank to the electromagnetic field. The precise definitions of chirality and charge conjugation should confirm our observation on the physical meanings of the constants s and t. It is also interesting to incorporate interactions other gauge fields lying in space-time and consider interactions between particles. We hope to address the aforementioned issues in the near future.

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Appendix A

Poincaré symmetry and Pauli-Lubanski pseudovector

In this appendix, we consider the Poincaré symmetry and the Pauli-Lubanski pseudovector within the framework of the gauged twistor formulation.

We can easily show that the GGS action (3.3.9) remains invariant under the infinitesimal Poincaré transformation (or more accurately, the infinitesimal $SL(2,\mathbb{C})\ltimes\mathbb{R}^{1,3}$ transformation)

$$\varrho_i^{\alpha} \to \varrho_i'^{\alpha} = \varrho_i^{\alpha} - \varepsilon^{\alpha}{}_{\beta}\varrho_i^{\beta} - i\varepsilon^{\alpha\dot{\beta}}\varpi_{i\dot{\beta}}\,, \tag{A.1 a}$$

$$\bar{\varrho}^{i\dot{\alpha}} \to \bar{\varrho}'^{i\dot{\alpha}} = \bar{\varrho}^{i\dot{\alpha}} - \bar{\varepsilon}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\varrho}^{i\dot{\beta}} + i\varepsilon^{\beta\dot{\alpha}}\bar{\varpi}^{i}_{\beta}\,, \tag{A.1 b}$$

$$\varpi_{i\dot{\alpha}} \to \varpi'_{i\dot{\alpha}} = \varpi_{i\dot{\alpha}} + \bar{\varepsilon}_{\dot{\alpha}}{}^{\dot{\beta}}\varpi_{i\dot{\beta}}\,, \tag{A.1 c}$$

$$\bar{\varpi}^{i}_{\alpha} \to \bar{\varpi}'^{i}_{\alpha} = \bar{\varpi}^{i}_{\alpha} + \varepsilon_{\alpha}{}^{\beta}\bar{\varpi}^{i}_{\beta}\,. \tag{A.1 d}$$

Here, $\varepsilon^{\alpha\beta}$ and $\bar{\varepsilon}^{\dot{\alpha}\dot{\beta}}\big(:=\overline{\varepsilon^{\alpha\beta}}\big)$ are parameters of the infinitesimal Lorentz transformation (or more accurately, the infinitesimal $SL(2,\mathbb{C})$ transformation), satisfying the symmetric properties $\varepsilon^{\alpha\beta}=\varepsilon^{\beta\alpha}$ and $\bar{\varepsilon}^{\dot{\alpha}\dot{\beta}}=\bar{\varepsilon}^{\dot{\beta}\dot{\alpha}}$, while $\varepsilon^{\alpha\dot{\beta}}$ is a parameter of the infinitesimal translation, satisfying the Hermiticity $\overline{\varepsilon^{\alpha\dot{\beta}}}=\varepsilon^{\beta\dot{\alpha}}$. The fields h, $\bar{h}$, a, and b^r are assumed to be Poincaré invariant. Since the GGS action is Poincaré invariant, we can derive conserved quantities by applying Noether's theorem. The conserved quantities corresponding to $\varepsilon^{\alpha\beta}$, $\bar{\varepsilon}^{\dot{\alpha}\dot{\beta}}$, and $\varepsilon^{\alpha\dot{\beta}}$ are found to be

$$\mu_{\alpha\beta} := \frac{i}{2}\big(\varrho_{i\alpha}\bar{\varpi}^{i}_{\beta} + \varrho_{i\beta}\bar{\varpi}^{i}_{\alpha}\big)\,, \tag{A.2 a}$$

$$\bar{\mu}_{\dot{\alpha}\dot{\beta}} := -\frac{i}{2}\big(\bar{\varrho}^{i}_{\dot{\alpha}}\varpi_{i\dot{\beta}} + \bar{\varrho}^{i}_{\dot{\beta}}\varpi_{i\dot{\alpha}}\big)\,, \tag{A.2 b}$$

$$p_{\alpha\dot{\beta}} := \bar{\varpi}^{i}_{\alpha}\varpi_{i\dot{\beta}}\,. \tag{A.2 c}$$

Substituting Eqs. (4.1.37a) and (4.1.37b) into Eqs. (A.2 a) and (A.2 b), respectively, we can rewrite $\mu_{\alpha\beta}$ and $\bar{\mu}_{\dot{\alpha}\dot{\beta}}$ as

$$\mu_{\alpha\beta} = \frac{i}{2}\big(\rho_{i\alpha}\bar{\varpi}^{i}_{\beta} + \rho_{i\beta}\bar{\varpi}^{i}_{\alpha}\big) , \tag{A.3 a}$$

$$\bar{\mu}_{\dot{\alpha}\dot{\beta}} = -\frac{i}{2}\big(\bar{\rho}^{i}_{\dot{\alpha}}\varpi_{i\dot{\beta}} + \bar{\rho}^{i}_{\dot{\beta}}\varpi_{i\dot{\alpha}}\big) . \tag{A.3 b}$$

The angular momentum tensor is given by

$$M_{\alpha\dot{\alpha}\beta\dot{\beta}} := \mu_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \bar{\mu}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} , \tag{A.4}$$

while the four-momentum vector is given by Eq. (A.2 c).

The Pauli-Lubanski pseudovector is defined by [3, 49]

$$W^{\alpha\dot{\alpha}} := \frac{1}{2}\epsilon^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} p_{\beta\dot{\beta}} M_{\gamma\dot{\gamma}\delta\dot{\delta}} , \tag{A.5}$$

which can be written as

$$W^{\alpha\dot{\alpha}} = -i\mu^{\alpha\beta}p_{\beta}{}^{\dot{\alpha}} + i\bar{\mu}^{\dot{\alpha}\dot{\beta}}p^{\alpha}{}_{\dot{\beta}} \tag{A.6}$$

by using the formula

$$\epsilon^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = i\Big(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\epsilon^{\dot{\alpha}\dot{\delta}}\epsilon^{\dot{\beta}\dot{\gamma}} - \epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}\Big) . \tag{A.7}$$

Using the identity

$$\epsilon^{\alpha\beta}\rho^{\gamma}_{i} + \epsilon^{\beta\gamma}\rho^{\alpha}_{i} + \epsilon^{\gamma\alpha}\rho^{\beta}_{i} = 0 \tag{A.8}$$

and its complex conjugate, we can express Eq. (A.6) with Eqs. (A.2 c) and (A.3) as

$$W^{\alpha\dot{\alpha}} = \Big(\rho^{\beta}_{i}\bar{\varpi}^{j}_{\beta} + \varpi_{i\dot{\beta}}\bar{\rho}^{j\dot{\beta}}\Big)\bar{\varpi}^{i\alpha}\varpi^{\dot{\alpha}}_{j} - \frac{1}{2}\Big(\rho^{\beta}_{i}\bar{\varpi}^{i}_{\beta} + \varpi_{i\dot{\beta}}\bar{\rho}^{i\dot{\beta}}\Big)\bar{\varpi}^{j\alpha}\varpi^{\dot{\alpha}}_{j} , \tag{A.9}$$

or concisely,

$$W^{\alpha\dot{\alpha}} = \bigg(\delta^{l}_{i}\delta^{j}_{k} - \frac{1}{2}\delta^{j}_{i}\delta^{l}_{k}\bigg)\bar{\mathsf{W}}^{k}_{B}\mathsf{W}^{B}_{l}\bar{\varpi}^{i\alpha}\varpi^{\dot{\alpha}}_{j} . \tag{A.10}$$

Here, W^{B}_{k} and $\bar{\mathsf{W}}^{k}_{B}$ are the twistors defined by $\mathsf{W}^{B}_{k} := \big(\rho^{\beta}_{k}, \varpi_{k\dot{\beta}}\big)$ and $\bar{\mathsf{W}}^{k}_{B} := \big(\bar{\varpi}^{k}_{\beta}, \bar{\rho}^{k\dot{\beta}}\big)$ (see right above Eq. (4.1.33)). Applying the formula

$$\frac{1}{2}\sigma_{ri}{}^{j}\sigma_{rk}{}^{l} = \delta^{l}_{i}\delta^{j}_{k} - \frac{1}{2}\delta^{j}_{i}\delta^{l}_{k} \tag{A.11}$$

valid for the Pauli matrices σ_r to Eq. (A.10), we obtain

$$W^{\alpha\dot{\alpha}} = \mathsf{T}_r \sigma_{ri}{}^{j} \bar{\varpi}^{i\alpha} \varpi_j^{\dot{\alpha}} \,, \tag{A.12}$$

with

$$\mathsf{T}_r := \frac{1}{2} \bar{\mathsf{W}}_B^k \sigma_{rk}{}^{l} \mathsf{W}_l^B \tag{A.13}$$

(see Eq. (4.1.39)). Equation (A.12) can be written in terms of the (original) twistors Z_l^B and $\bar{Z}_B^k$ as

$$W^{\alpha\dot{\alpha}} = T_r \sigma_{ri}{}^{j} \bar{\pi}^{i\alpha} \pi_j^{\dot{\alpha}} \,, \tag{A.14}$$

with

$$T_r := \frac{1}{2} \bar{Z}_B^k \sigma_{rk}{}^{l} Z_l^B \,. \tag{A.15}$$

Using the mass-shell constraints

$$\varpi_{i\dot{\alpha}} \varpi_j^{\dot{\alpha}} \approx \frac{m}{\sqrt{2}} \epsilon_{ij} e^{i\varphi}, \tag{A.16 a}$$

$$\bar{\varpi}_\alpha^i \bar{\varpi}^{j\alpha} \approx \frac{m}{\sqrt{2}} \epsilon^{ij} e^{-i\varphi} \tag{A.16 b}$$

equivalent, respectively, to Eqs. (4.1.11e) and (4.1.11f), and utilizing the formula $\sigma_2 \sigma_r \sigma_2 = -\sigma_r^{\mathrm{T}}$, we can show for Eq. (A.12) that

$$W_{\alpha\dot{\alpha}} W^{\alpha\dot{\alpha}} \approx -m^2 \mathsf{T}_r \mathsf{T}_r \,. \tag{A.17}$$

In our model, twistor quantization is performed with the commutation relations (4.2.2a) and (4.2.3), or equivalently,

$$\left[\hat{\rho}_{i\alpha}, \hat{\bar{\varpi}}_\beta^j\right] = -\delta_i^j \epsilon_{\alpha\beta} \,, \quad \left[\hat{\bar{\rho}}_{\dot{\alpha}}^i, \hat{\varpi}_{j\dot{\beta}}\right] = \delta_j^i \epsilon_{\dot{\alpha}\dot{\beta}} \,, \quad \text{all others} = 0 \,. \tag{A.18}$$

The operators corresponding to $\mu_{\alpha\beta}$ and $\bar{\mu}_{\dot{\alpha}\dot{\beta}}$ are defined by replacing the twistor variables in Eq. (A.3) with their corresponding operators and by obeying the Weyl ordering rule. After using the commutation relations in Eq. (A.18), we have

$$\hat{\mu}_{\alpha\beta} = \frac{i}{2} \left(\hat{\rho}_{i\alpha} \hat{\bar{\varpi}}_\beta^i + \hat{\rho}_{i\beta} \hat{\bar{\varpi}}_\alpha^i \right) , \tag{A.19 a}$$

$$\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}} = -\frac{i}{2} \left(\hat{\bar{\rho}}_{\dot{\alpha}}^i \hat{\varpi}_{i\dot{\beta}} + \hat{\bar{\rho}}_{\dot{\beta}}^i \hat{\varpi}_{i\dot{\alpha}} \right) . \tag{A.19 b}$$

The operator corresponding to $p_{\alpha\dot{\beta}}$ is found immediately from Eq. (A.2 c) to be

$$\hat{p}_{\alpha\dot{\beta}} = \hat{\bar{\varpi}}^{i}_{\alpha}\hat{\varpi}_{i\dot{\beta}}\,. \tag{A.20}$$

Using Eq. (A.18), we can calculate the commutation relations between $\hat{\mu}_{\alpha\beta}$, $\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}$, and $\hat{p}_{\alpha\dot{\beta}}$ to obtain

$$\Big[\hat{\mu}_{\alpha\beta}, \hat{\mu}_{\gamma\delta}\Big] = -\frac{i}{2}\Big(\epsilon_{\alpha\gamma}\hat{\mu}_{\beta\delta} + \epsilon_{\alpha\delta}\hat{\mu}_{\beta\gamma} + \epsilon_{\beta\gamma}\hat{\mu}_{\alpha\delta} + \epsilon_{\beta\delta}\hat{\mu}_{\alpha\gamma}\Big)\,, \tag{A.21 a}$$

$$\Big[\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}, \hat{\bar{\mu}}_{\dot{\gamma}\dot{\delta}}\Big] = -\frac{i}{2}\Big(\epsilon_{\dot{\alpha}\dot{\gamma}}\hat{\bar{\mu}}_{\dot{\beta}\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\delta}}\hat{\bar{\mu}}_{\dot{\beta}\dot{\gamma}} + \epsilon_{\dot{\beta}\dot{\gamma}}\hat{\bar{\mu}}_{\dot{\alpha}\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\delta}}\hat{\bar{\mu}}_{\dot{\alpha}\dot{\gamma}}\Big)\,, \tag{A.21 b}$$

$$\Big[\hat{\mu}_{\alpha\beta}, \hat{p}_{\gamma\dot{\delta}}\Big] = -\frac{i}{2}\Big(\epsilon_{\alpha\gamma}\hat{p}_{\beta\dot{\delta}} + \epsilon_{\beta\gamma}\hat{p}_{\alpha\dot{\delta}}\Big)\,, \tag{A.21 c}$$

$$\Big[\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}, \hat{p}_{\gamma\dot{\delta}}\Big] = -\frac{i}{2}\Big(\epsilon_{\dot{\alpha}\dot{\delta}}\hat{p}_{\gamma\dot{\beta}} + \epsilon_{\dot{\beta}\dot{\delta}}\hat{p}_{\gamma\dot{\alpha}}\Big)\,, \tag{A.21 d}$$

$$\text{all others} = 0\,. \tag{A.21 e}$$

These commutation relations specify together a spinor representation of the Poincaré algebra. The operators $\hat{\mu}_{\alpha\beta}$, $\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}$, and $\hat{p}_{\alpha\dot{\beta}}$ are thus established as the generators of $SL(2,\mathbb{C})\ltimes\mathbb{R}^{1,3}$. We can verify that $\hat{\mu}_{\alpha\beta}$, $\hat{\bar{\mu}}_{\dot{\alpha}\dot{\beta}}$, and $\hat{p}_{\alpha\dot{\beta}}$ commute with the generators $\hat{\mathsf{T}}_0$ and $\hat{\mathsf{T}}_r$ defined in Eq. (4.2.4). This implies that the Poincaré symmetry and the $U(1)_a \times SU(2)$ internal symmetry are not combined, so that the result is consistent with the Coleman-Mandula theorem [51, 52].

The Weyl ordered operator corresponding to the Pauli-Lubanski pseudovector $W^{\alpha\dot{\alpha}}$ can be simplified as

$$\hat{W}^{\alpha\dot{\alpha}} = \hat{\mathsf{T}}_r\sigma_{ri}{}^{j}\hat{\bar{\varpi}}^{i\alpha}\hat{\varpi}^{\dot{\alpha}}_{j} \tag{A.22}$$

by using the commutation relation

$$\Big[\hat{\mathsf{T}}_r, \sigma_{si}{}^{j}\hat{\bar{\varpi}}^{i\alpha}\hat{\varpi}^{\dot{\alpha}}_{j}\Big] = i\epsilon_{rst}\sigma_{ti}{}^{j}\hat{\bar{\varpi}}^{i\alpha}\hat{\varpi}^{\dot{\alpha}}_{j}\,. \tag{A.23}$$

Then, using the physical state conditions

$$\hat{\varpi}_{i\dot{\alpha}}\hat{\varpi}^{\dot{\alpha}}_{j}|F\rangle = \frac{m}{\sqrt{2}}\epsilon_{ij}e^{i\hat{\varphi}}|F\rangle, \tag{A.24 a}$$

$$\hat{\bar{\varpi}}^{i}_{\alpha}\hat{\bar{\varpi}}^{j\alpha}|F\rangle = \frac{m}{\sqrt{2}}\epsilon^{ij}e^{-i\hat{\varphi}}|F\rangle \tag{A.24 b}$$

equivalent, respectively, to Eqs. (4.2.3h) and (4.2.3i), we can show that

$$\hat{W}_{\alpha\dot{\alpha}}\hat{W}^{\alpha\dot{\alpha}}|F\rangle = -m^2\hat{\mathsf{T}}_r\hat{\mathsf{T}}_r|F\rangle\,. \tag{A.25}$$

This is precisely a quantum mechanical counterpart of Eq. (A.17). The Casimir invariants of the Poincaré algebra are given by $\hat{p}_{\alpha\dot{\beta}}\hat{p}^{\alpha\dot{\beta}}$ and $\hat{W}_{\alpha\dot{\alpha}}\hat{W}^{\alpha\dot{\alpha}}$. From Eq. (A.24), it follows that

$$\hat{p}_{\alpha\dot{\beta}}\hat{p}^{\alpha\dot{\beta}}|F\rangle = m^2|F\rangle\,. \tag{A.26}$$

Then it can be shown that [49, 50]

$$\hat{W}_{\alpha\dot{\alpha}}\hat{W}^{\alpha\dot{\alpha}}|F\rangle = -m^2 J(J+1)|F\rangle\,, \tag{A.27}$$

with J being the spin quantum number taking the values

$$J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots. \tag{A.28}$$

Here, $|F\rangle$ is assumed to be a simultaneous eigenvector of $\hat{W}_{\alpha\dot{\alpha}}\hat{W}^{\alpha\dot{\alpha}}$ and the other relevant operators $\hat{\mathsf{T}}_0$, $\hat{\mathsf{T}}_3$, $\hat{\mathsf{T}}_{\hat{\imath}}\hat{\mathsf{T}}_{\hat{\imath}}$, and $\hat{p}_{\alpha\dot{\beta}}\hat{p}^{\alpha\dot{\beta}}$ (see Eqs. (4.2.3e), (4.2.3f), and (4.2.3g)). This assumption holds true, because the generators of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ commute with those of $U(1)_a \times SU(2)$. The vector $|F\rangle$ turns out to be characterized by the set of quantum numbers $(s, I, t; m, J)$. In terms of $|F\rangle$, Eq. (4.2.20) reads

$$\hat{\mathsf{T}}_r\hat{\mathsf{T}}_r|F\rangle = I(I+1)|F\rangle\,. \tag{A.29}$$

Applying Eqs. (A.27) and (A.29) to Eq. (A.25), we eventually have

$$I = J\,. \tag{A.30}$$

This result is consistent with the fact that the number of $SU(2)$ indices of the spinor field Ψ, given in Eq. (4.3.1), is equal to the number of its spinor indices.

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