ALMOST SYMMETRIC NUMERICAL SEMIGROUPS WITH SMALL NUMBER OF GENERATORS

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Contents

Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* H is a submonoid of $\mathbb{N}$ whose complement in $\mathbb{N}$ is finite. We denote by

$$H = \langle a_1, ..., a_n \rangle := \{\lambda_1 a_1 + \cdots \lambda_n a_n \mid \lambda_1, ..., \lambda_n \in \mathbb{N}\}$$

if it is minimally generated by $a_1, ..., a_n > 0$. A numerical semigroup is an object which is related to various fields in mathematics. In particular, we are interested in the relationship to commutative algebra. For a numerical semigroup H, we define its semigroup ring as

$$k[H] := k[t^h \mid h \in H] \subset k[t],$$

or $k[[H]] := k[[t^h \mid h \in H]] \subset k[[t]]$ if we consider the local case, where k is a field and t an indeterminate. The ring $k[H]$ is a one-dimensional Cohen-Macaulay ring as a commutative algebra. It is natural that the properties and invariants of H correspond to those of $k[H]$. For example, E. Kunz [Ku] proved that H is symmetric if and only if $k[H]$ is Gorenstein, which is a classical and well-known result in this subject. Therefore we can observe the object of study from both numerical semigroups and commutative rings sides. This enables us to simplify problems since we can use many tools in commutative ring theory.

Our main object in this paper is almost symmetric numerical semigroups. The notion of almost symmetric numerical semigroups was introduced by V. Barucci and R. Fröberg [BF]. The concept is a generalization of those of symmetric and pseudo-symmetric numerical semigroups. In fact, both symmetric and pseudo-symmetric numerical semigroups are almost symmetric. Conversely, almost symmetric numerical semigroups with type 1 and 2 are symmetric and pseudo-symmetric, respectively. The notion of almost Gorenstein rings was also introduced in [BF] as the ring corresponding to almost symmetric numerical semigroups, that is, H is almost symmetric if and only if $k[[H]]$ is almost Gorenstein. They developed many interesting theory of almost symmetric numerical semigroups and almost Gorenstein rings. In [BF], they defined the notion of almost Gorenstein rings for one-dimensional analytically unramified local rings. A few years ago, S. Goto, N. Matsuoka and T. T. Phuong [GMP] gave a new definition of almost Gorenstein rings. In the definition, almost Gorenstein rings are defined for one-dimensional Cohen-Macaulay local rings. This definition is more general than that of [BF]. They also developed more deeper theory of almost Gorenstein rings and showed many interesting results. Furthermore, S. Goto, R. Takahashi and N. Taniguchi [GTT] recently extended the definition of almost Gorenstein rings for any dimensional Cohen-Macaulay rings. We expect that this could bring a new development to commutative ring theory.

Almost symmetric numerical semigroups produce many good examples of almost Gorenstein rings. Therefore the study of almost symmetric numerical semigroups could contribute to study of almost Gorenstein rings. Numerical semigroups are very explicit, and hence we can compute various invariants, which is an advantage to consider numerical semigroups. On the other hand, almost symmetric numerical semigroups enjoy very interesting properties itself in the view of numerical semigroup theory.

In this paper, we study almost symmetric numerical semigroups in the view of commutative ring theory. At first, we observe 3 and 4-generated cases (Chapter 2 and Chapter 3). We note that all 2-generated numerical semigroups are symmetric. In general, some problems are difficult when numerical semigroups have large number of minimal generators. So, we consider numerical semigroups which have a special system of minimal generators (Chapter 4). The highlight in this paper is Chapter 2. In this chapter, we give a characterization for 3-generated numerical semigroups to be almost symmetric by using their minimal free resolutions (or defining ideals). This is a new perspective of almost symmetric numerical semigroups. Let us explain the contents of each chapter in this paper as follows.

We start Chapter 1 by recalling some basic definitions and notations on numerical semigroups and numerical semigroup rings. We will follow the notations and terminologies of [RG4].

In Chapter 2, we investigate 3-generated almost symmetric numerical semigroups $H = \langle a, b, c\rangle$. We note that if H is not symmetric, then H is almost symmetric if and only if H is pseudo-symmetric. When H is symmetric, the structure of H was studied by [FGH], [He] and [Wa]. Hence we are interested in the case where H is not symmetric. Then it is known by [He] that the defining ideal of $k[H]$ is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix},$$

where α, β, γ, α', β', and γ' are positive integers. Then we prove that we can describe the genus of H, denoted by $\mathrm{g}(H)$, by α, β, γ, α', β', and γ':

Theorem 1 (Theorem 2.8)**.** *Let $H = \langle a, b, c\rangle$ be a numerical semigroup which is not symmetric. Then:*

(1) *if $\beta' b > \alpha a$, then $2 \cdot \mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$,*

(2) *if $\beta' b < \alpha a$, then $2 \cdot \mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha'\beta'\gamma'$.*

As a direct consequence of Theorem 1, we get the characterization of pseudo-symmetric numerical semigroups:

Corollary 2 (Corollary 2.9)**.** *Let H be as above. Then H is pseudo-symmetric if and only if either $\alpha = \beta = \gamma = 1$ or $\alpha' = \beta' = \gamma' = 1$*

As an application of Corollary 2, for any fixed even integer f, we can construct all the pseudo-symmetric numerical semigroups $H = \langle a, b, c \rangle$ whose Frobenius numbers are f. This Chapter is based on [NNW2].

In Chapter 3, we study 4-generated almost symmetric numerical semigroups. We have the following conjecture on the upper bound of the type of 4-generated almost symmetric numerical semigroups. We denote the type of H by $\mathrm{t}(H)$.

Conjecture 3 (Conjecture 3.1)**.** *If H is a 4-generated almost symmetric numerical semigroup, then $\mathrm{t}(H) \leq 3$.*

In general, it is known that there is no upper bound on type of numerical semigroups $H = \langle a_1, ..., a_n \rangle$ if $n \geq 4$ (see [FGH]). J. C. Rosales and P. A. García-Sánchez [RG2] recently proved that every almost symmetric numerical semigroup can be constructed by removing some minimal generators from an irreducible numerical semigroup with the same Frobenius number. Using this result, we explicitly construct 4-generated almost symmetric numerical semigroups from 2 or 3-generated irreducible numerical semigroups. For those special semigroups, we see that Conjecture 3 is true. The main results in this chapter are Theorems 3.6, 3.12, 3.16 and 3.20. This Chapter is based on [Nu1].

At the end of Chapter 3, we add a comment about the defining ideals of 4-generated almost symmetric numerical semigroups. If H is a 4-generated symmetric or pseudo-symmetric numerical semigroup, then the defining ideals I_H of $k[H]$ are completely determined by H. Bresinsky [Br] and R. Komeda [Ko], respectively. However, when H is almost symmetric but not symmetric and pseudo-symmetric, this problem is still open. We expect that the upper bound of the number of minimal generators of I_H is 7 if H is almost symmetric.

In Chapter 4, we consider numerical semigroups which have a special system of minimal generators, that is,

$$H = \langle a, sa + d, sa + 2d, ..., sa + nd \rangle,$$

where $s, a, d > 0$, $n \geq 2$ and $\gcd(a, d) = 1$. Then H is called a numerical semigroup generated by a *generalized arithmetic sequence* and it is called a numerical semigroup generated by an *arithmetic sequence* if $s = 1$. Numerical semigroups of those forms are studied by many authors (see [EL], [GSS], [Ju], [MS], [Ma]). In particular, the characterization for $H = \langle a, sa + d, sa + 2d, ..., sa + nd \rangle$ to be symmetric is given by M. Estrada, A. Lópes [EL] and G. L. Matthews [Ma]. When $s = 1$, this characterization was given by L. Juan [Ju]. G. L. Matthews [Ma] also gave a characterization for H to be pseudo-symmetric. We generalize this result for almost symmetric numerical semigroups:

Theorem 4 (Corollary 4.4)**.** *Let $H = \langle a, sa + d, ..., sa + nd \rangle$ be a numerical semigroup generated by a generalized arithmetic sequence. Then H is almost symmetric but not symmetric if and only if $a = n + 1$ and $s = 1$. In particular, H is pseudo-symmetric if and only if $H = \langle 3, 3 + d, 3 + 2d \rangle$.*

In Chapter 5, as an application in Chapter 4, we investigate Ulrich ideals of Gorenstein numerical semigroup rings which are generated by monomials. The notion of Ulrich ideals was introduced by S. Goto, K. Ozeki, R. Takahashi, K. -i. Watanabe and K. Yoshida [GOTWY]. Let us recall the definition of Ulrich ideals.

Definition ([GOTWY])**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R$ and I be an $\mathfrak{m}$-primary ideal. Then I is called an *Ulrich ideal* of R if the following two conditions hold true for a minimal reduction $Q \subset I$:

(1) $I^2 = QI$ and

(2) I/I^2 is R/I-free.

In [GOTWY], they also gave a characterization of Ulrich ideals of Gorenstein numerical semigroup rings which are generated by monomials (see Theorem 5.2). This characterization is the key to achieve our goal. When H is a numerical semigroup generated by an arithmetic sequence, we determine when $k[[H]]$ has Ulrich ideals generated by monomials. To be specific, we prove the following theorem:

Theorem 5 (Theorem 5.5)**.** *Let $H = \langle a, a + d, a + 2d, ..., a + nd \rangle$ be a symmetric numerical semigroup generated by an arithmetic sequence. Then $k[[H]]$ has an Urlich ideal generated by monomials if and only if $n = 2$.*

Chapter 4 and 5 are based on [Nu2]. Finally, we mention the case where $H = \langle a, b, c \rangle$ is a 3-generated symmetric numerical semigroup. In that case, the author in [Nu3] completely determine when $k[[H]]$ has Ulrich ideals generated by monomials. We refer to the result and give some remarks.

In Chapter 6, we introduce a sufficient condition for a numerical semigroup to be almost symmetric from [Nu4]. We expect that it is an important property of almost symmetric numerical semigroups.

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Chapter 1

Numerical semigroups and numerical semigroup rings

First, we recall some basic definitions and notations on numerical semigroups and numerical semigroup rings.

1.1 The definitions of numerical semigroups and numerical semigroup rings

Definition 1.1 (Numerical semigroups)**.** A *numerical semigroup* H is a subset of $\mathbb{N}$ which satisfies the following conditions:

(1) $0 \in H$ (contains the zero element).

(2) $H + H \subset H$ (closed under addition).

(3) $\#(\mathbb{N} \setminus H) < \infty$ (complement in $\mathbb{N}$ is finite).

A numerical semigroup has the unique system of minimal generators. If H is minimally generated by $a_1, ..., a_n > 0$, then we denote by

$$H = \langle a_1, ..., a_n \rangle := \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, ..., \lambda_n \geq 0\}.$$

We note that $\#(\mathbb{N} \setminus H) < \infty$ if and only if $\gcd(a_1, ..., a_n) = 1$.

Definition 1.2 (Numerical semigroup rings)**.** For a numerical semigroup H, we define its *semigroup ring* as

$$k[H] := k[t^h \mid h \in H] \subset k[t]$$

where k is a field and t is an indeterminate.

A numerical semigroup ring $k[H] = k[t^{a_1}, ..., t^{a_n}]$ is:

- a subring of a polynomial ring $k[t]$,

- a one-dimensional Cohen-Macaulay domain, and
- isomorphic to a quotient ring $k[X_1, ..., X_n]/I_H$, where I_H is the kernel of the k-algebra surjective homomorphism $\phi : k[X_1, ..., X_n] \to k[H]$, where $X_i \to t^{a_i}$ for each $1 \leq i \leq n$. Then I_H is called the *defining ideal* of $k[H]$.

We usually regard $k[H]$ as a $\mathbb{Z}$-graded ring in the natural way. Then $k[H]$ has the unique homogeneous maximal ideal $\mathfrak{m} = (t^{a_1}, ..., t^{a_n})$. If we define as $\deg(X_i) = a_i$ for each $1 \leq i \leq n$ in $k[X_1, ..., X_n]$, then the map ϕ is homogeneous of degree 0.

1.2 Some invariants

The notion of Apéry sets is a very useful tool in numerical semigroup theory.

Definition 1.3 (Apéry sets)**.** Let H be a numerical semigroup and take $0 \neq a \in H$. The *Apéry set* of a in H is

$$\mathrm{Ap}(H, a) = \{h \in H \mid h - a \notin H\}.$$

From the definition, we can easily see the following.

Lemma 1.4 ([RG4, Lemma 2.4])**.** *Let H be a numerical semigroup and take $0 \neq a \in H$. Then*

$$\mathrm{Ap}(H, a) = \{0 = w(0), w(1), ..., w(a-1)\},$$

where $w(i) = \min\{h \in H \mid h \equiv i \pmod a\}$ for each $0 \leq i \leq a-1$.

Let us recall some basic and important invariants of numerical semigroups.

Definition 1.5 (Some invariants of numerical semigroups)**.** Let H be a numerical semigroup.

(1) $\mathrm{F}(H) = \max(\mathbb{Z} \setminus H)$, the *Frobenius number* of H.

(2) $\mathrm{PF}(H) = \{x \in \mathbb{Z} \setminus H \mid x + h \in H, 0 \neq \forall h \in H\}$, the set of *pseudo-Frobenius numbers* of H.

(3) $\mathrm{t}(H) = \#\,\mathrm{PF}(H)$, the *type* of H.

(4) $\mathrm{G}(H) = \mathbb{N} \setminus H$, the set of *gaps* of H.

(5) $\mathrm{g}(H) = \#G(H)$, the *genus* of H.

(6) $\mathrm{e}(H) = \min(H \setminus \{0\})$, the *multiplicity* of H.

(7) $\mathrm{emb}(H) = n$, the *embedding dimension* of H

By definition, $\mathrm{F}(H) \in \mathrm{PF}(H)$. When $\mathrm{emb}(H) = n$, we often say that H is *n-generated* for simplicity. We discuss relations between those invariants for a while.

It is easily seen that $\mathrm{e}(H)$ and $\mathrm{emb}(H)$ correspond to the multiplicity and embedding dimension of $k[H]$, respectively. We can easily prove the following result. In the view of commutative ring theory, however, it is a special case of the result in [Ab].

Proposition-Definition 1.6 ([Ab], [RG4, Proposition 2.4])**.** Let H be a numerical semigroup. Then the following inequality holds:

$$\mathrm{emb}(H) \le \mathrm{e}(H).$$

If equality holds true, then we say that H has *maximal embedding dimension*.

Let $R = k[H]$ be a numerical semigroup ring. From the short exact sequence $0 \to R \to k[t, t^{-1}] \to k[t, t^{-1}]/k[H] \to 0$, we get the isomorphism

$$\mathrm{H}^1_{\mathfrak{m}}(R) \cong k[t, t^{-1}]/R, \tag{1.1}$$

where $\mathrm{H}^1_{\mathfrak{m}}(R)$ is the first local cohomology module of R. Hence $x \in \mathrm{PF}(H)$ if and only if $t^x \in \mathrm{Soc}(\mathrm{H}^1_{\mathfrak{m}}(R))$, the socle of $\mathrm{H}^1_{\mathfrak{m}}(R)$. It follows that the Cohen-Macaulay type of R, denoted by $\mathrm{r}(R)$, is equal to the type of H. The *a-invariant* of R is defined by $a(R) = \max\{i \in \mathbb{Z} \mid [\mathrm{H}^1_{\mathfrak{m}}(R)]_i \ne 0\}$ (see [GW]). Therefore $a(R) = \mathrm{F}(H)$ by (1.1). We summarize these results as follows.

Proposition 1.7 ([GW])**.** *Let $R = k[H]$ be a numerical semigroup ring. Then:*

(1) $\mathrm{t}(H) = \mathrm{r}(R)$.

(2) $\mathrm{F}(H) = a(R)$.

We can compute pseudo-Frobenius numbers by using Apéry sets as follows. For a numerical semigroup H, we define the order $\le_H$ over $\mathbb{Z}$: $x \le_H y$ if $y - x \in H$. We note that the set $\{t^w \mid w \in \mathrm{Ap}(H, a)\}$ is a k-basis of the quotient ring $T = k[H]/(t^a)$. Hence maximal elements in $\mathrm{Ap}(H, a)$ with respect to $\le_H$ correspond to the generators of $\mathrm{Soc}(T)$.

Proposition 1.8 ([RG4, Proposition 2.20])**.** *Let H be a numerical semigroup and take $0 \ne a \in H$. Then*

$$\mathrm{PF}(H) = \{w - a \mid w \textit{ is a maximal element in } \mathrm{Ap}(H, a) \textit{ with respect to } \le_H\}.$$

In particular, $\mathrm{F}(H) = \max \mathrm{Ap}(H, a) - a$.

Let H be a numerical semigroup. If $h \in H$, then $\mathrm{F}(H) - h \notin H$. This implies that there exists an injection from the set $\{h \in H \mid h < \mathrm{F}(H)\}$ to $\mathrm{G}(H)$. From this, we get the following relation between $\mathrm{F}(H)$ and $\mathrm{g}(H)$.

Proposition 1.9 ([RG4, Lemma 2.14])**.** *Let H be a numerical semigroup. Then the following inequality holds true:*

$$2 \cdot \mathrm{g}(H) \ge \mathrm{F}(H) + 1.$$

1.3 Symmetric numerical semigroups

Definition 1.10 (Symmetric numerical semigroups)**.** Let H be a numerical semigroup. We say H is *symmetric* if for any $x \in \mathbb{Z}$, either $x \in H$ or $\mathrm{F}(H) - x \in H$.

It it clear that if H is symmetric, then $\mathrm{F}(H)$ is odd. There are some characterization of symmetric numerical semigroups.

Proposition 1.11 ([RG4, Chapter 3])**.** *Let H be a numerical semigroup, $0 \neq a \in H$ and $\mathrm{Ap}(H, a) = \{0 = w_1 < w_2 < \cdots < w_a\}$. Then the following conditions are equivalent:*

(1) *H is symmetric.*

(2) *$w_i + w_{a-i} = w_a$ for all $1 \leq i \leq a - 1$.*

(3) *$\mathrm{t}(H) = 1$.*

(4) *$\mathrm{PF}(H) = \{\mathrm{F}(H)\}$.*

(5) *$2 \cdot \mathrm{g}(H) = \mathrm{F}(H) + 1$.*

By Proposition 1.7 and 1.11, we can see that H is symmetric if and only if $k[H]$ is Gorenstein. This result was originally proved by E. Kunz [Ku].

1.4 Pseudo-symmetric numerical semigroups

Definition 1.12 ([BDF])**.** A numerical semigroup H is *pseudo-symmetric* if $\mathrm{F}(H)$ is even and for any $x \in \mathbb{Z}, x \neq \mathrm{F}(H)/2$, either $x \in H$ or $\mathrm{F}(H) - x \in H$.

Note that if H is pseudo-symmetric, $\mathrm{F}(H)/2 + a \in \mathrm{Ap}(H, a)$.

Proposition 1.13 ([BDF], [RG4, Chapter 3])**.** *Let H be a numerical semigroup with even Frobenius number, $0 \neq a \in H$ and $\mathrm{Ap}(H, a) = \{0 = w_1 < w_2 < \cdots < w_{a-1} = \mathrm{F}(H) + a\} \cup \{\mathrm{F}(H)/2 + a\}$. Then the following conditions are equivalent:*

(1) *H is pseudo-symmetric.*

(2) *$w_i + w_{a-1-i} = w_{a-1}$ for all $1 \leq i \leq a - 1$.*

(3) *$\mathrm{PF}(H) = \{\mathrm{F}(H)/2, \mathrm{F}(H)\}$.*

(4) *$2 \cdot \mathrm{g}(H) = \mathrm{F}(H) + 2$.*

We remark that a numerical semigroup with type 2 is not always pseudo-symmetric (see Example 1.16)

1.5 Almost symmetric numerical semigroups

The notion of almost symmetric numerical semigroups was introduced by V. Barucci and R. Fröberg. For a numerical semigroup H, we define

$$\mathrm{L}(H) = \{x \in \mathbb{Z} \setminus H \mid \mathrm{F}(H) - x \notin H\}.$$

Definition 1.14 ([BF]). A numerical semigroup H is *almost symmetric* if $\mathrm{L}(H) \subset \mathrm{PF}(H)$.

By definition, we can see that that H is symmetric (resp. pseudo-symmetric) if and only if $\mathrm{L}(H) = \emptyset$ (resp. $\mathrm{L}(H) = \{\mathrm{F}(H)/2\}$), and hence symmetric and pseudo-symmetric numerical semigroups are almost symmetric. Conversely, almost symmetric numerical semigroups with type 2 are pseudo-symmetric. Therefore, the concept of almost symmetric numerical semigroups is a generalization of those of symmetric and pseudo-symmetric numerical semigroups.

H. Nari gave a characterization of almost symmetric numerical semigroups. This result is an analogue of Proposition 1.11 and 1.13.

Theorem 1.15 ([Na], cf. [BF]). *Let H be a numerical semigroup and $\mathrm{PF}(H) = \{f_1 < f_2 < \cdots < f_t = \mathrm{F}(H)\}$. Then the following conditions are equivalent:*

(1) *H is almost symmetric.*

(2) *$f_i + f_{t-i} = \mathrm{F}(H)$ for any $1 \leq i \leq t-1$.*

(3) *$2 \cdot \mathrm{g}(H) = \mathrm{F}(H) + \mathrm{t}(H)$.*

1.6 Examples

We give some examples of symmetric, pseudo-symmetric and almost symmetric numerical semigroups, respectively.

Example 1.16.

(1) All 2-generated numerical semigroups are symmetric. In particular, $\langle 2, a\rangle$ is symmetric for every $a \geq 3$.

(2) Let $3 < a < b$ and $H = \langle 3, a, b\rangle$. Then H is pseudo-symmetric if and only if $b = 2a - 3$ (cf. Chapter 2). Hence, for example, $\langle 3, 7, 8\rangle$ is not pseudo-symmetric. Moreover, H is never symmetric for any a and b,

(3) $\langle 4, 5, 6\rangle$ is symmetric, and $\langle 4, 5, 7\rangle$ is pseudo-symmetric. However $\langle 4, 5, 6, 7\rangle$ is almost symmetric but not pseudo-symmetric or symmetric.

(4) In general, $\langle a, a+1, ..., 2a-1\rangle = \{0, a \ \rightarrow\}$ is almost symmetric but not pseudo-symmetric or symmetric if $a \geq 4$.

1.7 Minimal graded free resolutions of numerical semigroup rings

Now we observe minimal graded free resolutions of numerical semigroup rings. Let $H = \langle a_1, ..., a_n \rangle$, $S = k[X_1, ..., X_n]$ and $R = k[H]$. Since R is a one-dimensional Cohen-Macaulay ring, the minimal graded free resolution of R has length $n-1$ by Auslander-Buchsbaum formula:

$$0 \to \bigoplus_j S(-m_{n-1,j})^{\beta_{n-1,j}} \to \cdots \to \bigoplus_j S(-m_{1j})^{\beta_{1j}} \to S \to R \to 0,$$

where $\beta_{ij} > 0$ for each i, j. Note that $K_S \cong S(-\omega)$, where K_S is the canonical module of S and $\omega = \sum_{i=1}^n a_i$. Taking $\mathrm{Hom}_S(*, K_S) \cong \mathrm{Hom}_S(*, S(-\omega))$, we have

$$0 \to S(-\omega) \to \bigoplus_j S(m_{1j} - \omega)^{\beta_{1j}} \to \cdots \to \bigoplus_j S(m_{n-1,j} - \omega)^{\beta_{n-1,j}} \to K_S \to 0.$$

Since $K_S \cong \mathrm{Ext}_S^{n-1}(R, K_S)$ is generated by the elements of degree $-\mathrm{PF}(H)$, we have

$$\mathrm{PF}(H) = \{m_{n-1,j} - \omega \mid 1 \le j \le t\},$$

where $t = \mathrm{r}(R) = \mathrm{t}(H)$. We also note that $\beta_{n-1,j} = 1$ for each $1 \le j \le t$.

Chapter 2

Almost symmetric numerical semigroups generated by three elements

We investigate 3-generated numerical semigroups $H = \langle a, b, c \rangle$. First, let us mention the symmetric case. In that case, the structure of H is well known by [FGH], [He] and [Wa].

Theorem 2.1 (Herzog [He], cf. [FGH], [Wa])**.** *Let $H = \langle a, b, c \rangle$ be a numerical semigroup. Then the following conditions are equivalent:*

(1) *H is symmetric.*

(2) *Changing order of a, b and c if necessary, we can write $a = a'd, b = b'd$ where* $\gcd(a, b) = d > 1$ *and $c \in \langle a', b' \rangle$, $c \neq a', b'$. In this case, we denote by $H = \langle d \langle a', b' \rangle, c \rangle$.*

This is a very useful characterization of 3-generated symmetric numerical semigroups. Let us show some examples.

Example 2.2.

(1) Both $\langle 4, 5, 6 \rangle$ and $\langle 6, 10, 11 \rangle$ are symmetric. In fact, we can write as $\langle 4, 5, 6 \rangle = \langle 2 \langle 2, 3 \rangle, 5 \rangle$ and $\langle 6, 10, 11 \rangle = \langle 2 \langle 3, 5 \rangle, 11 \rangle$, respectively.

(2) Both $\langle 7, 10, 12 \rangle$ and $\langle 9, 11, 13 \rangle$ are not symmetric.

(3) In general, if any pairs of minimal generators of $H = \langle a, b, c \rangle$ are relatively coprime, then H is not symmetric.

Therefore, we are interested in the case where $H = \langle a, b, c \rangle$ is not symmetric. It is shown in [FGH] that if H is 3-generated, then $\mathrm{t}(H) \leq 2$. Hence the following is easily verified by definition.

Proposition 2.3. *Let $H = \langle a, b, c \rangle$ be a numerical semigroup which is not symmetric. Then H is almost symmetric if and only if it is pseudo-symmetric.*

Thus, we study 3-generated pseudo-symmetric numerical semigroups in the following.

2.1 Characterization of pseudo-symmetric numerical semigroups generated by three elements

Let $R = k[H] \cong k[X,Y,Z]/I_H$ be the semigroup ring of $H = \langle a,b,c\rangle$. Then it is known by [He] that the ideal I_H of $S = k[X,Y,Z]$ is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}, \tag{2.1}$$

where α, β, γ, α', β', and γ' are positive integers. We define the grading on S by $\deg(X) = a$, $\deg(Y) = b$, $\deg(Z) = c$.

In another word, this corresponds to the following assertion.

Proposition 2.4. *If $H = \langle a,b,c\rangle$ is not symmetric, then*

(1) $(\alpha+\alpha')a = \beta' b + \gamma c$ *and* $\alpha+\alpha' = \min\{n > 0 \mid an \in \langle b,c\rangle\}$,

(2) $(\beta+\beta')b = \alpha a + \gamma' c$ *and* $\beta+\beta' = \min\{n > 0 \mid bn \in \langle a,c\rangle\}$,

(3) $(\gamma+\gamma')c = \alpha' a + \beta b$ *and* $\gamma+\gamma' = \min\{n > 0 \mid cn \in \langle a,b\rangle\}$.

Since $k[H]/(t^a) \cong k[Y,Z]/(Y^{\beta+\beta'}, Y^{\beta'}Z^{\gamma}, Z^{\gamma+\gamma'})$, the defining ideal of $k[H]/(t^a)$ is generated by the maximal minors of the matrix $\begin{pmatrix} 0 & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & 0 \end{pmatrix}$. Since $a = \dim_k k[H]/(t^a) = \dim_k k[Y,Z]/(Y^{\beta+\beta'}, Y^{\beta'}Z^{\gamma}, Z^{\gamma+\gamma'})$, and likewise for b, c, we get the equations

$$\begin{aligned} a &= \beta\gamma + \beta'\gamma + \beta'\gamma', \\ b &= \gamma\alpha + \gamma'\alpha + \gamma'\alpha', \\ c &= \alpha\beta + \alpha'\beta + \alpha'\beta'. \end{aligned} \tag{2.2}$$

We put $l = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta}$, $m = X^{\alpha+\alpha'} - Y^{\beta'}Z^{\gamma}$, and $n = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma'}$. There are obvious relations

$$X^{\alpha}l + Y^{\beta}m + Z^{\gamma}n = Y^{\beta'}l + Z^{\gamma'}m + X^{\alpha'}n = 0.$$

We put $p = \deg(l) = c(\gamma+\gamma'), q = \deg(m) = a(\alpha+\alpha'), r = \deg(n) = b(\beta+\beta')$. Also we put $s = a\alpha + p, t = b\beta' + p$. Then we get the minimal graded free resolution of R over S as follows:

$$0 \to S(-s) \oplus S(-t) \to S(-p) \oplus S(-q) \oplus S(-r) \to S \to R \to 0.$$

Note that $K_S \cong S(-\omega)$ with $\omega = a+b+c$. Taking $\mathrm{Hom}_S(*, K_S) = \mathrm{Hom}_S(*, S(-\omega))$, we get

$$0 \to S(-\omega) \to S(p-\omega) \oplus S(q-\omega) \oplus S(r-\omega) \to S(s-\omega) \oplus S(t-\omega) \to K_R \to 0.$$

From this exact sequence, we have that $\mathrm{PF}(H) = \{s-\omega, t-\omega\}$. We put $f = s-\omega$ and $f' = t - \omega$.

In conclusion, we obtain the following results.

Proposition 2.5. *If $H = \langle a, b, c\rangle$ is not symmetric, then $\mathrm{PF}(H) = \{f, f'\}$ where*

(1) $f = \alpha a + (\gamma + \gamma')c - (a + b + c)$,

(2) $f' = \beta' b + (\gamma + \gamma')c - (a + b + c)$.

Remark 2.6. Above formulas related to our results can be found in [RG2], [RG3], [RG4].

The following is the key lemma to prove our main theorem.

Lemma 2.7. *Let $H = \langle a, b, c\rangle$ be as in the previous section.*

(1) *If $\beta' b > \alpha a$, or equivalently, $f' > f$, then*

  (i) *for $p, q, r \in \mathbb{N}$, $f' - f + pa + qb + rc \notin H$ if and only if $p < \alpha, q < \beta$ and $r < \gamma$.*

  (ii) $\#\{h \in H \mid f' - f + h \notin H\} = \alpha\beta\gamma$.

(2) *If $\beta' b < \alpha a$, or equivalently, $f' < f$, then*

  (i) *for $p, q, r \in \mathbb{N}$, $f - f' + pa + qb + rc \notin H$ if and only if $p < \alpha', q < \beta'$ and $r < \gamma'$.*

  (ii) $\#\{h \in H \mid f - f' + h \notin H\} = \alpha'\beta'\gamma'$.

Proof. We assume $\beta' b > \alpha a$. Since $f' - f + \alpha a = \beta' b, f' - f + \beta b = \gamma' c, f' - f + \gamma c = \alpha' a \in H$, $f' - f + pa + qb + rc \in H$ if $p \geq \alpha$ or $q \geq \beta$ or $r \geq \gamma$. Conversely, assume $p < \alpha, q < \beta$ and $r < \gamma$ and $f' - f + pa + qb + rc = ua + vb + wc \in H$ for some $u, v, w \in \mathbb{N}$. Then we have $(\beta' + q - v)b = (\alpha - p + u)a + (w - r)c$. If $\beta' + q - v \leq 0$, then $(r - w)c = (\alpha - p + u)a + (v - \beta' - q)b$, which implies $r - w \geq \gamma + \gamma'$. This is a contradiction since $r - w < \gamma$. Thus we have $\beta' + q - v > 0$. If $w \geq r$, then this contradicts Proposition 2.4 (2). If $r > w$, we have $(\alpha - p + u)a = (\beta' + q - v)b + (r - w)c$. Then by Proposition 2.4 (1), we must have $u - p \geq \alpha'$. This means $X^{\alpha-p+u} - Y^{\beta'+q-v}Z^{r-w} \in I_H$, which is impossible by (2.1) and Proposition 2.4 (1), since $r - w < \gamma$. This finishes the proof of (i), and (ii) is a direct consequence of (i). □

Theorem 2.8. *Let $H = \langle a, b, c\rangle$ be a numerical semigroup. Then*

(1) *if $\beta' b > \alpha a$, then $2 \cdot \mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$,*

(2) *if $\beta' b < \alpha a$, then $2 \cdot \mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha'\beta'\gamma'$.*

Proof. We may assume $\beta' b > \alpha a$. Then by Proposition 2.5, $\mathrm{F}(H) = f'$. Since for $h \in H$, $f - h \notin f' - H$ if and only if $f' - (f - h) \notin H$, using Lemma 2.7 we have

$$\#[(f - H) \cap \mathbb{N} \setminus (f' - H)] = \#\{h \in H \mid f' - f + h \notin H\} = \alpha\beta\gamma.$$

Since $\mathbb{N} \setminus H = ((f' - H) \cap \mathbb{N}) \cup ((f - H) \cap \mathbb{N})$, we get

$$\mathrm{g}(H) = \#[(f' - H) \cap \mathbb{N}] + \#[[(f - H) \cap \mathbb{N}] \setminus (f' - H)]$$

hence

$$\mathrm{g}(H) = (\mathrm{F}(H) + 1 - \mathrm{g}(H)) + \alpha\beta\gamma.$$

□

As a corollary, we find a characterization of 3-generated pseudo-symmetric numerical semigroups.

Corollary 2.9. *H is pseudo-symmetric if and only if*

(1) *if $\beta' b > \alpha a$, then $\alpha = \beta = \gamma = 1$ and*

(2) *if $\beta' b < \alpha a$, then $\alpha' = \beta' = \gamma' = 1$.*

Proof. We may assume that $\beta' b > \alpha a$. By Theorem 2.8, $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$. Since H is pseudo-symmetric if and only if $2\,\mathrm{g}(H) = \mathrm{F}(H) + 2$ by Proposition 1.13, we obtain that $\alpha\beta\gamma = 1$, or equivalently, $\alpha = \beta = \gamma = 1$. □

2.2 The structure of a pseudo-symmetric numerical semigroup generated by three elements

In this section, we assume that $H = \langle a, b, c \rangle$ is a pseudo-symmetric numerical semigroup. Our purpose is to classify, for any fixed even integer f, all the pseudo-symmetric numerical semigroups $H = \langle a, b, c \rangle$ with $\mathrm{F}(H) = f$. For example, it is shown in Exercise 10.8 of [RG4] that there is no pseudo-symmetric numerical semigroup $H = \langle a, b, c \rangle$ with $\mathrm{F}(H) = 12$. Actually, we can now give many examples of such an even integer f for which there does not exist a pseudo-symmetric numerical semigroup $H = \langle a, b, c \rangle$ with $\mathrm{F}(H) = f$. (It is shown in [RGG] that every even integer is the Frobenius number of some numerical semigroup generated by at most 4 elements.)

As is mentioned before, the defining ideal I_H is generated by the maximal minors of the matrix as in (2.1) and by Corollary 2.9, we can always assume that $\alpha = \beta = \gamma = 1$. Recall that in this case we have by (2.2),

$$a = \beta'\gamma' + \beta' + 1, \quad b = \gamma'\alpha' + \gamma' + 1, \quad c = \alpha'\beta' + \alpha' + 1. \qquad (2.3)$$

The following is the key for our goal.

Theorem 2.10. *Let $H = \langle a, b, c \rangle$ be a pseudo-symmetric numerical semigroup and assume that I_H is generated by the maximal minors of the matrix* $\begin{pmatrix} X & Y & Z \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$. *Then we have*

$$\alpha'\beta'\gamma' = \frac{\mathrm{F}(H)}{2} + 1.$$

Proof. From our hypothesis and Corollary 2.9, we have $f < f'$. Thus by Proposition 2.5 and (2.3), $\mathrm{F}(H) = f' = \beta' b + (1 + \gamma')c - (a + b + c) = 2\alpha'\beta'\gamma' - 2$. □

Now, given a positive even integer f, we can list all possibilities of the set $\{\alpha', \beta', \gamma'\}$ by prime factorization of $\frac{\mathrm{F}(H)}{2} + 1$.

Remark 2.11. Let σ be a permutation of $\{\alpha', \beta', \gamma'\}$. Then it is easy to see that if σ is an even permutation, then the set $\{a, b, c\}$ obtained by $\{\sigma(\alpha'), \sigma(\beta'), \sigma(\gamma')\}$ as in (4.1.1) is the same and hence the semigroup $H = \langle a, b, c \rangle$ does not change.

But if σ is an odd permutation, then the set $\{a, b, c\}$ does change. So, from the factorization of $\dfrac{\mathrm{F}(H)}{2} + 1$, we get 2 different semigroups in general.

Example 2.12. For example, let us classify all pseudo-symmetric semigroup $H = \langle a, b, c \rangle$ with $\mathrm{F}(H) = f = 18$. Since we have $\alpha'\beta'\gamma' = f/2 + 1 = 10$ by Theorem 2.10, we have $\{\alpha', \beta', \gamma'\} = \{10, 1, 1\}$ or $\{5, 2, 1\}$. But if we put $\{\alpha', \beta', \gamma'\} = \{10, 1, 1\}$ in any order to (4.1.1), a, b, c are all multiple of 3 and we don't get a numerical semigroup.

Thus we get 2 semigroups with $\mathrm{F}(H) = 18$; if $(\alpha', \beta', \gamma') = (5, 2, 1)$ we get $H = \langle 5, 7, 16 \rangle$ and if $(\alpha', \beta', \gamma') = (5, 1, 2)$, then we get $H = \langle 4, 11, 13 \rangle$.

If f is an even integer not divisible by 12, then there is a pseudo-symmetric semigroup $H = \langle a, b, c \rangle$ with $\mathrm{F}(H) = f$ by [RGG].

Proposition 2.13 (Rosales, García-Sánchez, García-García [RGG])**.** *Let $H = \langle a, b, c \rangle$ be a numerical semigroup and $\mathrm{F}(H) = f$. Then*

(1) *If f is an even integer not divisible by* 3*, then*

$$H = \left\langle 3, \frac{f}{2} + 3, f + 3 \right\rangle$$

is a pseudo-symmetric numerical semigroup with Frobenius number f.

(2) *If f is a multiple of* 6 *and not a multiple of* 12*, then*

$$H = \left\langle 4, \frac{f}{2} + 2, \frac{f}{2} + 4 \right\rangle,$$

is a pseudo-symmetric numerical semigroup with Frobenius number f.

If f is divisible by 12, there are many cases such that there does not exist pseudo-symmetric numerical semigroup $H = \langle a, b, c \rangle$ with $\mathrm{F}(H) = f$.

Proposition 2.14. *We suppose* $12 \mid f$*. If there exists a pseudo-symmetric numerical semigroup* $H = \langle a, b, c\rangle$ *with* $\mathrm{F}(H) = f$*, then* $f/2+1$ *has a prime factor of the form* $3k + 2$ $(k \geq 1)$*.*

Proof. Otherwise, since α', β', γ' are divisors of $f/2 + 1$, we get $\alpha' \equiv \beta' \equiv \gamma' \equiv 1 \pmod 3$. Then by (2.3), we see that a, b, c are divisible by 3 and $H = \langle a, b, c\rangle$ is not a numerical semigroup. □

Example 2.15. Let f be an integer divisible by 12.

(1) By Proposition 2.14, there is no pseudo-symmetric semigroup $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = 12, 24, 36, 60, 72, 84, 96, 120, 132, 144, 156, 180, 192$.

(2) On the other hand, there exists pseudo-symmetric semigroups $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = 48, 108, 168$. Actually, $H = \langle 7, 11, 31\rangle$ is the unique pseudo-symmetric semigroup generated by 3 elements, with $\mathrm{F}(H) = 48$, and $H = \langle 11, 19, 103\rangle$ is the unique pseudo-symmetric semigroup generated by 3 elements with $\mathrm{F}(H) = 168$. Furthermore, both $\langle 11, 13, 67\rangle$ and $\langle 7, 23, 61\rangle$ are pseudo-symmetric numerical semigroups generated by 3 elements with $\mathrm{F}(H) = 108$.

(3) The converse of Proposition 2.14 is not true. Indeed, If $f = 1596$, then $f/2 + 1 = 799 = 17 \times 47$ has a prime factor which is congruent to 2 mod 3. But if we substitute $(\alpha', \beta', \gamma') = (17, 47, 1)$ (resp. $(47, 17, 1)$) in (4.1.1), then we get $(a, b, c) = (95, 19, 817)$ (resp. $(35, 49, 847)$). These are not numerical semigroups since (a, b, c) have common prime factor. It is not difficult to show that $f = 1596$ is the smallest of such examples.

2.3 Simple numerical semigroups

In this section, we give a characterization of 3-generated simple numerical semigroups. Let us recall the definition of simple numerical semigroups.

Let $H = \langle a_1, ..., a_n\rangle$ be a numerical semigroup. We assume that a_1 is the least positive integer in H. For every $i \in \{1, ..., n\}$, set

$$\delta_i := \min\{k \in \mathbb{N} \setminus \{0\} \mid ka_i \in \langle \{a_1, ..., a_n\} \setminus \{a_i\}\rangle\}.$$

The notion of simple numerical semigroup was defined in Exercise 10.3 of [RG4].

Definition 2.16 (Simple numerical semigroups)**.** We say that H is *simple* if $a_1 = (\delta_2 - 1) + (\delta_3 - 1) + \cdots + (\delta_n - 1) + 1$.

Proposition 2.17. *Let* $H = \langle a_1, a_2, ..., a_n\rangle$ *be a simple numerical semigroup. Then the type of* H *is* $n - 1$*. Hence if* H *is simple with* $n \geq 3$*, then* H *is not symmetric.*

Proof. By definition of pseudo-Frobenius number, we have that

$$\mathrm{PF}(H) = \{(\delta_2 - 1)a_2 - a_1, (\delta_3 - 1)a_3 - a_1, ..., (\delta_n - 1)a_n - a_1\},$$

that is, H has type $n - 1$. □

The following is the main result in this section.

Theorem 2.18. *Let $H = \langle a, b, c \rangle$ be a numerical semigroup defined by the matrix as in (2.1). If we assume that a is the least positive integer in H. H, then H is simple if and only if $\beta' = \gamma = 1$.*

Proof. Since $a = \beta\gamma + \beta'\gamma + \beta'\gamma'$, and since we have $\delta_2 = \beta + \beta', \delta_3 = \gamma + \gamma$, H is simple if and only if

$$\beta\gamma + \beta'\gamma + \beta'\gamma' = \beta + \beta' + \gamma + \gamma' - 1$$

or, equivalently,

$$(\beta - 1)(\gamma - 1) + (\beta' - 1)(\gamma' - 1) + (\beta'\gamma - 1) = 0.$$

Since $\beta, \beta', \gamma, \gamma'$ are positive integers, the latter equation is equivalent to $\beta' = \gamma = 1$. □

Chapter 3

Almost symmetric numerical semigroups generated by four elements

Next, we study 4-generated almost symmetric numerical semigroups. As is mentioned in Chapter 2, if $H = \langle a, b, c \rangle$ is not symmetric, then $\mathrm{t}(H) = 2$, and hence it is almost symmetric if and only if it is pseudo-symmetric (Proposition 2.3). Hence the first interesting case is the 4-generated case. In particular, we are interested in the upper bound of the type of 4-generated almost symmetric numerical semigroups.

Conjecture 3.1. *If H is a 4-generated almost symmetric numerical semigroup, then $\mathrm{t}(H) \leq 3$.*

In [NNW3], the authors classified 4-generated almost symmetric numerical semigroups of multiplicity 5. In particular, we proved that the type of such numerical semigroups is at most 3. In general, we know that there is no upper bound on type of $H = \langle a_1, ..., a_n \rangle$ for $n \geq 4$ (see [FGH]).

A numerical semigroup is *irreducible* if it cannot be expressed as an intersection of two numerical semigroups properly containing it. It is known that an irreducible numerical semigroup is either symmetric or pseudo-symmetric ([RG4, Chapter 3]). J. C. Rosales and P. A. García-Sánchez proved that every almost symmetric numerical semigroup can be constructed by removing some minimal generators from an irreducible numerical semigroup with the same Frobenius number.

Theorem 3.2 (Rosales, García-Sánchez [RG5])**.** *Let H_1 be a numerical semigroup. Then H_1 is almost symmetric if and only if there exists an irreducible numerical semigroup H with $\mathrm{F}(H) = \mathrm{F}(H_1)$ such that $H_1 = H \setminus \mathcal{A}$, where $\mathcal{A}$ is a set of minimal generators of H_1 such that*

$$\mathcal{A} \subset [\mathrm{F}(H)/2, \mathrm{F}(H)] \text{ and } x + y - \mathrm{F}(H) \notin H_1 \text{ for any } x, y \in \mathcal{A}. \qquad (*)$$

When this is the case, $\mathrm{t}(H_1) = 2 \cdot \#(\mathcal{A}) + \mathrm{t}(H)$.

In this chapter, we explicitly construct 4-generated almost symmetric numerical semigroups from 2 or 3-generated irreducible numerical semigroups by using Theorem 3.2. Then we see that Conjecture 3.1 holds true for those almost symmetric numerical semigroups.

3.1 The case where H is 2-generated

We remark that all 2-generated numerical semigroups are symmetric. Let $H = \langle a, b\rangle$ be a numerical semigroup and $H_1 = H \setminus \{b\}$. Then since $H_1 = \langle a, a+b, 2b, 3b\rangle$, we see that $\mathrm{emb}(H_1) \leq 4$ and $a, a+b$ are always minimal generators of H_1.

Lemma 3.3. *Let $H = \langle a, b\rangle$ be a numerical semigroup and $H_1 = H \setminus \{b\}$. Then $\mathrm{emb}(H_1) = 4$ if and only if $a \geq 4$. In this case, $H_1 = \langle a, a+b, 2b, 3b\rangle$.*

Proof. It is easily seen that if $\mathrm{emb}(H_1) = 4$, then $H_1 = \langle a, a+b, 2b, 3b\rangle$. If $H_1 = \langle a, a+b, 2b, 3b\rangle$, then $2b, 3b \notin \langle a\rangle$ and hence $a \geq 4$. Conversely, if $a \geq 4$, then $2b, 3b \notin \langle a\rangle$. This implies $2b \notin \langle a, a+b, 3b\rangle$ and $3b \notin \langle a, a+b, 2b\rangle$. Hence $H_1 = \langle a, a+b, 2b, 3b\rangle$ and $\mathrm{emb}(H_1) = 4$. □

Proposition 3.4. *Let $H = \langle a, b\rangle$ be a numerical semigroup and $\mathcal{A} \subset \{a, b\}$ with $\#A = 1$. If $\mathrm{emb}(H \setminus \mathcal{A}) = 4$ and the set $\mathcal{A}$ satisfies Condition $(*)$, then $H = \langle 2, 5\rangle$ and $\mathcal{A} = \{2\}$, or $H = \langle 3, 4\rangle$ and $\mathcal{A} = \{3\}$.*

Proof. We may assume that $\mathcal{A} = \{b\}$. Since $\mathrm{F}(H) = ab - a - b$, we have $(a-3)b < a$ from $\mathrm{F}(H)/2 < b$. Since $a \geq 4$ by Lemma 3.3, we get $a = 5$ and $b = 2$, or $a = 4$ and $b = 3$. Hence $H = \langle 2, 5\rangle$ or $H = \langle 3, 4\rangle$. Then $\mathcal{A} = \{b\}$ satisfies Condition $(*)$, respectively. □

Next, we consider the case of removing 2-generators from $H = \langle a, b\rangle$.

Proposition 3.5. *Let $H = \langle a, b\rangle$ be a numerical semigroup and $\mathcal{A} = \{a, b\}$. If the set $\mathcal{A}$ satisfies Condition $(*)$, then $H = \langle 3, 4\rangle$, and $H \setminus \mathcal{A} = \langle 6, 7, 8, 9, 10\rangle$.*

Proof. We may assume that $2 \leq a < b$. Since $\mathrm{F}(H) = ab - a - b$, Condition $(*)$ implies that

$$\frac{ab - a - b}{2} < a < b < ab - a - b.$$

Then we have that $ab - 3a - b < 0$ and $ab - a - 2b > 0$. Hence $2(b-3)a < 2b < (b-1)a$. This yields that $b \leq 4$ and thus $(a, b) = (3, 4)$. □

Theorem 3.6. *4-generated almost symmetric numerical semigroups that are constructed from $H = \langle a, b\rangle$ under Condition $(*)$ are*

$$\langle 4, 5, 6, 7\rangle \ \text{ and } \ \langle 4, 6, 7, 9\rangle\,.$$

Proof. By Proposition 3.4 and Proposition 3.5, the pairs of $H = \langle a, b\rangle$ and $\mathcal{A} \subset \{a, b\}$ that satisfy Condition $(*)$ are $H = \langle 2, 5\rangle$ and $\mathcal{A} = \{2\}$, or $H = \langle 3, 4\rangle$ and $\mathcal{A} = \{3\}$. Then $H \setminus \mathcal{A} = \langle 4, 5, 6, 7\rangle$ or $\langle 4, 6, 7, 9\rangle$, which are almost symmetric with $\mathrm{t}(H) = 3$ by Theorem 3.2. □

By Theorem 3.6, we conclude that we cannot construct 4-generated almost symmetric numerical semigroups whose type are bigger than 3 from $H = \langle a, b \rangle$.

3.2 The case where H is 3-generated

Next, we consider the case of $H = \langle a, b, c \rangle$. Then the embedding dimension of $H_1 = H \setminus \{b\}$ is at most 6, that is $H_1 = \langle a, c, a+b, b+c, 2b, 3b \rangle$.

Lemma 3.7. *Let $H = \langle a, b, c \rangle$ be a numerical semigroup and $H_1 = H \setminus \{b\}$. If $\mathrm{emb}(H_1) = 4$, then one of the following conditions holds:*

(*a*) $2b \in \langle a, c \rangle$, $a+b \notin \langle c \rangle$, $b+c \notin \langle a \rangle$ *and* $H_1 = \langle a, c, a+b, b+c \rangle$.

(*b*) $2b \notin \langle a, c \rangle$, $3b \in \langle a, c \rangle$, $a+b \in \langle c \rangle$, $b+c \notin \langle a \rangle$ *and* $H_1 = \langle a, c, b+c, 2b \rangle$.

(*c*) $2b \notin \langle a, c \rangle$, $3b \in \langle a, c \rangle$, $a+b \notin \langle c \rangle$, $b+c \in \langle a \rangle$ *and* $H_1 = \langle a, c, a+b, 2b \rangle$ (*If we change a and c in* (*b*)*, then we get case* (*c*)).

Proof. It is easy to check that H_1 is either one of the above forms since a and c are always minimal generators of H_1. $\square$

First, we consider the symmetric case. We recall the characterization of 3-generated symmetric numerical semigroups in Theorem 2.1. Then we can classify symmetric numerical semigroups H with $\mathrm{emb}(H_1) = 4$.

Theorem 3.8. *Let $H = \langle a, b, c \rangle$ be a symmetric numerical semigroup and $H_1 = H \setminus \{b\}$. If $\mathrm{emb}(H_1) = 4$, then, changing a and c if necessary, H is one of the following form:*

(1) $\langle d \langle 2, b' \rangle, c \rangle$, *where* $a = 2d, b = b'd$ *and* $c \neq 2 + b'$.

(2) $\langle 2 \langle a', c' \rangle, b \rangle$, *where* $a = 2a'$ *and* $c = 2c'$.

(3) $\langle d \langle 3, b' \rangle, c \rangle$, *where* $a = 3d, b = b'd$ *and* $c = 3 + b'$.

Proof. By Lemma 3.7, H must satisfy one of condtions (*a*), (*b*) or (*c*).

First, we assume that H satisfies the condition (*a*). Then one of the following two cases occurs:

(i) If $2b \in \langle a \rangle$, then we see that $H = \langle d \langle 2, b' \rangle, c \rangle$, where $a = 2d, b = b'd$. In this situation, it holds that $b + c \notin \langle a \rangle$ since otherwise, a, b and c are dividisible by d. Since $c \in \langle 2, b' \rangle$, it follows that $a + b \notin \langle c \rangle$ if and only if $c \neq 2 + b'$. Hence we see that H is as in (1).

(ii) If $2b \in \langle a, c \rangle$ and $2b \notin \langle a \rangle, \langle c \rangle$, then $H = \langle 2 \langle a', c' \rangle, b \rangle$, where $a = 2a', c = 2c'$. Then $a + b \notin \langle c \rangle$ and $b + c \notin \langle a \rangle$. Hence in this case, we see that H is as in (2).

Next, we assume that H satisfies the condition (b) (or (c) if changing a and c). Then we should also consider following two cases:

(i) If $3b \in \langle a \rangle$, we know that $H = \langle d\langle 3, b'\rangle, c\rangle$, where $a = 3d, b = b'd$. In this case, $b + c \notin \langle a \rangle$ since otherwise, a, b, c are divisible by d. It is also easily seen that $a + b \in \langle c \rangle$ if and only if $c = 3 + b'$. Hence H is as in (3).

(ii) If $3b \in \langle a, c\rangle$ and $3b \notin \langle a \rangle, \langle c \rangle$, we guess that $H = \langle 3\langle a', c'\rangle, b\rangle$, where $a = 3a', c = 3c'$. But in this case, $a + b \notin \langle c \rangle$ and $b + c \notin \langle a \rangle$ since otherwise, a, b, c are divisible by 3. So we see that $\mathrm{emb}(H_1) > 4$, which is a contradiction.

Thus, we conclude that H is the one of forms of (1), (2) or (3) if $\mathrm{emb}(H_1) = 4$. $\square$

In each case of Theorem 3.8, if $\mathrm{F}(H)/2 < b$, then $H = \langle a, b, c\rangle$ and $\mathcal{A} = \{b\}$ satisfy Condition $(*)$. In this case, $\mathrm{t}(H \setminus \mathcal{A}) = 3$ by Theorem 3.2.

Next, we consider the case of removing 2 or 3 generators from $H = \langle a, b, c\rangle$.

Lemma 3.9. *Let $H = \langle a, b, c\rangle$ be a symmetric numerical semigroup. Assume that $H = \langle d\langle a', b'\rangle, c\rangle$, where $a = a'd$ and $b = b'd$. Then*

(1) *if $\mathrm{F}(H)/2 < b$, then $a + b > c(d-1)$ if $a' = 2$, or $a > c(d-1)$ if $a' \geq 3$.*

(2) *if $\mathrm{F}(H)/2 < c$, then $d = 2$.*

Proof. (1) Since $\mathrm{F}(H) = d(a'b' - a' - b') + c(d-1) = a'b - a - b + c(d-1) < 2b$, we get $a - c(d-1) > (a'-3)b$. Hence the assertion follows from this inequality.

(2) From $\mathrm{F}(H) < 2c$, we have $d\,\mathrm{F}(H') < (3-d)c$, where $H' = \langle a', b'\rangle$. So we get $d = 2$. $\square$

Lemma 3.10. *Let $H = \langle a, b, c\rangle$ be a symmetric numerical semigroup. If $\mathrm{F}(H)/2 < x$ and $\mathrm{F}(H)/2 < y$ for some $x, y \in \{a, b, c\}, x \neq y$, then $H = \langle 4, b, c\rangle = \langle 2\langle 2, b'\rangle, c\rangle$, where $b - 4 < c$.*

Proof. We may assume that $H = \langle d\langle a', b'\rangle, c\rangle$, where $a = a'd, b = b'd$ and $a' < b'$. First, we assume that $\mathrm{F}(H)/2 < a, b$.

(i) If $a' \geq 3$ and $b' \geq 3$, then $c < a/(d-1) =$ and $c < b/(d-1)$ by Lemma 3.9. Since $a = a'd, b = b'd$ and $d > 1$, we get the following inequalities;

$$c < \frac{a}{d-1} = \frac{d}{d-1}a' \leq 2a',\ c < \frac{b}{d-1} = \frac{d}{d-1}b' \leq 2b'. \tag{3.1}$$

Then we see that $c = a' + b'$ since $c \in \langle a', b'\rangle$. But it is a contradiction since we get $b' < a'$ and $a' < b'$ by (3.1).

(ii) If $a' = 2$, then $\mathrm{F}(H) = d(b'-2) - c + dc$. Since $\mathrm{F}(H)/2 < a$, we get

$$c < \frac{3a - b}{d-1} = \frac{d}{d-1} \cdot (6 - b') \leq 2(6 - b'). \tag{3.2}$$

So we have $b' = 3$ since $c > 3$, and hence $c = 4$ or 5.

If $c = 4$, then $d = 2$ or 3 by (3.2). When $d = 2$, it is a contradiction since $c = 4$. If $d = 3$, then we see that $H = \langle 3\langle 2,3\rangle, 4\rangle = \langle 6,9,4\rangle$. Note that we can also write as $H = \langle 4,6,9\rangle = \langle 2\langle 2,3\rangle, 9\rangle$.

If $c = 5$, then we get $d = 2$ by (3.2). Hence we see that $H = \langle 2\langle 2,3\rangle, 5\rangle = \langle 4,6,5\rangle$.

Next we assume that $\mathrm{F}(H)/2 < b$ and $\mathrm{F}(H)/2 < c$. Then we may also assume that $b' \geq 3$ and we have $d = 2$ by Lemma 3.9. If $a' > 2$, then we get $c < 6$ since $\mathrm{F}(H) = 2(a'b' - a' - b') + c \geq 2(3b' - 3 - b') + c$. This is a contradiction since $c \in \langle a', b'\rangle$ and $a', b' > 2$. Hence we have $a' = 2$. Then $b - 4 < c$ since $\mathrm{F}(H) = 2(b' - 2) + c < 2c$. So in this case, we see that $H = \langle 2\langle 2, b'\rangle, c\rangle$, where $b - 4 < c$.

Hence, we conclude that $H = \langle 2\langle 2, b'\rangle, c\rangle$, where $b - 4 < c$ if $\mathrm{F}(H)/2 < x$ and $\mathrm{F}(H)/2 < y$ for some $x, y \in \{a, b, c\}, x \neq y$. □

Lemma 3.11. *Let $H = \langle a,b,c\rangle$ be a symmetric numerical semigroup. If $\mathrm{F}(H)/2 < z$ for any $z \in \{a,b,c\}$, then $H = \langle 4,5,6\rangle$.*

Proof. We may assume that $H = \langle d\langle a', b'\rangle, c\rangle$, where $a = a'd$, $b = b'd$ and $a' < b'$. Then we have $d = 2$ by Lemma 3.9.

(i) If $a' = 2$ and $b' \geq 3$, then it follows that $b = 6$ and $c = 5$ since $\mathrm{F}(H) = b - 4 + c < 2a = 8$. Hence $H = \langle 4,6,5\rangle$.

(ii) If $a', b' \geq 3$, then $c < a$ and $c < b$ by Lemma 3.9. Since $c \in \langle a', b'\rangle$, we write $c = \lambda_1 a' + \lambda_2 b'$, where $\lambda_1, \lambda_2 \geq 0$. So we have that $\lambda_2 b' < (2 - \lambda_1)a'$ and $\lambda_1 a' < (2 - \lambda_2)b'$. Then it must be $\lambda_1 = \lambda_2 = 1$ since $c \neq a', b'$. Hence we get $b' < a'$ and $a' < b'$, which is a contradiction.

Hence we conclude that $H = \langle 4,5,6\rangle$. □

Theorem 3.12. *Let $H = \langle a,b,c\rangle$ be a symmetric numerical semigroup and $\mathcal{A} \subset \{a,b,c\}$ with $\#\mathcal{A} \geq 2$. If $H \neq \langle 4,5,6\rangle$, then the set $\mathcal{A}$ never satisfies Condition $(*)$.*

Proof. If $\#\mathcal{A} = 3$, then it follows by Lemma 3.11.

If $\#\mathcal{A} = 2$, then $H = \langle 4,b,c\rangle = \langle 2\langle 2,b'\rangle, c\rangle$ by Lemma 3.10. Then we see that $\mathrm{F}(H)/2 < b, c$ by the proof of Lemma 3.11. Since $\mathrm{F}(H) = b + c - 4$, it follows that $b + c - \mathrm{F}(H) = 4 \in H \setminus \{b,c\}$, which implies that $\mathcal{A} = \{b,c\}$ does not satisfy Condition $(*)$. □

Theorem 3.12 implies that $\langle 4,5,6\rangle$ is the only symmetric numerical semigroup with embedding dimension 3 which constructs almost symmetric numerical semigroups by removing 2 or 3 elements under Condition $(*)$. Thus, the following example (1) shows that if $H = \langle a,b,c\rangle$ is symmetric and $\mathrm{emb}(H \setminus \mathcal{A}) = 4$, then $\mathrm{t}(H \setminus \mathcal{A}) = 3$.

Example 3.13.

(1) Let $H = \langle 4, 5, 6 \rangle$. Then all almost symmetric numerical semigroups constructed from H are $H \setminus \{5\} = \langle 4, 6, 9, 11 \rangle$, $H \setminus \{4\} = \langle 5, 6, 8, 9 \rangle$, $H \setminus \{4, 5\} = \langle 6, 8, 9, 10, 11, 13 \rangle$ and $H \setminus \{4, 5, 6\} = \langle 8, 9, 10, 11, 12, 13, 14, 15 \rangle$.

(2) Let $H = \langle 5 \langle 2, 5 \rangle, 8 \rangle$. Then $H \setminus \{25\} = \langle 8, 10, 33, 35 \rangle$ is almost symmetric and $\mathrm{PF}(H \setminus \{25\}) = \{22, 25, 47\}$.

(3) Let $H = \langle 6, 8, 11 \rangle = \langle 2 \langle 3, 4 \rangle, 11 \rangle$. Then $H \setminus \{11\} = \langle 6, 8, 17, 19 \rangle$ is almost symmetric with $\mathrm{PF}(H \setminus \{11\}) = \{10, 11, 21\}$.

Now, we consider the case where $H = \langle a, b, c \rangle$ is pseudo-symmetric. Let $H_1 = H \setminus \{b\}$. Then we prove that if $\mathrm{emb}(H_1) = 4$, then H_1 has maximal embedding dimension. This means that $\mathcal{A} = \{b\}$ never satisfies Condition $(*)$. We recall the characterization of pseudo-symmetric numerical semigroup (see Corollary 2.9 in Chapter 2).

Lemma 3.14. *Let $H = \langle a, b, c \rangle$ be a numerical semigroup. If H is pseudo-symmetric, the followings hold:*

(1) *If $2b \in \langle a, c \rangle$, then $b + c \in \langle a \rangle$ or $a + b \in \langle c \rangle$.*

(2) *If $a + b \in \langle c \rangle$, then $2a \in \langle b, c \rangle$ or $2b \in \langle a, c \rangle$.*

(3) *If $b + c \in \langle a \rangle$, then $2c \in \langle a, b \rangle$ or $2b \in \langle a, c \rangle$.*

Proof. Since H is pseudo-symmetric, $\alpha = \beta = \gamma = 1$ or $\alpha' = \beta' = \gamma' = 1$ in the matrix of (2.1) by Corollary 2.9.

(1) First, assume that $\alpha = \beta = \gamma = 1$. Since $2b \in \langle a, c \rangle$, it must be $\beta' = 1$ and hence $b + c \in \langle a \rangle$. Next, we assume that $\alpha' = \beta' = \gamma' = 1$. Then we have $\beta = 1$ since $2b \in \langle a, c \rangle$. This implies that $a + b \in \langle c \rangle$.

(2) If $\alpha = \beta = \gamma = 1$, then $\alpha' = 1$ since $a + b \in \langle c \rangle$, which implies $2a \in \langle b, c \rangle$. If $\alpha' = \beta' = \gamma' = 1$, then we have $\beta = 1$ since $a + b \in \langle c \rangle$, and hence $2b \in \langle a, c \rangle$.

(3) Changing a and c in (2), we get the assertion. □

Remark 3.15. In Lemma 3.14, we need to assume pseudo-symmetric for H. For example, let $H = \langle 5, 8, 6 \rangle, H' = \langle 5, 7, 6 \rangle$, both of which are not pseudo-symmetric and do not satisfy (1), (2), respectively.

Theorem 3.16. *Let $H = \langle a, b, c \rangle$ be a pseudo-symmetric numerical semigroup and $A \subset \{a, b, c\}$ with $\#\mathcal{A} = 1$. If $\mathrm{emb}(H \setminus \mathcal{A}) = 4$, then $H \setminus \mathcal{A}$ has maximal embedding dimension. In particular, the set $\mathcal{A}$ never satisfies Condition $(*)$.*

Proof. We may assume that $\mathcal{A} = \{b\}$. We use the classification of Lemma 3.7.

First, assume that $H_1 = H \setminus \mathcal{A}$ is as in the case of (a) in Lemma 3.7. Then $b + c \in \langle a \rangle$ or $a + b \in \langle c \rangle$ by Lemma 3.14 since $2b \in \langle a, c \rangle$. Hence $\mathrm{emb}(H_1) < 4$, which is a contradiction.

Next, we consider the case where H_1 is as in the case (b).

(i) If $\alpha = \beta = \gamma = 1$, we have $\alpha' = 1$ since $a + b \in \langle c \rangle$, and $\beta' = 2$ since $2b \notin \langle a, c \rangle$ and $3b \in \langle a, c \rangle$. Then it necessary holds that $2a \in \langle b, c \rangle$ by Lemma 3.14. By (2.2), we get $a = 2\gamma' + 3$, $b = 2\gamma' + 1$ and $c = 4$. Hence $H_1 = \langle 4, 2\gamma' + 3, 2\gamma' + 5, 4\gamma' + 2, \rangle$ has maximal embedding dimension.

(ii) If $\alpha' = \beta' = \gamma' = 1$, then $\beta = 2$ since $2b \notin \langle a, c \rangle$ and $3b \in \langle a, c \rangle$. Then $a + b \notin \langle c \rangle$, which contradicts the condition of (b).

Lastly, we consider H_1 is as in the case (c).

(i) If $\alpha = \beta = \gamma = 1$, then $\beta' = 2$ from the condition of (c). In this case, $b + c \notin \langle a \rangle$, which contradicts the condition of (c).

(ii) If $\alpha' = \beta' = \gamma' = 1$, we have $\beta = 2$ and $\gamma = 1$ by the condition of (c) and Lemma 3.14. Then we see that $a = 4$, $b = 2\alpha + 1$ and $c = 2\alpha + 3$ from (2.2). Hence $H_1 = \langle 4, 2\alpha + 3, 2\alpha + 5, 4\alpha + 2 \rangle$ has maximal embedding dimension.

In conclusion, H_1 has maximal embedding dimension if $\mathrm{emb}(H_1) = 4$. Then $\mathrm{t}(H_1) = 3$ and hence $\mathcal{A}$ does not satisfy Condition $(*)$. □

Next, we consider the case of removing 2 or 3 elements.

Lemma 3.17. *Let $H = \langle a, b, c \rangle$ be a pseudo-symmetric numerical semigroup and $\mathcal{A} \subset \{a, b, c\}$ with $\#\mathcal{A} = 2$. If $\mathrm{F}(H)/2 < x$ and $\mathrm{F}(H)/2 < y$ for $x, y \in \mathcal{A}, x \neq y$, then H is either*

$$\left\langle 3, \frac{f}{2} + 3, f + 3 \right\rangle, \ \textit{or} \ \left\langle 4, \frac{f}{2} + 2, \frac{f}{2} + 4 \right\rangle,$$

where $\mathrm{F}(H) = f$.

Proof. By Corollary 2.9, we may assume that $\alpha' = \beta' = \gamma' = 1$. Then $\mathrm{F}(H) = 2\alpha\beta\gamma - 2$. We may also assume that $\mathrm{F}(H)/2 < a$ and $\mathrm{F}(H)/2 < b$ by considering of changing order of a, b, c. Then we get the following inequalities by (2.2).

$$\begin{aligned} \alpha\beta\gamma &< \beta\gamma + \gamma + 2, \\ \alpha\beta\gamma &< \gamma\alpha + \alpha + 2. \end{aligned} \tag{3.3}$$

The pairs (α, β, γ) which satisfy (3.3) are, changing order of α, β, γ if necessary, $(1, 1, \gamma)$ and $(2, 1, \gamma)$, where γ is any positive integer. Then we see that H is the above form by (2.2). □

Lemma 3.18. *Let $H = \langle a, b, c \rangle$ be a pseudo-symmetric numerical semigroup and $\mathcal{A} = \{a, b, c\}$. If $\mathrm{F}(H)/2 < z$ for any $z \in \mathcal{A}$, then $H = \langle 3, 4, 5 \rangle$, $\langle 3, 5, 7 \rangle$, or $\langle 4, 5, 7 \rangle$.*

Proof. We may assume that $\alpha' = \beta' = \gamma' = 1$. By the proof of Lemma 3.17, we may also assume that $(\alpha, \beta) = (1, 1)$ or $(2, 1)$. In both cases, we have $\gamma < 3$ from $\mathrm{F}(H)/2 < c$ and (2.2). Then the pairs (α, β, γ) which H is to be a numerical semigroup are $(1, 1, 2)$, $(1, 1, 3)$ and $(1, 2, 2)$. By (2.2), we see that $H = \langle 3, 4, 5 \rangle$, $\langle 3, 5, 7 \rangle$ or $\langle 4, 5, 7 \rangle$. □

Note that $\langle 3,4,5\rangle$, $\langle 3,5,7\rangle$ and $\langle 4,5,7\rangle$ do not satisfy Condition $(*)$ since a minimal generator of those numerical semigroups is more than $\mathrm{F}(H)$. Hence we have the following proposition.

Proposition 3.19. *Let $H = \langle a,b,c\rangle$ be a pseudo-symmetric numerical semigroup and $\mathcal{A} = \{a,b,c\}$. Then the set $\mathcal{A}$ never satisfies Condition $(*)$.*

Now we can prove our main theorem.

Theorem 3.20. *Let $H = \langle a,b,c\rangle$ be a pseudo-symmetric numerical semigroup and $\mathcal{A} \subset \{a,b,c\}$ with $\#\mathcal{A} \geq 2$. If $H \neq \langle 4,5,7\rangle$, then the set $\mathcal{A}$ never satisfies Condition $(*)$.*

Proof. If $\#\mathcal{A} = 3$, then the assertion follows from Proposition 3.19.

If $\#\mathcal{A} = 2$, then $H = \langle 3, f/2+3, f+3\rangle$ or $\langle 4, f/2+2, f/2+4\rangle$ by Lemma 3.17. First, assume that $H = \langle 3, f/2+3, f+3\rangle$. We may assume that $f \geq 8$ and $\mathcal{A} = \{f/2+3, f+3\}$. But, then $f+3 > f = \mathrm{F}(H)$, which implies the set $\mathcal{A}$ does not satisfy Condition $(*)$.

Next we assume that $H = \langle 4, f/2+2, f/2+4\rangle$. Since $H \neq \langle 4,5,7\rangle$, we may also assume that $f \geq 10$. Then we guess that $\mathcal{A} = \{f/2+2, f/2+4\}$. But, $2b - \mathrm{F}(H) = 4\alpha + 2 - 4\alpha + 2 = 4 \in H \setminus A$, which implies that $\mathcal{A}$ does not satisfy Condition $(*)$. □

By Theorem 3.20, we conclude that if $H = \langle a,b,c\rangle$ is pseudo-symmetric and if the set $\mathcal{A}$ satisfies Condition $(*)$, then $H = \langle 4,5,7\rangle$ and $\mathcal{A} = \{4,5\}$. Since $H \setminus \mathcal{A} = \langle 7,8,9,10,11,12,13\rangle$ in this case, we cannot construct 4-generated almost symmetric numerical semigroups from 3-generated pseudo-symmetric semigroups. Also, we conclude that all 4-generated almost symmetric semigroups obtained from 3-generated irreducible semigroups by the method of J. C. Rosales and P. A. García-Sánchez [RG5] have type ≤ 3.

3.3 The defining ideals of 4-generated almost symmetric numerical semigroups

Finally, we mention the defining ideals of 4-generated almost symmetric numerical semigroups. Let H be a 4-generated numerical semigroup. When H is symmetric, H. Bresinsky [Br] completely determined the defining ideal I_H of $k[H]$. In particular, he proved that $\mu(I_H) = 3$ or 5, where $\mu(I_H)$ is the number of minimal generators of I_H. When H is pseudo-symmetric, J. Komeda [Ko] gave a complete characterization of the defining ideal I_H, and he proved that $\mu(I_H) = 5$. Therefore, it is natural to ask if $\mu(I_H)$ is bounded in the case where H is almost symmetric. We expect that the following conjecture is true about this.

Conjecture 3.21. *If H is 4-generated almost symmetric, then $\mu(I_H) \leq 7$.*

Remark 3.22. The authors in [NNW3] proved that if H is 4-generated almost symmetric with multiplicity 5, then $\mu(I_H) \leq 6$.

Chapter 4

Numerical semigroups generated by generalized arithmetic sequences

In this chapter, we study numerical semigroups which are in the form of the following:

Definition 4.1. We say that H is a numerical semigroup generated by a *generalized arithmetic sequence* if $H = \langle a, sa+d, sa+2d, ..., sa+nd \rangle$, where $a, s, d > 0$, $n \geq 2$ and $\gcd(a, d) = 1$. When $s = 1$, H is said to be generated by an *arithmetic sequence*.

Our main aim is to give a characterization for H to be almost symmetric. Apéry sets play a central roll in this chapter and next chapter.

Let $H = \langle a, sa+d, sa+2d, ..., sa+nd \rangle$ be a numerical semigroup generated by a generalized arithmetic sequence. Put $a = qn + r$, $0 \leq r < n$. We define the subset A_i of H for $1 \leq i \leq q$ as the following:

$$A_i := \{isa + ld \mid (i-1)n + 1 \leq l \leq in\}.$$

Then we determine the Apéry set of a in H and the type of H.

Theorem 4.2 (Matthews [Ma]). *Let H be as above.*

(1) If $r = 0$, then

$$\mathrm{Ap}(H, a) = \{0\} \cup A_1 \cup A_2 \cup \cdots \cup A_{q-1} \cup (A_q \setminus \{qsa + qnd\}).$$

Then $\mathrm{PF}(H) = \{\omega - a \mid \omega \in A_q \setminus \{qsa + qnd\}\}$ *and* $\mathrm{t}(H) = n - 1$.

(2) If $r = 1$, then

$$\mathrm{Ap}(H, a) = \{0\} \cup A_1 \cup \cdots \cup A_q.$$

Then $\mathrm{PF}(H) = \{\omega - a \mid \omega \in A_q\}$ *and* $\mathrm{t}(H) = n$.

(3) Otherwise ($r \neq 0, 1$),

$$\mathrm{Ap}(H, a) = \{0\} \cup A_1 \cup \cdots \cup A_q \cup \{(qs+1)a + ld \mid qn + 1 \leq l \leq qn + r - 1\}.$$

Then $\mathrm{PF}(H) = \{(qs+1)a + ld - a \mid qn + 1 \leq l \leq qn + r - 1\}$ *and* $\mathrm{t}(H) = r - 1$.

Proof. This theorem is shown by G. L. Matthews (see [Ma, Proof of Lemma 2.7]). But we reproduce the proof for the convenience of readers. We know that

$$\mathrm{Ap}(H,a) = \{0 = w(0) < w(d) < w(2d) < \cdots < w((a-1)d)\},$$

where $w(jd) \equiv jd \pmod{a}$ for all $1 \le j \le a-1$ since $\gcd(a,d) = 1$. Furthermore, $w(jd) = (p+1)sa + jd$ if $j = pn + u$, where $0 < u \le n$. Therefore, the theorem easily follows from the definition of Apéry sets and Proposition 1.8. □

The following result was firstly shown by M. Estrada and A. López [EL] (when $s = 1$, it was shown by L. Juan [Ju]). We obtain this result as a corollary of Theorem 4.2 since H is symmetric if and only if $\mathrm{t}(H) = 1$ by Proposition 1.11.

Corollary 4.3 (Estrada, López [EL], cf. [Ma])**.** *Let $H = \langle a, sa+d, ..., sa+nd\rangle$ be a numerical semigroup generated by a generalized arithmetic sequence. Then H is symmetric if and only if $a \equiv 2 \pmod{n}$.*

G. L. Matthews gave a characterization for H to be pseudo-symmetric (see [Ma]). We generalize this result for almost symmetric numerical semigroups.

Corollary 4.4. *Let $H = \langle a, sa+d, ..., sa+nd\rangle$ be a numerical semigroup generated by a generalized arithmetic sequence. Then H is almost symmetric but not symmetric if and only if H has maximal embedding dimension and $s = 1$. In particular, H is pseudo-symmetric if and only if $H = \langle 3, 3+d, 3+2d\rangle$.*

Proof. If H has maximal embedding dimension and $s = 1$, then $\mathrm{PF}(H) = \{d, 2d, ..., nd\}$ by Proposition 1.8, and hence H is almost symmetric by Theorem 1.15.

Conversely, assume that H is almost symmetric with $\mathrm{PF}(H) = \{f_1 < f_2 < \cdots f_t = \mathrm{F}(H)\}$, where $t \ge 2$. Then it holds that $f_i + f_{t-i} = \mathrm{F}(H)$ for all $1 \le i \le t-1$ by Theorem 1.15. By Theorem 4.2, we see that this condition holds if and only if $q = 1$, $r = 1$ and $s = 1$. Hence H has maximal embedding dimension. The last statement of Corollary easily follows from Corollary 1.13 □

Remark 4.5. When H is a numerical semigroup generated by an arithmetic sequence, the explicit formula of the Betti numbers of $k[H]$ is given by P. Gimenez, I. Sengupta and H. Srinivasan [GSS]. In [EL] and [Ma], the definition of generalized arithmetic sequences does not except the case of $n = 1$. Therefore, Corollary 4.3 is slightly different from that in [EL] or [Ma].

Example 4.6.

(1) Let $H = \langle 11, 25, 28, 31, 34\rangle$. Then

$$\mathrm{Ap}(H, 15) = \{0, \quad 25, 28, 31, 34, \quad 59, 62, 65, 68, \quad 93, 96\}.$$

Thus $\mathrm{PF}(H) = \{82, 85\}$ and $\mathrm{t}(H) = 2$.

(2) Let $H = \langle 6, 11, 16, 21, 26, 31\rangle$. Then H is almost symmetric. In fact, since $\mathrm{PF}(H) = \{5, 10, 15, 20, 25\}$, we see that H is almost symmetric.

(3) If $H = \langle 6, 17, 22, 27, 32, 37 \rangle$, then H has maximal embedding dimension, but which is not almost symmetric. Indeed, we have $\mathrm{PF}(H) = \{11, 16, 21, 26, 31\}$, which implies that H is not almost symmetric. This example show that the condition "$s = 1$" in Corollary 4.4 is essential.

(4) If $H = \langle 17, 56, 61, 66, 71, 76 \rangle$, then H is symmetric. In fact, since

$$\mathrm{Ap}(H, 17) = \{0,\ \ 56, 61, 66, 71, 76,\ \ 132, 137, 142, 147, 152,\ \ 208, 213, 218, 223, 228,\ \ 284\},$$

we have $\mathrm{PF}(H) = \{267\}$, and hence H is symmetric.

Chapter 5

Ulrich ideals of some Gorenstein numerical semigroup rings

In this chapter, we study Ulrich ideals of Gorenstein numerical semigroup rings which are generated by monomials. First, we recall the definition of Ulrich ideals from [GOTWY].

Definition 5.1 (Goto, et al [GOTWY]). Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 0$ and I be an $\mathfrak{m}$-primary ideal of R. Suppose that I contains a parameter ideal $Q = (a_1, ..., a_d)$ of R as a minimal reduction. Then I is called an *Ulrich ideal* of R if the following two conditions hold true:

(1) $I^2 = QI$ and

(2) R/I-module I/I^2 is free.

By definition, any parameter ideal of R is clearly Ulrich. For convenience, in this paper, we except parameter ideals whenever we refer to Ulrich ideals. We consider the case of $R = k[[H]] \subset k[[t]]$, a formal power series ring. Let χ_R^g denote the set of Ulrich ideals of R which are generated by monomials in t. The following is the key theorem to achieve our goal.

Theorem 5.2 ([GOTWY]). *Suppose that $R = k[[H]]$ is a Gorenstein numerical semigroup ring (equivalently, H is symmetric) and let I be a nonzero ideal of R. Then following condition are equivalent.*

(1) $I \in \chi_R^g$.

(2) $I = (t^\alpha, t^\beta)$ $(\alpha, \beta \in H, \alpha < \beta)$ *and if we put $x = \beta - \alpha$, the following conditions hold.*

  (i) $x \notin H$, $2x \in H$.

  (ii) *the numerical semigroup $H_1 = H + \langle x \rangle$ is symmetric, and*

  (iii) $\alpha = \min\{h \in H \mid h + x \in H\}$.

In particular, we note that $\chi_R^g \neq \emptyset$ if and only if there is an integer $x \in \mathbb{Z}$ which satisfies that conditions (i) and (ii) above. We show examples to see how to use Theorem 5.2.

Example 5.3.

(1) Let $H = \langle 4, 5 \rangle = \{0, 4, 5, 8, 9, 10, 12 \rightarrow\}$. We can find the integers which satisfy the condition (i):
$$x = 2, 6, 7, 11.$$
Among these integers, 2 and 6 just satisfy the condition (ii). Therefore we have
$$\chi_{k[[H]]}^g = \{(t^8, t^{10}), (t^4, t^{10})\}$$
by the condition (iii).

(2) If $H = \langle 3, 5 \rangle$, then $\chi_{k[[H]]}^g = \emptyset$ since we can check that there are no integers which satisfy the conditions (i) and (ii).

Remark 5.4. When H is a 2-generated numerical semigroup, the set $\chi_{k[[H]]}^g$ is completely described in [GOTWY].

5.1 The case where H is generated by an arithmetic sequence

Let $H = \langle a, a+d, ..., a+nd \rangle$ be a numerical semigroup generated by an arithmetic sequence. In that case, we completely determine when $\chi_{k[[H]]}^g$ is empty or not.

Our purpose is to show the following theorem.

Theorem 5.5. *Let $H = \langle a, a+d, a+2d, ..., a+nd \rangle$ be a symmetric numerical semigroup generated by an arithmetic sequence. Then $\chi_{k[[H]]}^g \neq \emptyset$ if and only if $n = 2$.*

We provide some lemmas to prove this theorem.

Proposition 5.6. *Let H be a numerical semigroup and $x \in \mathbb{Z}$ be an integer such that $x \notin H$ and $2x \in H$. Put $H_1 = H + \langle x \rangle$. Then:*

(1) Every element $y \in H_1$ which is not in H can be written as $y = x + h$ for some $h \in H$.

(2) Let $0 \neq a \in H$. Every element $\omega' \in \mathrm{Ap}(H_1, a)$ which is not in $\mathrm{Ap}(H, a)$ can be written as $\omega' = x + \omega$ for some $\omega \in \mathrm{Ap}(H, a)$.

Proof. (1) It is clear since $2x \in H$.

(2) If $\omega' \in \mathrm{Ap}(H_1, a)$ and $\omega' \notin \mathrm{Ap}(H, a)$, then $\omega' \notin H$, and hence $\omega' = x + \omega$ for some $\omega \in H$ by (1). Then it is easily seen that $\omega \in \mathrm{Ap}(H, a)$ since otherwise $\omega' \notin \mathrm{Ap}(H_1, a)$. □

Now let $H = \langle a, a+d, ..., a+nd \rangle$ be a numerical semigroup generated by an arithmetic sequence and put

$$\mathrm{Ap}(H, a) = \{0 = w(0) < w(d) < w(2d) < \cdots < w((a-1)d)\}, \tag{5.1}$$

where $w(id) \equiv id \pmod a$ for all $1 \le i \le a-1$.

Lemma 5.7. *Let H and $\mathrm{Ap}(H,a)$ be as above, and let $x \in \mathbb{Z}$ be an integer such that $x \notin H$ and $2x \in H$, and set $H_1 = H + \langle x \rangle$. If $x \equiv rd \pmod a$ for some $1 \le r \le a-1$, then*

$$\begin{aligned} \mathrm{Ap}(H_1, a) = \{0 = w(0) < w(d) < w(2d) < \cdots < w((r-1)d)\} \\ \cup \{w'((r+i)d) := x + w(id) \mid 0 \le i < a-r\}, \end{aligned} \tag{5.2}$$

where $w'((r+i)d) \equiv (r+i)d \pmod a$. Furthermore, if $w'((r+i)d) \in H$, then $w'((r+i)d) = w((r+i)d)$, and otherwise $w'((r+i)d) < w((r+i)d)$.

Proof. First, we claim that $x + w(id) \le w((r+i)d)$ for all $0 \le i < a-r$. It suffices to show this for $i = 1$ by induction. Since $x \equiv w(rd) \pmod a$ and $x \notin H$, $w(rd) - x \ge a$, and which implies $x + (a+d) \le w(rd) + d$. Thus we see that $x + w(d) \le w((r+1)d)$ since $w((r+1)d) = w(rd) + d$ or $w(rd) + (a+d)$ by Theorem 4.2.

By Proposition 5.6, every element $\omega' \in \mathrm{Ap}(H_1, a)$ is in $\mathrm{Ap}(H,a)$ or the form of $\omega' = x + \omega$ for some $\omega \in \mathrm{Ap}(H,a)$. Thus we should only prove that $w(jd) \in \mathrm{Ap}(H_1, a)$ for all $1 \le j \le r-1$ since the last statement of Lemma is clear from the claim in previous paragraph. Now we fix the j. Then there exists an integer $0 \le i < a-1$ such that $r + i \equiv j \pmod a$ with $r + i \ge a$. Thus we have $x + w(id) > w(jd)$ since $w(id) > w(jd)$. This implies $w(jd) \in \mathrm{Ap}(H_1, a)$. □

Example 5.8. Let $H = \langle 14, 17, 20, 23, 26 \rangle$. By Theorem 4.2, we have

$$\mathrm{Ap}(H, 14) = \{0, \quad 17, 20, 23, 26, \quad 43, 46, 49, 52, \quad 69, 72, 75, 78, \quad 95\}.$$

We can take $x = 7, 38, 50, 58$ and so on. Then Apéry sets in $H + \langle x \rangle$ are as follows.

$$\begin{aligned}
\mathrm{Ap}(H + \langle 7 \rangle, 14) &= \{0, \quad 17, 20, 23, 26, \quad 43, 46, 7, 24, \quad 27, 30, 33, 50, \quad 53\}. \\
\mathrm{Ap}(H + \langle 38 \rangle, 14) &= \{0, \quad 17, 20, 23, 26, \quad 43, 46, 49, 38, \quad 55, 58, 61, 64, \quad 81\}. \\
\mathrm{Ap}(H + \langle 50 \rangle, 14) &= \{0, \quad 17, 20, 23, 26, \quad 43, 46, 49, 52, \quad 69, 72, 75, 50, \quad 67\}. \\
\mathrm{Ap}(H + \langle 58 \rangle, 14) &= \{0, \quad 17, 20, 23, 26, \quad 43, 46, 49, 52, \quad 69, 58, 75, 78, \quad 81\}.
\end{aligned}$$

Remark 5.9. Lemma 5.7 is not true for numerical semigroups generated by generalized arithmetic sequences. Indeed, if $H = \langle 8, 17, 18, 19 \rangle$ and $x = 9$, then $\mathrm{Ap}(H + \langle 9 \rangle, 8)$ is not in the form of (5.2) since $\mathrm{Ap}(H, 8) = \{0, 17, 18, 19, 36, 37, 38, 55\}$ and $\mathrm{Ap}(H + \langle 9 \rangle, 8) = \{0, 9, 18, 19, 28, 37, 38, 47\}$.

Observing carefully Example 5.8, we show Theorem 5.5.

Proof of Theorem 5.5. We assume that $\chi^g_{k[[H]]} \ne \emptyset$. Then there exists an integer $x \in \mathbb{Z}$ such that $x \notin H, 2x \in H$ and $H_1 := H + \langle x \rangle$ is symmetric by Theorem 5.2. Since H is symmetric, we have

$$a \equiv 2 \pmod n \tag{5.3}$$

by Corollary 4.3. Let $\mathrm{Ap}(H,a)$ be as in (5.1). We assume that $x \equiv rd \pmod{a}, 1 \le r < a$, and let $\mathrm{Ap}(H_1,a) = \{0 = \omega'_0 < \omega'_1 < \cdots < \omega'_{a-1} = \mathrm{F}(H_1)+a\}$. Then, by Proposition 1.11, it holds that

$$\omega'_i + \omega'_{a-1-i} = \omega'_{a-1} \text{ for all } 1 \le i \le a-1. \tag{5.4}$$

By Lemma 5.7, $\max \mathrm{Ap}(H_1,a) = \omega'_{a-1}$ is either $w((r-1)d)$ or $x + w((a-r-1)d)$.

(i) If $\max \mathrm{Ap}(H_1,a) = x + w((a-r-1)d)$, then

$$r \equiv 0 \pmod{n} \tag{5.5}$$

since otherwise $\mathrm{t}(H_1) > 1$, and hence H_1 is not symmetric. And to satisfy the condition (5.4), we must have $r-1 = a-r-1$, and hence we get

$$a = 2r. \tag{5.6}$$

Now solving (5.3), (5.5) and (5.6), we obtain $n = 2$.

(ii) If $\max \mathrm{Ap}(H_1,a) = w((r-1)d)$, then we have

$$r \equiv 2 \pmod{n} \tag{5.7}$$

since otherwise $\mathrm{t}(H_1) > 1$. Furthermore, since equations (5.4) hold in the set $\{x + w(id) \mid 0 \le i < a-r\}$, we get the condition (5.5). By solving (5.4), (5.5) and (5.7), we obtain $n = 2$.

In both cases, we have $n = 2$ as desired.

Conversely, we assume that $n = 2$, that is, $H = \langle a, a+d, a+2d \rangle$. Then a is even by Corollary 4.3, and so we put $a = 2m$, where $m \ge 2$. We note that $m \notin H$ but $a = 2m \in H$, and $m+d \notin H$ but $a+2d = 2m+2d \in H$. We claim $H + \langle m \rangle$ or $H + \langle m+d \rangle$ is symmetric.

(i) If m is even, then $H + \langle m \rangle = \langle 2\langle m', m+d \rangle, 2m+d \rangle$, where $m = 2m'$. Hence $H + \langle m \rangle$ is symmetric by Theorem 2.1.

(ii) If m is odd, then $H + \langle m+d \rangle = \langle 2\langle m, (m+d)/2 \rangle, 2m+d \rangle$ since d is odd, and hence which is symmetric.

Thus we conclude that $\chi^g_{k[[H]]} \ne \emptyset$ if $n = 2$ by Theorem 5.2. This complete the proof. □

Example 5.10.

(1) If $H = \langle 10, 11, 12 \rangle$, then $\chi^g_{k[[H]]} \ne \emptyset$. Actually, we can easily check that $H + \langle 6 \rangle$ is symmetric by using Theorem 2.1.

(2) Let $H = \langle 16, 19, 22 \rangle$. Then $H + \langle 8 \rangle$ is symmetric, and hence $\chi^g_{k[[H]]} \ne \emptyset$.

(3) When $H = \langle 8, 11, 14, 17 \rangle$, we can check directly that there is no integers x such that $x \notin H$, $2x \in H$ and $H + \langle x \rangle$ is symmetric. Hence $\chi^g_{k[[H]]} = \emptyset$.

For the reasons mentioned in Remark 5.9, we cannot apply the proof of Theorem 5.5 for numerical semigroups generated by generalized arithmetic sequences.

Question 5.11. When H is a numerical semigroup generated by a generalized arithmetic sequence, does Theorem 5.5 hold true ?

5.2 Other cases

Now we mention the case where H is 3-generated symmetric numerical semigroups. The author in [Nu3] completely determined when Ulrich ideals generated by monomials of $k[[H]]$ are exist. We refer to the result from [Nu3].

Theorem 5.12 ([Nu3]). *Let $H = \langle a, b, c\rangle$ be a symmetric numerical semigroup and assume that $H = \langle d\langle a', b'\rangle, c\rangle$. We put $R = k[[H]]$, $H_1 = \langle a', b'\rangle$ and $R_1 = k[[H_1]]$. Then the following assertions hold true.*

(1) *If d and c are odd, then*

$$\chi_R^g = \{(t^{d\alpha}, t^{d\beta}) \mid \alpha, \beta \in H_1 \text{ such that } (t^\alpha, t^\beta) \in \chi_{R_1}^g\}.$$

In particular, $\#\chi_R^g = \#\chi_{R_1}^g$.

(2) *If a, b and c are odd, then $\chi_R^g = \emptyset$.*

In the following, we assume that a' and b' are odd.

(3) *If d is odd and c is even, then*

(i) *$\chi_R^g \neq \emptyset$ if and only if $H + \langle c/2\rangle$ is symmetric.*

(ii) *if $\chi_R^g \neq \emptyset$, then*

$$\chi_R^g = \{(t^{\frac{c}{2}l}, t^{\frac{c}{2}d}) \mid \ l \text{ is even with } \ 1 < l < d\}.$$

In particular, $\#\chi_R^g = (d-1)/2$.

(4) *If d is even and c is odd, then $H + \langle da'/2\rangle$ or $H + \langle db'/2\rangle$ is symmetric. In particular, $\chi_R^g \neq \emptyset$. Furthermore, the number of χ_R^g does not depend on d.*

Finally, we give a remark. For $H = \langle a, b, c\rangle$, it is symmetric if and only if it is a complete intersection (see [He]). When $H = \langle a, a+d, a+2d, ..., a+nd\rangle$ is a symmetric numerical semigroups generated by an arithmetic sequence, it is a complete intersection if and only if $n = 2$ (see [MS]). Hence we may expect that if $k[[H]]$ is Gorenstein but not a complete intersection, then it has no Ulrich ideals generated by monomials. However, unfortunately, there exist a counter example.

Example 5.13 ([Nu3]). A numerical semigroup $H = \langle 10, 12, 13, 14, 15\rangle$ is symmetric but not a complete intersection. However $H + \langle 5\rangle = \langle 5, 12, 13, 14\rangle$ is symmetric, and hence $\chi_{k[[H]]}^g \neq \emptyset$. In general, $H_m = \langle 2m, 2m+2, 2m+3, ..., 3m\rangle$ is symmetric but not a complete intersection if $m \geq 5$. Then we can check that $H + \langle m\rangle$ is symmetric. Therefore $\chi_{k[[H_m]]}^g \neq \emptyset$.

Chapter 6

Some properties of almost symmetric numerical semigroups

We conclude the paper by introducing some interesting properties of almost symmetric numerical semigroups. In particular, we refer to the result from [Nu4].

We recall the definition of gluing of numerical semigroups.

Definition 6.1 (Delorme [De], cf. [RG4, Chapter 8]). A *gluing* of two numerical semigroups $H_1 = \langle a_1, ..., a_n \rangle$ and $H_2 = \langle b_1, ..., b_m \rangle$ is defined by

$$\langle d_1 H_1, d_2 H_2 \rangle := \langle d_1 a_1, ..., d_1 a_n, d_2 b_1, ..., d_2 b_m \rangle ,$$

where $d_1 \in H_2 \setminus \{b_1, ..., b_m\}, d_2 \in H_1 \setminus \{a_1, ..., a_n\}$ and $\gcd(d_1, d_2) = 1$.

Theorem 6.2 (Delorme [De], cf. [RG4, Chapter 8]).

(1) *If $H = \langle d_1 H_1, d_2 H_2 \rangle$ is a gluing of H_1 and H_2, then H is symmetric (resp. a complete intersection) if and only if both H_1 and H_2 are symmetric (complete intersections).*

(2) *If $H \neq \mathbb{N}$ is a complete intersection, then H is a gluing of two numerical semigroups which are complete intersections.*

We see that Theorem 6.2 (2) is a generalization of Herzog's result (Theorem 2.1) since $H = \langle a, b, c \rangle$ is symmetric if and only if it is a complete intersection.

H. Nari proved that the almost symmetric property is not preserved by gluing.

Theorem 6.3 (Nari [Na]). *Let H be a gluing of two numerical semigroups and assume that H is not symmetric. Then H is never almost symmetric.*

The author in [Nu4] studied the relation between two numerical semigroups

$$\langle a_1, ..., a_n \rangle \quad \text{and} \quad \langle da_1, ..., da_{n-1}, a_n \rangle ,$$

where $d > 1$ and $\gcd(d, a_n) = 1$. Let us explain the motivation of this work. K -i. Watanabe [Wa] constructed the numerical semigroups $H = \langle da_1, ..., da_n, b \rangle$ for given

$H_1 = \langle a_1, ..., a_n \rangle$, where $d > 1$, $b \in H_1 \setminus \{a_1, ...a_{,n}\}$ and $\gcd(d, b) = 1$. We see that this construction is a special case of gluing of numerical semigroups, that is, the case of $H_2 = \mathbb{N}$ in Definition 6.1. In gluing, the condition that $b \in H_1 \setminus \{a_1, ..., a_n\}$ is essential. Then we want to know what happen if $b \notin H_1$.

Let $H_1 = \langle a_1, ..., a_n \rangle$ and $H = \langle da_1, ..., da_{n-1}, a_n \rangle$, where $d > 1$. Then we proved the following result which is an analogue of Watanabe's result (see [Wa, Lemma 1]).

Proposition 6.4 ([Nu4]). *Let H_1 and H be as above. Then the Betti numbers of $k[H]$ are equal to those of $k[H_1]$. In particular, H is symmetric (resp. a complete intersection) if and only if H_1 is symmetric (resp. a complete intersection).*

Furthermore, we also had an analogue of a special case of Theorem 6.3.

Proposition 6.5 ([Nu4]). *Let H_1 and H be as above. Assume that H is not symmetric (equivalently, H_1 is not symmetric). Then H is never almost symmetric.*

By this result, we get the following theorem.

Theorem 6.6 ([Nu4]). *If $H = \langle a_1, ..., a_n \rangle$ is almost symmetric which is not symmetric, then any $(n-1)$-tuples of $\{a_1, ..., a_n\}$ are relatively coprime.*

J. C. Rosales and P. A. García-Sánchez proved the following result. Therefore, we can regard Theorem 6.6 as an natural generalization of Theorem 6.7.

Theorem 6.7 (Rosales, García-Sánchez [RG4, Corollay 10.14]). *If $H = \langle a_1, a_2, a_3 \rangle$ is a pseudo-symmetric numerical semigroup, then any pairs of $\{a_1, a_2, a_3\}$ are relatively coprime*

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