On correlation function of a Wilson loop operator and a local operator in the gauge/gravity correspondence

ゲージ／重力対応における Wilson loop 演算子と局所演算子の相関関数について

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Contents

Chapter 1

Introduction

There are four interactions in nature; electromagnetic, weak, strong, and gravitational interactions. Three of them except the gravitational one are described by a gauge theory called *the Standard Model*. It is based on quantum mechanics and the special theory of relativity, so it does not include gravitational effects. Nevertheless, it succeeds in the explanation and prediction of various experimental facts of elementary particles. The reason is that the gravitational interaction is much weaker than the other ones at the energy scale which can be observed in present experiments.

When we study physics at higher energy scale beyond the Standard Model, such as the early universe, it is expected that all interactions become of the same order. Then we need to formulate a quantum theory of gravity and further construct a unified theory including all four interactions. The gravitational interaction is described classically by the general theory of relativity. However, when we attempt to quantize gravity with a standard framework of the field theory, we face serious difficulties. A notable one is *the non-renormalizability*; infinite quantities appear, which can not be absorbed by any redefinition of parameters in the theory.

String theory was originally proposed as a theory of hadrons in the beginning of 1970s. However, it was not regarded as realistic because its dimensionality is not 4 but 26 and it contains tachyons. In addition it was found that quantum chromodynamics (QCD) explains behavior of hadrons well. Because of such reasons, string theory as a theory of hadrons was abandoned. On the other hand, it was pointed out that spectra of strings contain states which correspond to gauge particles and gravitons. Moreover, it was found that certain string theories whose gauge symmetries are $SO(32)$ or $E_8 \times E_8$ have no gravitational anomaly. These gauge symmetries include $SU(3) \times SU(2) \times U(1)$, the one of the Standard Model. Therefore, it indicates that string theory has a possibility of being a consistent unified theory which includes all four interactions. Through these studies, string theory was revived as a candidate for a quantum gravity theory and a unified theory.

Fundamental objects in string theory are one-dimensional extended ones, strings. There are two types of fundamental strings; open strings, which have their endpoints and closed strings, which have no endpoints. We regard each string oscillation mode as each elementary particle in quantum field theories. In particular, the lowest energy mode of an open string corresponds to a gauge particle and the one of a closed string corresponds to a graviton. Hence, all the fundamental interactions can be included in string theory.

String theory includes not only strings but also p-dimensional extended objects, *Dp-branes* or shortly *D-branes*. It is known that endpoints of open strings are on the D-branes as a consequence of the T-duality, a peculiar nature of string theory. Then the open strings and the D-branes are closely connected. In the low-energy limit, dynamics of the D-branes is described by a gauge theory, because the gauge particle which is the lowest mode of the open string is dominant in the limit.

On the other hand, there is another description of the D-branes. Since the D-branes have energy, they are sources of a gravitational field which originates from the closed strings emitted from the D-branes. In the low-energy limit, their properties can be read from a geometry given by a classical solution of a supergravity theory which is the low-energy theory of string theory.

As we have seen so far, the low-energy D-branes can be described by two pictures, open and closed string pictures. The former is the description by the gauge theory and the latter is the description by the gravity theory. Since we see the same system by the two different theories, it leads us to a conjecture: the gauge/gravity correspondence. It asserts that there exists correspondence between gauge theories and gravity theories.

Moreover, as we will see later, one theory in a strong coupling regime corresponds to the other theory in a weak coupling regime. It indicates that if this conjecture is correct, we could analyze the theory in the strong coupling regime by studying the corresponding theory in the weak coupling regime. This is an attractive feature of the gauge/gravity correspondence. Exploiting this suggestion, some applications have been proposed; for example, an analytic treatment for QCD.

A direct proof of the correspondence is difficult because we need to analyze theories in the strong coupling regime. However, it is known that certain observables can be analyzed in the strong coupling regime because their quantum corrections are restricted due to their high symmetries. Using such observables, we can give a support for the correspondence. The main theme of this thesis is giving such a support by studying a correlation function between a Wilson loop operator and a local operator defined in the gauge theory side and its counterpart in the gravity theory side.

This thesis is organized as follows. In chapter 2, we review the best studied version of the gauge/gravity correspondence, the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. In chapter 3, we discuss a support for the correspondence by studying a Wilson loop operator. In chapter 4, we first present an analysis of a correlation function between a Wilson loop and a local operator in the gauge theory side. After that we evaluate a string amplitude which corresponds to the correlation function in the gravity theory side. The last chapter is devoted to a summary and discussions.

Chapter 2

The gauge/gravity (AdS/CFT) correspondence

The term "gauge/gravity correspondence" indicates various conjectures which assert dualities or correspondences between gauge theories and gravity theories. Through this thesis, we consider one which asserts the correspondence between four-dimensional $U(N)$ $\mathcal{N}=4$ super Yang-Mills (SYM) theory and type IIB superstring theory on $AdS_5 \times S^5$ geometry, a direct product of a five-dimensional anti de-Sitter space and a five-dimensional sphere. It is the best studied version of the correspondences originally proposed by Maldacena [1–3][1], which are generally called the AdS/CFT correspondences. "CFT" is an abbreviation for a conformal field theory. In the present case, it is often called the AdS$_5$/CFT$_4$ correspondence, where CFT$_4$ indicates four-dimensional $\mathcal{N}=4$ SYM theory.

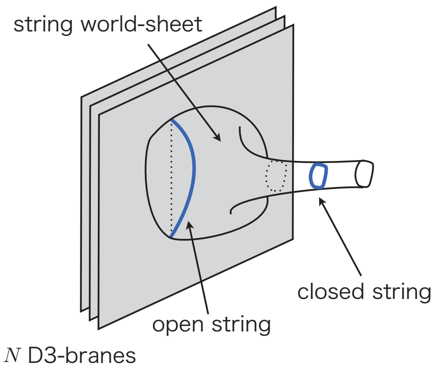

Figure 2.1: The interaction between an open string and a closed string

In order to present a circumstancial evidence for the AdS$_5$/CFT$_4$ correspondence, we consider a system of coincident N D3-branes, which are stable dynamical objects in

[1]Foundations, developments and applications of the AdS/CFT correspondence are discussed in [4, 5], for example.

type IIB superstring theory. Endpoints of open strings are attached on the D3-branes, while closed strings can propagate freely around them. Their interactions are expressed by a smooth string world-sheet as depicted in Fig.2.1. The correspondence is found by considering low-energy dynamics of the D3-branes from two different viewpoints, an open sting picture and a closed string picture.

String theory has one dimensionful parameter α', which is related to the string tension T as

$$T = \frac{1}{2\pi\alpha'}. \tag{2.1}$$

Then the dimension of α' is length squared and the square root of it corresponds to a typical length scale of string fluctuations. If an energy scale E of a system which we consider is much smaller than the typical energy scale of string oscillations $1/\sqrt{\alpha'}$, massive modes will not be excited. Hence massless oscillation modes are dominant in describing the low-energy dynamics of strings.

Now we focus on the open strings attached to the D3-branes. They interact with the closed strings propagating in *the bulk*, the whole space-time surrounding the D3-branes. In the low-energy limit, an effective theory on the D3-branes is a gauge theory while one in the bulk is a supergravity theory. The interactions between fields of the gauge theory and the supergravity theory vanish because a gravity coupling constant $\kappa = g_s\alpha'^2$ vanishes in the low-energy limit $E \ll 1/\sqrt{\alpha'}$. Here g_s denotes a dimensionless string coupling constant. Hence the supergravity theory is decoupled from the gauge theory.

We specify the gauge theory which emerges on the D3-branes by the following discussion. Each endpoint of the open string is on any one of the D3-branes. Hence degrees of freedom for the one endpoint is N, and that for an open string with two endpoints is N^2, which correspond to an adjoint representation of the $U(N)$ gauge symmetry on the D3-branes. On the other hand, type IIB superstring theory has $\mathcal{N} = 2$ supersymmetry in ten-dimensions. Supercharges in this case are given by two sets of ten-dimensional Majorana-Weyl spinors with a common chirality, so the total number of their degrees of freedom is 32. The D3-brane is a 1/2 BPS object, which preserves one half of the original supersymmetry. Then it has 16 degrees of freedom as preserved supersymmetry. In four-dimensional supersymmetric theories, the number of supercharges is counted by a four-dimensional Majorana spinor as a unit. Since a single Majorana spinor has 4 degrees of freedom, SYM theory on the D3-branes has $\mathcal{N} = 4$ supersymmetry. Finally, we conclude that the low-energy dynamics of the coincident N D3-branes is described by

$U(N)$ $\mathcal{N}=4$ SYM theory. The action is given by

$$S_{\mathcal{N}=4}=\frac{1}{2g_{\rm YM}^2}\int \mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}-D_\mu\Phi_I D^\mu\Phi_I+\frac{1}{2}[\Phi_I,\Phi_J]^2-i\bar{\Psi}\gamma^\mu D_\mu\Psi-\bar{\Psi}\gamma^{3+I}[\Phi_I,\Psi]\right]d^4x, \tag{2.2}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]$ $(\mu,\nu=0,1,2,3)$ is a four-dimensional field strength, $D_\mu=\partial_\mu-i[A_\mu,\ ]$ is a covariant derivative, and $(\gamma^\mu,\gamma^{3+I})$ are ten-dimensional gamma matrices. This theory contains a gauge field A_μ, and six scalar fields Φ_I $(I=1,2,\cdots,6)$ as bosonic fields, while four four-dimensional Majorana spinors are described as a ten-dimensional Mojorana-Weyl spinor Ψ.

Next we consider the same system, the low-energy D3-branes, from the closed string picture. As stated before, the D-branes have energy and their properties can be read from the geometry generated by themselves. In the case of the coincident N D3-branes, it is known that there is a classical solution of type IIB supergravity theory, which is the low-energy effective theory of type IIB superstring theory:

$$\begin{aligned}
&ds^2=H^{-\frac{1}{2}}(-dx_0^2+dx_1^2+dx_2^2+dx_3^2)+H^{\frac{1}{2}}(dr^2+r^2d\Omega_5^2),\\
&F_5=(1+*)dx_0\wedge dx_1\wedge dx_2\wedge dx_3\wedge d(H^{-1}),\\
&H=1+\frac{L^4}{r^4}.
\end{aligned} \tag{2.3}$$

r denotes a radial coordinate of the six-dimensional space surrounding the D3-branes and $d\Omega_5^2$ denotes a line element of a unit five-dimensional sphere. L is a typical length scale of this space-time, defined by

$$L=(4\pi g_sN)^{\frac{1}{4}}\sqrt{\alpha'}. \tag{2.4}$$

If we take the limit $r\gg L$, getting far away from the D3-branes, (2.3) becomes ten-dimensional Minkowski space-time, $\mathbb{R}^{1,9}$. On the other hand, if we take the opposite limit $L\gg r$, which is called *the near horizon limit*, it becomes

$$\begin{aligned}
&ds^2=\frac{r^2}{L^2}(-dx_0^2+dx_1^2+dx_2^2+dx_3^2)+\frac{L^2}{r^2}dr^2+L^2d\Omega_5^2,\\
&F_5=(1+*)dx_0\wedge dx_1\wedge dx_2\wedge dx_3\wedge d(H^{-1}).
\end{aligned}$$

Performing a coordinate transformation $u=L^2/r$, we finally obtain

$$ds^2=L^2\left(\frac{u^2+(-dx_0^2+dx_1^2+dx_2^2+dx_3^2)}{u^2}+d\Omega_5^2\right). \tag{2.5}$$

This line element represents $AdS_5\times S^5$, a direct product of *five-dimensional anti-de Sitter space* and a *five-dimensional sphere*. Therefore, the space-time which we consider approaches to flat space-time in the region far away from the D3-branes, while it becomes $AdS_5 \times S^5$ in the neighborhood of them. Since the $AdS_5 \times S^5$ geometry is realized deep inside of the gravity potential, the dynamics on $AdS_5 \times S^5$ is decoupled from the asymptotic region in the low-energy limit. We are interested in the counterpart of SYM theory on the D3-branes, so we focus on the neighborhood of them. Therefore, the low-energy dynamics of the coincident N D3-branes in the closed string picture is described by type IIB supergravity theory on $AdS_5 \times S^5$.

We have obtained two different pictures of the D3-branes system so far. The one is $U(N)$ $\mathcal{N} = 4$ SYM theory and the other is type IIB supergravity theory on $AdS_5 \times S^5$. Since these two theories describe the same system, we are led to a conjecture which asserts the correspondence between $U(N)$ $\mathcal{N} = 4$ SYM theory and type IIB supergravity theory on $AdS_5 \times S^5$. This is the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence.

In fact it is easy to see that the global symmetries in both theories correspond to each other. $\mathcal{N} = 4$ SYM theory has the conformal symmetry $SO(4, 2)$ and the R-symmetry $SO(6)$. The latter symmetry exchanges the six scalar fields. On the other hand, $AdS_5 \times S^5$ has the isometry group $SO(4, 2)$ in AdS_5 and the rotational symmetry $SO(6)$ in S^5. In addition the degrees of freedom of the supersymmetry in both theories agree. $\mathcal{N} = 4$ SYM theory has the Poincaré supersymmetry and the conformal supersymmetry. Each of them has 16 degrees of freedom, so the total amount is 32. On the other hand, a Killing spinor of $AdS_5 \times S^5$ is constructed by a pair of ten-dimensional Majorana-Weyl spinors. Since a ten-dimensional Majorana-Weyl spinor has 16 components, the total number of degrees of freedom is also 32.

Let us discuss the parameter range in which the above proposal is expected to be valid. String theory has two parameters, α' and g_s. Since the above arguments are based on weak coupling string theory, we are restricted to the regime,

$$g_s \ll 1. \tag{2.6}$$

We also described the properties of the D3-branes approximately with the classical solution of type IIB supergravity theory, which is valid only if all the stringy excited modes are suppressed. This requires that the scale of string fluctuations $\sqrt{\alpha'}$ must be much smaller than L, which is the typical scale of the geometry defined by (2.4),

$$\sqrt{\alpha'} \ll L. \tag{2.7}$$

This can be rewritten as

$$1 \ll 4\pi g_s N. \tag{2.8}$$

Combining (2.6) and (2.7), we find the following condition for the closed string picture to be valid:

$$1 \ll 4\pi g_s N \ll N. \tag{2.9}$$

On the other hand, $U(N)$ $\mathcal{N} = 4$ SYM theory has also two parameters, the gauge coupling constant $g_{\rm YM}$ and the rank of the gauge group N. By definition, N is equivalent to the number of the D3-branes. It is known that $g_{\rm YM}$ and g_s are related to each other as

$$4\pi g_s = g_{\rm YM}^2. \tag{2.10}$$

Using this relation, (2.9) can be translated into

$$1 \ll \lambda \ll N, \tag{2.11}$$

where $\lambda := g_{\rm YM}^2 N$. λ is called *the 't Hooft coupling constant.* This condition indicates that, in the gauge theory side, we take *the 't Hooft limit*, which is defined by

$$1 \ll N, \quad \lambda = \text{fixed}. \tag{2.12}$$

From (2.11) and (2.12), we find that $\mathcal{N} = 4$ SYM theory needs to be in the strong 't Hooft coupling limit in the present case. As explained in Appendix A, calculations of gauge theory simplify in the 't Hooft limit.

Now we reach a more precise statement of the AdS/CFT correspondence; it is a conjecture which asserts a correspondence between $U(N)$ $\mathcal{N} = 4$ SYM theory in the strong 't Hooft coupling limit and type IIB supergravity theory on $AdS_5 \times S^5$. This is a weaker version of the AdS$_5$/CFT$_4$ correspondence. The stronger one asserts that the correspondence between $U(N)$ $\mathcal{N} = 4$ SYM theory and type IIB superstring theory on $AdS_5 \times S^5$ holds in all corresponding parameter region.

If this conjecture is correct, we could analyze one theory in the strong coupling regime using the dual theory in the weak coupling regime. Generally, physics in the strong coupling regime is difficult to analyze, so this suggestion is attractive. However, we need to investigate in detail whether the correspondence is correct because the complete proof has not been given yet. We will present a support for the correspondence in the next chapter.

Chapter 3

Wilson loop operators in the AdS/CFT correspondence

As stated in Chapter 1, we can give a support for the gauge/gravity correspondence by using special observables which can be analyzed in the strong coupling regime. In the rest of this thesis, we mainly deal with a Wilson loop operator, an observable defined in gauge theories. In this chapter, first we see how the Wilson loop operator and its counterpart are read from the context of the AdS_5/CFT_4 correspondence. Next we present a nontrivial support for the correspondence by studying a circular *BPS Wilson loop operator*.

3.1 Wilson loop operators in QCD

The Wilson loop operator was originally introduced in order to explain the quark confinement in QCD and it is defined by

$$\begin{aligned} W_C &= \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left[\int_C iA_\mu \dot{x}^\mu d\sigma\right] \\ &= \frac{1}{N}\mathrm{Tr}\left[1+\int_0^{2\pi} d\sigma_1 (iA_\mu \dot{x}_1^\mu) + \int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 (iA_\mu \dot{x}_1^\mu)(iA_\nu \dot{x}_2^\nu)\right. \\ &\qquad \left. + \int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3 (iA_\mu \dot{x}_1^\mu)(iA_\nu \dot{x}_2^\nu)(iA_\rho \dot{x}_3^\rho) + \cdots \right]. \end{aligned} \tag{3.1}$$

Here A_μ is a gauge field and $x^\mu = x^\mu(\sigma)$ ($x_1^\mu := x^\mu(\sigma_1)$, etc.) denotes coordinates of the path C along which a quark and an anti-quark propagate.[1] The symbol $\mathcal{P}$ in the first line

[1]The signature is taken to be Euclidean in this chapter.

represents a path ordering, which is defined by the second and the third line. We assume that the quark and the anti-quark are heavy and non dynamical, i.e., their paths are not affected by interactions with the gauge field. An expectation value of the Wilson loop operator $\langle W_C \rangle$ corresponds to contributions of the gauge field to a transition amplitude for the propagation of the quark and the anti-quark in a gauge field background, see Fig.3.1.

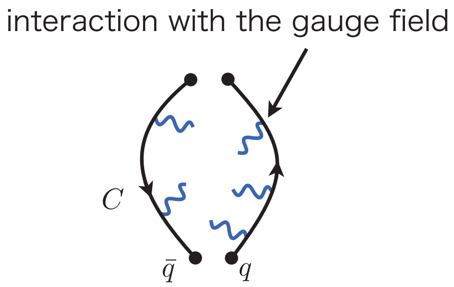

Figure 3.1: Wilson loop in QCD

In the next section, we explain that the Wilson loop operator and its counterpart are read from some D3-brane system in the context of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [6, 7].

3.2 Wilson loop operators in the AdS/CFT correspondence

Let us introduce a Wilson loop operator in four-dimensional $\mathcal{N}=4$ SYM theory. Fields of quarks in QCD belong to the fundamental representation of the gauge group. However, in $\mathcal{N}=4$ SYM theory, all fields are in the adjoint representation and there are no fields which belong to the fundamental one. In order to introduce such fields, we start from $U(N+1)$ gauge symmetry and break it to $U(N)\times U(1)$ [8]. By this symmetry breaking, the $U(N+1)$ fields $\hat{A}_\mu$ and $\hat{\Phi}_I$ are decomposed as follows:

$$\hat{A}_\mu = \begin{pmatrix} A_\mu & w_\mu \\ w_\mu^\dagger & a_\mu \end{pmatrix}, \quad \hat{\Phi}_I = \begin{pmatrix} \Phi_I & w_I \\ w_I^\dagger & m\Theta_I + \phi_I \end{pmatrix} \quad (\mu = 1,2,3,4;\ I = 1,2,\cdots,6). \tag{3.2}$$

Here $m\Theta_I$ ($\Theta_I^2 = 1$) are vacuum expectation values for the $U(1)$ part of the scalar fields, which break the symmetry.

Applying the $U(N)\,(\subset U(N+1))$ gauge transformation to (3.2), we find that A_μ and Φ_I belong to the adjoint representation of $U(N)$, while w_μ and w_I are the fundamentals:

$$\hat{U}(\partial_\mu + i\hat{A}_\mu)\hat{U}^\dagger = \begin{pmatrix} U(\partial_\mu + iA_\mu)U^\dagger & Uw_\mu \\ w_\mu^\dagger U^\dagger & a_\mu \end{pmatrix}, \quad \hat{U}\hat{\Phi}_I\hat{U}^\dagger = \begin{pmatrix} U\Phi_I U^\dagger & Uw_I \\ w_I^\dagger U^\dagger & m\Theta_I + \phi_I \end{pmatrix}.$$

Here the unitary matrix $\hat{U}$ is assumed to have the following from:

$$\hat{U} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}.$$

The fundamentals w_μ and w_I acquire large mass of order m in the process. Since they are coupled to not only the gauge filed A_μ but also the scalar fields Φ_I, the Wilson loop operator which arises from the propagation of these fields is given by

$$W_C = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left[\int_C (iA_\mu \dot{x}^\mu + \Theta_I\Phi_I|\dot{x}|)d\sigma\right]. \tag{3.3}$$

Note that the scalar fields Φ_I appear in the operator.

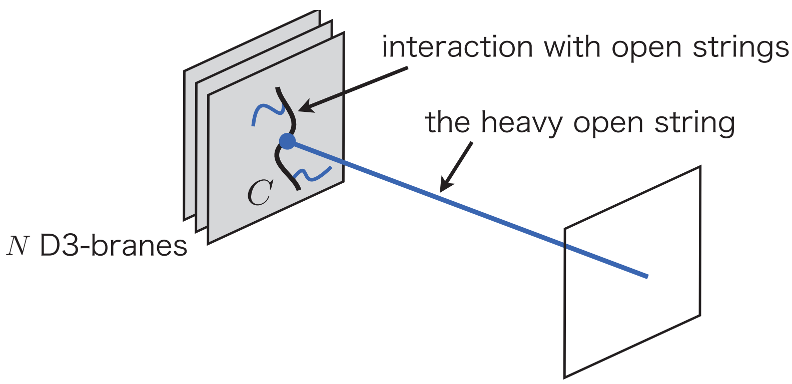

Figure 3.2: The interaction as open strings

Let us interpret the above discussion from the string theory viewpoint. $U(N+1)$ $\mathcal{N}=4$ SYM theory emerges on the coincident $N+1$ D3-branes and the symmetry breaking $U(N+1) \to U(N) \times U(1)$ corresponds to a procedure to separate a single D3-brane from the coincident N D3-branes at an infinite distance. Then an open string stretched between the N D3-branes and the separated D3-brane acquire large mass because the mass of the string is proportional to its length. We call it the heavy open string. Its one endpoint is attached on the N D3-branes, so it belongs to the fundamental representation of $U(N)$. On the other hand, the open strings whose two endpoints are attached on the N D3-branes belong to the adjoint representation of $U(N)$. Therefore, the heavy open string corresponds to w_μ and w_I while the open strings attached on the N D3-branes correspond to A_μ and Φ_I. Finally, when the heavy open string moves along an arbitrary path C on the N D3-branes, it interacts with open strings attached on them, namely, A_μ and Φ_I, see Fig.3.2. The expectation value of the Wilson loop operator (3.3), $\langle W_C \rangle$, arises from the interactions.

Next we consider the same system in the closed string picture. As discussed in the previous chapter, interactions with the gauge fields and the scalar fields are now replaced by the interactions with the gravity fields in the neighborhood of the N D3-branes. Hence we focus on the propagation of a part of the heavy open string inside the $AdS_5 \times S^5$ geometry. Then the path C is regarded to be on the boundary of the neighborhood, namely the boundary of the AdS_5 (Fig.3.3). The contribution to the heavy open string transition amplitude is given by the path integral

$$\int \mathcal{D}X e^{-S_{\text{string}}}. \tag{3.4}$$

Here S_{string} denotes the type IIB superstring action on $AdS_5 \times S^5$ and the boundary conditions for the path integral are specified by the loop C.

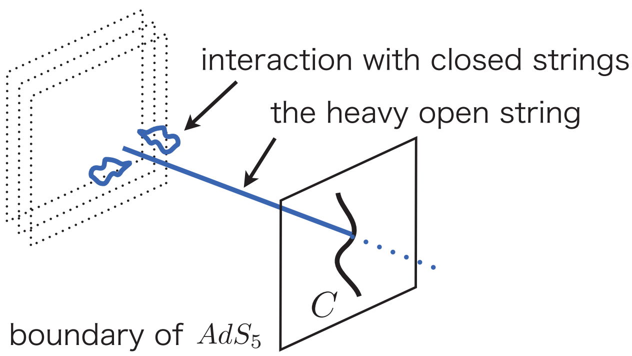

Figure 3.3: The interaction as closed strings (the neighborhood of N D3-branes)

Therefore, we obtain the two different pictures of the heavy open string propagation. In the open string picture, the heavy open string interacts with gauge fields and we obtain the factor $\langle W_C \rangle$. In the closed string picture, it interacts with gravity fields and the factor is given by the string amplitude in $AdS_5 \times S^5$. These two pictures describe the same system, so we expect that there exist the following correspondence:

$$\langle W_C \rangle = \int \mathcal{D}X e^{-S_{\text{string}}}. \tag{3.5}$$

3.3 A circular BPS Wilson loop operator : a support for the AdS/CFT correspondence

In this section, we present a support for the AdS/CFT correspondence by comparing both sides of (3.5) for a circular BPS Wilson loop operator.

We start with calculation in the gauge theory side [9, 10]. We set the Wilson loop operator (3.3) as follows:

$$\begin{aligned} C &: x^\mu = (r\cos\sigma, r\sin\sigma, 0, 0), \\ \Theta_I &= (0, 0, 1, 0, 0, 0). \end{aligned} \tag{3.6}$$

Then C represents a circle with radius r and only a single scalar field Φ_3 is taken into account. This is a circular 1/2 BPS Wilson loop operator, which preserves half of the original supersymmetry of $\mathcal{N} = 4$ SYM theory. To perform perturbative calculation, we recall that W_C is defined by path ordering,

$$\begin{aligned} W_C = \frac{1}{N}\mathrm{Tr}\Big[1 &+ \int_0^{2\pi} d\sigma_1 (iA_\mu \dot{x}_1^\mu + r\Phi_3) + \int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 (iA_\mu \dot{x}_1^\mu + r\Phi_3)(iA_\nu \dot{x}_2^\nu + r\Phi_3) \\ &+ \int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3 (iA_\mu \dot{x}_1^\mu + r\Phi_3)(iA_\nu \dot{x}_2^\nu + r\Phi_3)(iA_\rho \dot{x}_3^\rho + r\Phi_3) + \cdots \Big]. \end{aligned} \tag{3.7}$$

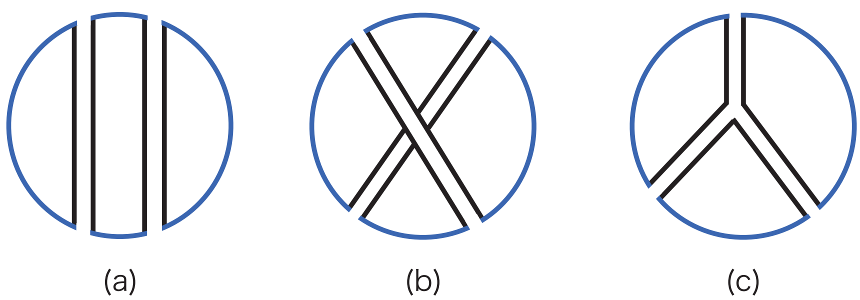

Figure 3.4: Diagrams for the Wilson loop operator, (a) planar ladder, (b) nonplanar, (c) a vertex involved (black double lines represent the propagation of gauge and scalar fields and blue single lines represent the one of "quarks")

In (3.7) when we contract each field, planar diagrams such as Fig.3.4 (a) and (c) give leading contributions, while nonplanar ones such as Fig.3.4 (b) are suppressed because we take the 't Hooft limit.[2] Moreover, we assume that the diagrams which involve vertices such as Fig.3.4 (c) cancel out due to the supersymmetry. Then we consider only the planar ladder diagrams such as Fig.3.4 (a). Since odd power terms vanish, we pick up and calculate only even power ones. Each calculation of the first three terms is as follows:

$$\text{quadratic:}\quad \frac{1}{N}\int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \mathrm{Tr}\langle (iA_{\mu_1}\dot{x}_1^{\mu_1}+r\Phi_3)(iA_{\mu_2}\dot{x}_2^{\mu_2}+r\Phi_3)\rangle = \frac{\lambda}{4}\cdot\frac{1}{(2\cdot 1)!}\cdot 1,$$

$$\begin{aligned}\text{quartic:}\quad &\frac{1}{N}\int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3 \int_0^{\sigma_3} d\sigma_4 \times \\ &\times \mathrm{Tr}\langle (iA_{\mu_1}\dot{x}_1^{\mu_1}+r\Phi_3)\cdots(iA_{\mu_4}\dot{x}_4^{\mu_4}+r\Phi_3)\rangle = \left(\frac{\lambda}{4}\right)^2\cdot\frac{1}{(2\cdot 2)!}\cdot 2,\end{aligned}$$

$$\begin{aligned}\text{sextic:}\quad &\frac{1}{N}\int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int_0^{\sigma_2} d\sigma_3 \int_0^{\sigma_3} d\sigma_4 \int_0^{\sigma_4} d\sigma_5 \int_0^{\sigma_5} d\sigma_6 \times \\ &\times \mathrm{Tr}\langle (iA_{\mu_1}\dot{x}_1^{\mu_1}+r\Phi_3)\cdots(iA_{\mu_6}\dot{x}_6^{\mu_6}+r\Phi_3)\rangle = \left(\frac{\lambda}{4}\right)^3\cdot\frac{1}{(2\cdot 3)!}\cdot 5,\end{aligned}$$

where λ is the 't Hooft coupling constant. Here the gauge and scalar field propagators are given by

$$\langle A_\mu(x_1)A_\nu(x_2)\rangle = \frac{\lambda}{4\pi^2}\frac{\delta_{\mu\nu}}{(x_1-x_2)^2}\cdot\frac{I}{2},\quad \langle \Phi_I(x_1)\Phi_J(x_2)\rangle = \frac{\lambda}{4\pi^2}\frac{\delta_{IJ}}{(x_1-x_2)^2}\cdot\frac{I}{2},$$

respectively, where I is an $N\times N$ unit matrix. A $U(N)$ gauge group generator T^a is normalized as $\mathrm{tr}(T^aT^b) = (1/2)\cdot\delta^{ab}$. Repeating similar calculations, we find that the $2n$-th order term is given by

$$\left(\frac{\lambda}{4}\right)^n\cdot\frac{1}{(2n)!}\cdot A_n.$$

A_n is a number of planar ladder diagrams in $2n$-th order term and it satisfies

$$A_{n+1} = \sum_{k=0}^{n} A_{n-k}A_k. \tag{3.8}$$

The first term A_0, the number of diagrams with no gauge and scalar fields propagation is 1. To confirm this relation, we consider the $2(n+1)$-th order diagram depicted in Fig.3.5.

[2]For the explanation of 't Hooft limit, see Appendix A.

There are $2(n+1)$ field operators on the circle C and we number them as $1, 2, \cdots, 2(n+1)$. An operator at the point 1 needs to be contracted with another operator at the point with any even number, $2(k+1)$ $(k = 0, 1, \cdots, n)$ so that the diagram becomes a planar ladder diagram. By this contraction, the $2(n+1)$-th order diagram is divided into two segments. The left side one has $2k$ operators while the right side one has $2(n-k)$ operators. There are no contraction between left and right in the planar diagram. Then, the number of planar diagrams for each segment is equal to A_k and A_{n-k}, respectively. Finally A_{n+1} can be calculated by summing up $A_{n-k}A_k$ for all possible values of k and we obtain (3.8).

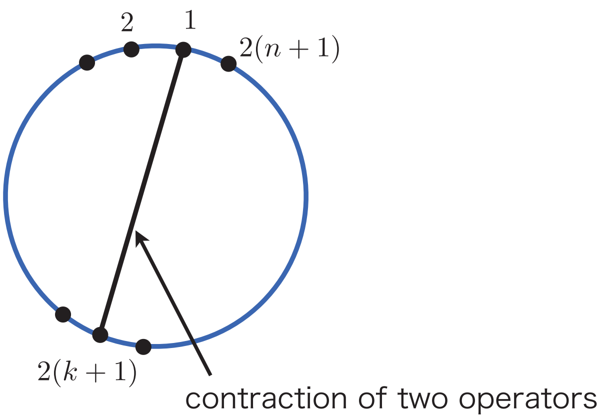

Figure 3.5: The $2(n+1)$-th order diagram

To obtain explicit form of A_n, we introduce a generating function

$$f(z) = \sum_{n=0}^{\infty} A_n z^n. \tag{3.9}$$

This satisfies the following relation:

$$z f^2(z) = \sum_{N=0}^{\infty} \sum_{m=0}^{N} A_{N-m} A_m z^{N+1} = \sum_{N=0}^{\infty} A_{N+1} z^{N+1} = f(z) - 1.$$

Here we used $A_0 = 1$ in the last step. Solving this equation for $f(z)$, we find

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} z^n$$

and obtain

$$A_n = \frac{(2n)!}{(n+1)!n!}.$$

We take the sign in front of the square root in $f(z)$ so that the value of $f(0)$ is finite. Therefore, $\langle W_C \rangle$ is evaluated as

$$\langle W_C \rangle = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left(\frac{\sqrt{\lambda}}{2} \right)^{2n}. \tag{3.10}$$

On the other hand, the infinite series representation of the modified Bessel function $I_1(x)$ is given by

$$I_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left(\frac{x}{2} \right)^{2n} \cdot \frac{x}{2},$$

so we obtain

$$\langle W_C \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}). \tag{3.11}$$

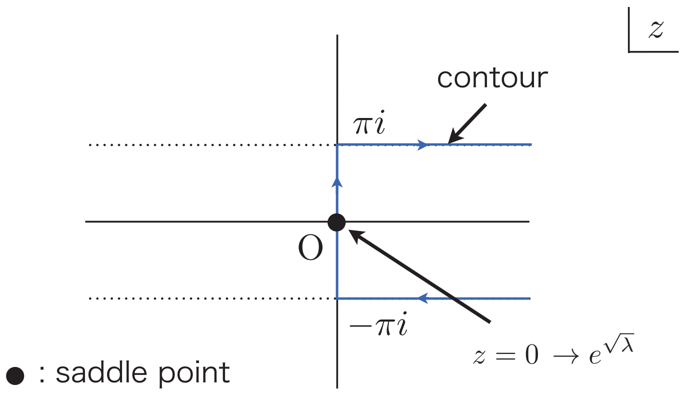

Figure 3.6: The method of steepest descent for $I_1(\sqrt{\lambda})$

As explained in section 2.1, we need to take the limit $\lambda \to \infty$ to compare with the gravity theory side.[3] In this limit, the behavior of $I_1(\sqrt{\lambda})$ is evaluated by the method of steepest descent using the integral representation [12]

$$I_1(\sqrt{\lambda}) = \frac{1}{2\pi i} \int_{\infty - \pi i}^{\infty + \pi i} \exp(\sqrt{\lambda} \cosh z - z) dz. \tag{3.12}$$

[3]Although the perturbative expression (3.10) is valid only for a small λ, we may assume that, after summing up all order contribution, the modified Bessel function allows analytic continuation to strong coupling regime. In fact the computation here can be mapped to a simple Gaussian matrix model, which can be solved exactly. Furthermore, the matrix model can be derived by using the localization theorem by Pestun [11].

Fig.3.6 depicts the contour of (3.12) and its saddle point which gives dominant contributions to the evaluation, i.e. under the limit $\lambda \to \infty$, the integrand of (3.12) has a sharp peak at $z = 0$ and the integral is evaluated locally at this point:

$$I_1(\sqrt{\lambda}) \sim e^{\sqrt{\lambda}}.$$

Finally, the result in the gauge theory side is

$$\langle W_C \rangle \sim e^{\sqrt{\lambda}} \quad (\lambda \to \infty). \tag{3.13}$$

Supergravity approximation of string theory enables us to evaluate the string path integral (3.4) with a classical string solution because stringy effects are suppressed. In the present case, we take S_{string} to be the Nambu-Goto action with Euclidean signature,

$$S_{\text{string}} = \frac{1}{2\pi\alpha'} \int \sqrt{\det G_{MN} \partial_a X^M \partial_b X^N} d\tau_{\text{E}} d\sigma, \tag{3.14}$$

where G_{MN} is the metric of $AdS_5 \times S^5$ and τ_{E} and σ are the Euclidean world-sheet coordinates. We take the Poincaré coordinates for AdS_5,

$$ds^2 = L^2 \frac{dY^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2}{Y^2}. \tag{3.15}$$

The path of the Wilson loop operator C is regarded as the boundary condition of the string path integral. Now we consider the case of a circle with radius r. Such a classical solution is given by

$$X_1 = \sqrt{r^2 - Y^2} \cos \tau_{\text{E}},\ X_2 = \sqrt{r^2 - Y^2} \sin \tau_{\text{E}},\ Y = \sigma, \tag{3.16}$$

where the range of Y is $0 \leq Y \leq r$ [13]. Fig.3.7 depicts the world-sheet of (3.16). Substituting (3.16) into (3.14), it becomes

$$S_{\text{string}} = \frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\tau_{\text{E}} \int_{\varepsilon}^{r} dY \frac{r}{Y^2} = \sqrt{\lambda}\left(\frac{r}{\varepsilon} - 1\right).$$

Here we introduced the cutoff ε because the metric diverges on the boundary of AdS_5 which is located at $Y = 0$. Subtracting the divergent term, we obtain

$$\int \mathcal{D}X e^{-S_{\text{string}}} \sim e^{-S_{\text{string}}}\Big|_{\text{classical}} = e^{\sqrt{\lambda}}. \tag{3.17}$$

Therefore, this result reproduces the large λ behavior of (3.13).

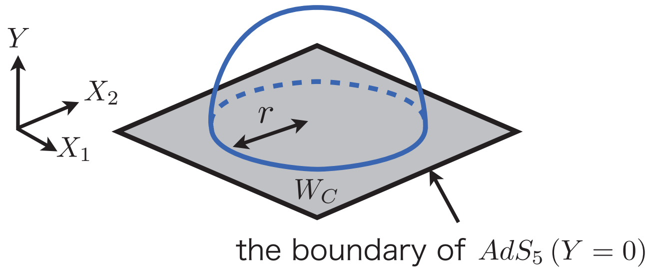

Figure 3.7: The world-sheet of (3.16)

Chapter 4

Correlation function as a support for the AdS/CFT correspondence

Our purpose of this thesis is giving a support for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence studying a correlation function between a 1/4 BPS Wilson loop and a local operator. The correlation function is calculated in [14, 15], and its counterpart in the gravity theory side, i.e, the string path integral is calculated in [16].

In this chapter, we present a result of calculation of the correlation function in the first section and then explain a semi-classical evaluation of the string path integral in the remaining sections.

4.1 The correlation function between a circular Wilson loop and a local operator

First we present the operators. The circular 1/4 BPS Wilson loop is given by setting

$$\begin{aligned} &C : x^\mu = (r\cos\sigma, r\sin\sigma, 0, 0), \\ &\Theta_I = (\sin\theta_0\cos\sigma, \sin\theta_0\sin\sigma, \cos\theta_0, 0, 0, 0) \end{aligned} \tag{4.1}$$

in (3.3) [17]. The local operator in the present case is given by

$$O_J = \mathrm{Tr}(\Phi_3 - i\Phi_4)^J, \tag{4.2}$$

where Φ_3 and Φ_4 are the scalar fields. This operator carries an R-charge J, which corresponds to a string angular momentum J in S^5 [18]. Fig.4.1 depicts the configuration of these operators.

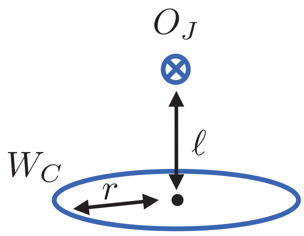

Figure 4.1: The configuration of the operators

Summing up planar ladder diagrams, the correlation function is given by

$$\langle W_C O_J \rangle \sim \left(\frac{r}{r^2 + \ell^2} \right)^J I_J(\sqrt{\lambda'}), \tag{4.3}$$

where $I_J(\sqrt{\lambda'})$ is the modified Bessel function and $\lambda' = \lambda \cos^2 \theta_0$ [15].[1]

We need to take the limit $\lambda \to \infty$ in order to compare with the gravity theory side. In addition we take the limit $J \to \infty$ with keeping $j = J/\sqrt{\lambda}$ fixed. The behavior of (4.3) under this limit is studied by the method of steepest descent with the integral representation [12]

$$I_J(\sqrt{\lambda'}) = \frac{1}{2\pi i} \int_{\infty - \pi i}^{\infty + \pi i} \exp(\sqrt{\lambda'} \cosh z - Jz) dz. \tag{4.4}$$

The result is

$$I_J(\sqrt{\lambda'}) \sim \exp[\sqrt{\lambda'}(\sqrt{j'^2 + 1} + j' \ln(\sqrt{j'^2 + 1} - j'))], \tag{4.5}$$

where $j' = j/\cos\theta_0$. The contour for (4.4), the steepest descent path P and the saddle point giving (4.5) are shown in Fig.4.2.

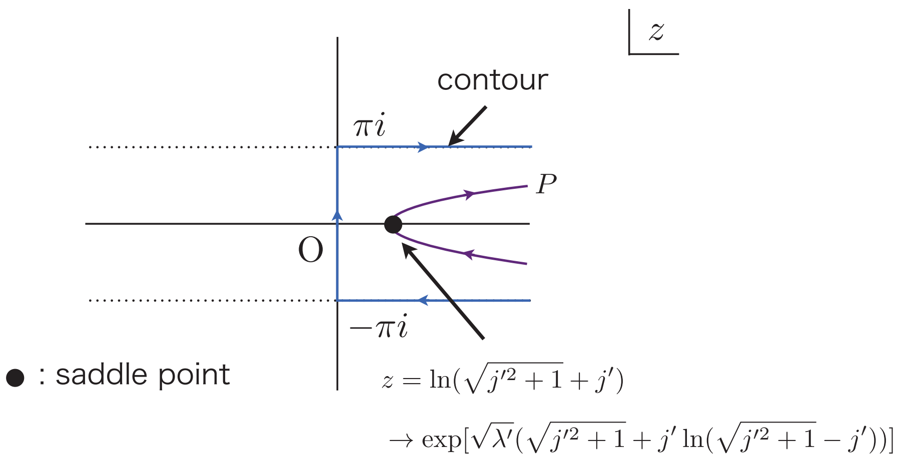

Figure 4.2: The method of steepest descent for $I_J(\sqrt{\lambda'})$

[1] Exact calculation of $\langle W_C O_J \rangle$ applying the localization theorem was studied in [19].

Finally, the correlation function becomes

$$\langle W_C O_J \rangle \sim \left(\frac{r}{r^2+\ell^2}\right)^J \exp[\sqrt{\lambda'}(\sqrt{j'^2+1}+j'\ln(\sqrt{j'^2+1}-j'))]. \tag{4.6}$$

If we set $\cos\theta_0 = 1$, (4.6) reproduces large λ behavior of a correlation function between a 1/2 BPS Wilson loop operator and a local operator (4.2) [14].

4.2 A classical solution

In the rest of this chapter, we evaluate the path integral for the transition amplitude of the string, which is expected to reproduce the correlation function given in the previous section. As stated repeatedly, we consider the case in which SYM theory in the strong 't Hooft coupling regime corresponds to type IIB supergravity theory, so we can evaluate the path integral by a classical string solution. Then we first look for the solution which satisfies boundary conditions coming from the gauge theory operators. Next we investigate whether preserved supersymmetries of the solution agrees with the ones of the gauge theory operators in the previous section. Finally, we evaluate the path integral and discuss the result.

Since our discussion in the gauge theory side is done by taking the Euclidean signature, we want to deal with the Euclidean AdS space represented by the Poincaré coordinates. However, we start with the Lorentzian AdS represented by the global coordinates and obtain a solution in the Euclidean global AdS . The reasons are as follows. First, the Lorentzian signature is more useful to investigate the preserved supersymmetries of the classical string solution. Second, a counterpart of the local operator (4.2) in the global AdS is well known. Third, if we obtain the solution in the Euclidean global AdS, the one in the Euclidean Poincaré AdS can be easily constructed by coordinate transformations.

We take the line element of the Lorentzian $AdS_5 \times S^5$ space as

$$\begin{aligned} ds^2 &= G_{MN} dX^M dX^N \\ &= L^2[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\varphi_1^2 + \sin^2\varphi_1 d\varphi_2^2 + \cos^2\varphi_1 d\varphi_3^2) \\ &\qquad + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta(d\chi_1^2 + \sin^2\chi_1 d\chi_2^2 + \cos^2\chi_1 d\chi_3^2)]. \end{aligned} \tag{4.7}$$

In the present case, the type IIB superstring action on $AdS_5 \times S^5$ becomes

$$S = -\frac{1}{4\pi\alpha'}\int h^{ab} G_{MN}\partial_a X^M \partial_b X^N \sqrt{-h}\, d\tau d\sigma. \tag{4.8}$$

By the information of the gauge theory operators, the expected string motion is as follows. The counterpart of the local operator (4.2) is a localized closed string motion which is located at $\rho = 0$ while rotating along a great circle of S^5 with an angular momentum J [18]. On the other hand, as for the boundary condition coming from the Wilson loop operator, we assume that the string world-sheet is attached to a circle on the boundary $\rho = \infty$ at $t = 0$ for example. This is because it can be mapped to a circle with radius r on the boundary of the Poincaré AdS by coordinate transformations as we will see in section 4.4. We set W_C at $t|_{\tau=0} = 0$ and O_J at $t|_{\tau=\infty} = \infty$ as initial and final state, respectively. Therefore, we assume an ansatz, in which a smooth world-sheet connects between W_C and O_J. Fig.4.3 depicts the ansatz for the string motion in the AdS_5.

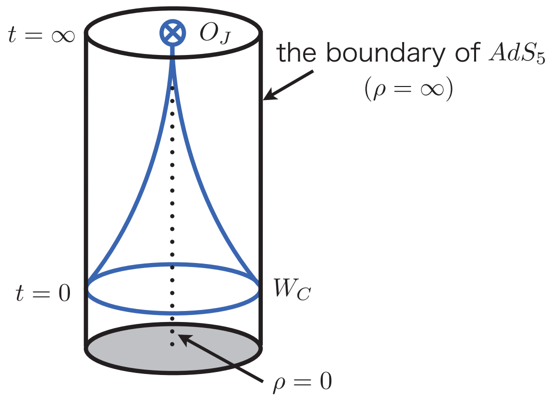

Figure 4.3: Ansatz for string motion in AdS_5 represented by the global coordinates

Since W_C contains several scalar fields, the string motion in S^5 is a little complicated. To explain it, we introduce the embedding coordinates:

$$Z_1 + iZ_2 = \sin\theta e^{i\phi}, \quad Z_3 + iZ_4 = \cos\theta \sin\chi_1 e^{i\chi_2}, \quad Z_5 + iZ_6 = \cos\theta\cos\chi_1 e^{i\chi_3},$$

where $Z_1, Z_2, \cdots, Z_6$ denote the flat six-dimensional coordinates in which S^5 is embedded.

On the Z_1-Z_2 plane, we assume that the string forms a circle with a radius $\sin\theta_0$ at $\tau = 0$, and then it shrinks to the origin at $\tau = \infty$. On the other hand, on the Z_3-Z_4 plane, we assume that the string is localized at a point $\cos\theta_0$ at $\tau = 0$, while it goes on rotating along a great circle with an angular momentum J at $\tau = \infty$. The initial condition of the string is determined by Θ_I contained in the Wilson loop operator shown in (4.1) and the final state, rotating with an angular momentum J, is determined by O_J. Figure 4.4 depicts the assumed string motion in S^5.

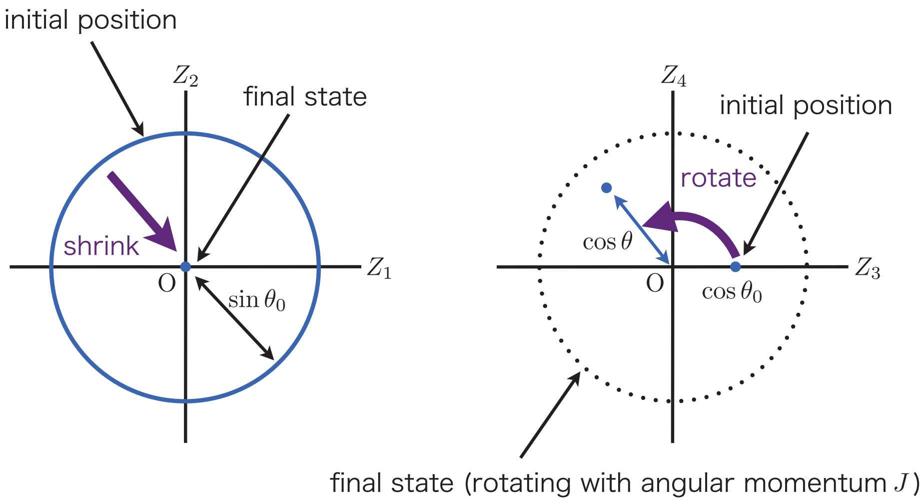

Figure 4.4: The assumed string motion in S^5

Hence we take the following ansatz,[2]

$$\begin{aligned} &t = t(\tau), \quad \rho = \rho(\tau), \quad \varphi_1 = \frac{\pi}{2}, \quad \varphi_2 = \sigma, \\ &\theta = \theta(\tau), \quad \phi = \sigma, \quad \chi_1 = \frac{\pi}{2}, \quad \chi_2 = \chi_2(\tau) \end{aligned} \tag{4.9}$$

and impose the boundary conditions for t, ρ, θ, and χ_2 which is summarized in Table 4.1. Here the conditions for $Z_1 + iZ_2$ and $Z_3 + iZ_4$ should be regarded the ones for θ and χ_2.

Table 4.1: The initial and final condition for the string motion

$\tau = 0$	$\tau = \infty$
$\rho = \infty$	$\rho = 0$
$t = 0$	$t = \infty$
$Z_1 + iZ_2 = \sin\theta_0 e^{i\sigma}$	$Z_1 + iZ_2 = 0$
$Z_3 + iZ_4 = \cos\theta_0$	$Z_3 + iZ_4 = \cos\theta(\tau)e^{i\chi_2(\tau)}$ (rotating)

Finally, (4.8) becomes

$$S = \frac{\sqrt{\lambda}}{2}\int(-\cosh^2\rho\dot{t}^2 + \dot{\rho}^2 - \sinh^2\rho + \dot{\theta}^2 - \sin^2\theta + \cos^2\theta\dot{\chi}_2^2)d\tau, \tag{4.10}$$

[2]This ansatz is classified into the $AdS_3 \times S^3$ one which was studied in [20].

where we used the relation $L^4/\alpha'^2 = \lambda$. Since λ is the overall factor of (4.10), we can deal with a string path integral semi-classically in the $\lambda \to \infty$ limit.

The equations of motion for the string are

$$\frac{d}{d\tau}(\cosh^2\rho\dot{t}) = 0, \tag{4.11}$$

$$\frac{d}{d\tau}(\cos^2\theta\dot{\chi}_2) = 0, \tag{4.12}$$

$$\ddot{\rho} + \sinh\rho\cosh\rho(\dot{t}^2 + 1) = 0, \tag{4.13}$$

$$\ddot{\theta} + \sin\theta\cos\theta(\dot{\chi}_2^2 + 1) = 0. \tag{4.14}$$

On the other hand, by variation of (4.8), the equation of motion for the world-sheet metric gives the Virasoro constraints

$$G_{MN}\partial_a X^M \partial_b X^N - \frac{1}{2}h^{cd}G_{MN}\partial_c X^M \partial_d X^N h_{ab} = 0.$$

Using the ansatz (4.9) and taking the conformal gauge, we obtain

$$\dot{\rho}^2 + \sinh^2\rho - \cosh^2\rho\dot{t}^2 + \dot{\theta}^2 + \sin^2\theta + \cos^2\theta\dot{\chi}_2^2 = 0. \tag{4.15}$$

From the equations of motion for the string, we obtain four constants of motion

$$\cosh^2\rho\dot{t} = C_1, \tag{4.16}$$

$$\cos^2\theta\dot{\chi}_2 = C_2, \tag{4.17}$$

$$\dot{\rho}^2 + \sinh^2\rho - \cosh^2\rho\dot{t}^2 = C_3, \tag{4.18}$$

$$\dot{\theta}^2 + \sin^2\theta + \cos^2\theta\dot{\chi}_2^2 = C_4. \tag{4.19}$$

We notice here that the Virasoro constraint (4.15) imposes the condition, $C_3 + C_4 = 0$. Although (4.16) and (4.17) are trivial, it needs some steps to derive (4.18) and (4.19). To derive (4.18), we multiply $2\dot{\rho}$ on both sides of (4.13), then it becomes

$$\frac{d}{d\tau}(\dot{\rho}^2 + \sinh^2\rho) + \left(\frac{d}{d\tau}\sinh^2\rho\right)\dot{t}^2 = 0. \tag{4.20}$$

Next, from (4.16), we obtain the following equation:

$$\begin{aligned}\frac{d}{d\tau}(\cosh^2\rho\dot{t}) = 0 &\Longleftrightarrow \dot{t}\frac{d}{d\tau}[(\sinh^2\rho + 1)\dot{t}] = 0\\ &\Longleftrightarrow \left(\frac{d}{d\tau}\sinh^2\rho\right)\dot{t}^2 + \cosh^2\rho\dot{t}\frac{d}{d\tau}\dot{t} = 0\\ &\Longleftrightarrow \left(\frac{d}{d\tau}\sinh^2\rho\right)\dot{t}^2 = -\frac{d}{d\tau}(\cosh^2\rho\dot{t}^2).\end{aligned}$$

Therefore, (4.13) is rewritten as

$$\frac{d}{d\tau}(\dot{\rho}^2 + \sinh^2\rho - \cosh^2\rho\dot{t}^2) = 0,$$

and then this leads to (4.18). In a similar way, we can derive (4.19) from (4.12) and (4.14).

C_1 is an energy and C_2 is an angular momentum of the string. In terms of the AdS/CFT correspondence, they correspond to the conformal weight and the R-charge of the local operator O_J, respectively [18]. Then it is natural to set $C_1 = C_2 = j := J/\sqrt{\lambda}$ and from the assumption $\rho \to 0$, $\sin\theta \to 0\,(\tau \to \infty)$, we can set $C_3 = -j^2, C_4 = j^2$. Therefore, (4.16), (4.17), (4.18) and (4.19) are

$$\cosh^2\rho\dot{t} = j, \tag{4.21}$$

$$\cos^2\theta\dot{\chi}_2 = j, \tag{4.22}$$

$$\dot{\rho}^2 + \sinh^2\rho - \cosh^2\rho\dot{t}^2 = -j^2, \tag{4.23}$$

$$\dot{\theta}^2 + \sin^2\theta + \cos^2\theta\dot{\chi}_2^2 = j^2. \tag{4.24}$$

Substituting (4.21) into (4.23) and (4.22) into (4.24), we obtain two equations

$$\left(\frac{d\rho}{d\tau}\right)^2 = -\sinh^2\rho - j^2\tanh^2\rho,$$

$$\left(\frac{d\theta}{d\tau}\right)^2 = -\sin^2\theta - j^2\tan^2\theta.$$

These equations indicate that there are no real solutions in the Lorentzian signature. Therefore, we perform the Wick rotation $\tau \to -i\tau_{\mathrm{E}}$, $t \to -it_{\mathrm{E}}$ and look for Euclidian solutions. After the Wick rotation, equations for ρ and θ become

$$\left(\frac{d\rho}{d\tau_{\mathrm{E}}}\right)^2 = \sinh^2\rho + j^2\tanh^2\rho, \tag{4.25}$$

$$\left(\frac{d\theta}{d\tau_{\mathrm{E}}}\right)^2 = \sin^2\theta + j^2\tan^2\theta. \tag{4.26}$$

The solutions for these equations which satisfy the boundary conditions are given by

$$\sinh\rho = \frac{\sqrt{j^2+1}}{\sinh\sqrt{j^2+1}\tau_{\mathrm{E}}}, \tag{4.27}$$

$$\sin\theta = \frac{\sqrt{j^2+1}}{\cosh\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)}. \tag{4.28}$$

Substituting (4.27) and (4.28) into Wick rotated versions of (4.21) and (4.22), we obtain

$$t_{\rm E} = j\tau_{\rm E} - \frac{1}{2}\ln\left(\frac{\cosh(\sqrt{j^2+1}\tau_{\rm E}+\xi)}{\cosh(\sqrt{j^2+1}\tau_{\rm E}-\xi)}\right), \tag{4.29}$$

$$\chi_2 = -ij\tau_{\rm E} + \frac{i}{2}\ln\left(\frac{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)+\xi)\sinh(\sqrt{j^2+1}\tau_0-\xi)}{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)-\xi)\sinh(\sqrt{j^2+1}\tau_0+\xi)}\right), \tag{4.30}$$

where $\xi := \ln(j+\sqrt{j^2+1})$. Here we introduce a constant τ_0 which is related to θ_0 as

$$\sin\theta_0 = \frac{\sqrt{j^2+1}}{\cosh\sqrt{j^2+1}\tau_0} \tag{4.31}$$

so that $\sin\theta$ satisfies the boundary condition $\sin\theta|_{\tau_{\rm E}=0} = \sin\theta_0$. The calculation to obtain these solutions is shown in Appendix B.

4.3 The BPS condition

If the AdS/CFT correspondence is correct, the supersymmetries in both sides need to agree. In this section, we investigate the preserved supersymmetries of the string solution which we derived in the previous section and check whether they agree with the ones in the gauge theory operators.

First we present notations. We take vielbeins on the $AdS_5\times S^5$, e^a_M $(a=0,1,\cdots,9)$ as follows:

$$\begin{aligned}
&e^0_0 = L\cosh\rho, \quad e^1_1 = L, \quad e^2_2 = L\sinh\rho, \quad e^3_3 = L\sinh\rho\sin\varphi_1, \quad e^4_4 = L\sinh\rho\cos\varphi_1,\\
&e^5_5 = L, \quad e^6_6 = L\sin\theta, \quad e^7_7 = L\cos\theta, \quad e^8_8 = L\cos\theta\sin\chi_1, \quad e^9_9 = L\cos\theta\cos\chi_1.
\end{aligned}$$

$\hat\Gamma_M$ is defined by $\hat\Gamma_M = e^a_M\Gamma_a$ with 10-dimensional gamma matrices Γ_a. ϵ_1 and ϵ_2 denote ten-dimensional Majorana-Weyl spinors. We combine them into a column vector ϵ,

$$\epsilon := \begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}.$$

σ_3 and $\varepsilon = i\sigma_2$ denote 2×2 matrix which acts on ϵ as

$$\sigma_3\epsilon = \begin{pmatrix}\epsilon_1\\ -\epsilon_2\end{pmatrix}, \quad \varepsilon\epsilon = \begin{pmatrix}\epsilon_2\\ -\epsilon_1\end{pmatrix},$$

respectively.

The BPS condition, which determines the preserved supersymmetries of a string solution is given by

$$\frac{1}{\sqrt{-g}}\partial_\tau X^M \partial_\sigma X^N \hat{\Gamma}_M \hat{\Gamma}_N \sigma_3 \epsilon = \epsilon, \tag{4.32}$$

where g is the induced metric on the world-sheet [21]. ϵ is the Killing spinor on $AdS_5 \times S^5$ and it represents total degrees of the supersymmetry of $AdS_5 \times S^5$ and satisfies

$$\left(D_M - \frac{\varepsilon}{2L}\Gamma_\star \hat{\Gamma}_M\right)\epsilon = \epsilon.$$

Here D_M is a covariant derivative on $AdS_5 \times S^5$, $\Gamma_\star$ is defined by $\Gamma_\star = \Gamma^{01234}$. In the present analysis, we can assume the following form by considering only the relevant coordinates

$$\epsilon = e^{\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1} e^{\frac{t}{2}\varepsilon\Gamma_\star\Gamma_0} e^{\frac{\varphi_2}{2}\Gamma_{13}} e^{\frac{\theta}{2}\varepsilon\Gamma_\star\Gamma_5} e^{\frac{\phi}{2}\Gamma_{56}} e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8} \begin{pmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \end{pmatrix}, \tag{4.33}$$

where $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ are constant ten-dimensional Majorana-Weyl spinors [22]. Using the ansatz (4.9) and the Virasoro constraint (4.15), the BPS condition (4.32) becomes

$$\frac{1}{\sinh^2\rho + \sin^2\theta}(\dot{t}\cosh\rho\Gamma_0 + \dot{\rho}\Gamma_1 + \dot{\theta}\Gamma_5 + \dot{\chi}_2\cos\theta\Gamma_8)(\sinh\rho\Gamma_3 + \sin\theta\Gamma_6)\sigma_3\epsilon = \epsilon,$$

and then performing the Wick rotation, we obtain

$$\frac{i}{\sinh^2\rho + \sin^2\theta}(\dot{t}_{\mathrm{E}}\cosh\rho\Gamma_{\mathrm{E}} + \dot{\rho}\Gamma_1 + \dot{\theta}\Gamma_5 + \dot{\chi}_2\cos\theta\Gamma_8)(\sinh\rho\Gamma_3 + \sin\theta\Gamma_6)\sigma_3\epsilon = \epsilon. \tag{4.34}$$

Here $\Gamma_{\mathrm{E}} = -i\Gamma_0$ and dots represent derivatives with respect to τ_{E}. After the Wick rotation, $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ are no longer Majorana spinors, but they are complex. However, we do not consider that their degrees of freedom are doubled. We regard this procedure as an analytic continuation corresponding to the one for bosonic coordinates.

We obtain three projection conditions from (4.34):

$$(\Gamma_{13} + \Gamma_{56})\begin{pmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \end{pmatrix} = 0, \tag{4.35}$$

$$(\Gamma_{\mathrm{E}} - i\Gamma_8)\begin{pmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \end{pmatrix} = 0, \tag{4.36}$$

$$-i(\sin\theta_0\Gamma_{16} + \cos\theta_0\Gamma_{13})\sigma_3\begin{pmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \end{pmatrix} = \begin{pmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \end{pmatrix}, \tag{4.37}$$

which commute with each other. Therefore, the solution which we found in the previous section preserve 1/8 of the original supersymmetry. This agrees with the number of the supersymmetry preserved by the gauge theory operators. The calculation to obtain the conditions (4.35), (4.36) and (4.37) is shown in Appendix B.

4.4 AdS_5 solution with the Poincaré coordinates

We used the global AdS space to derive the classical string solution and investigated its preserved supersymmetry. As stated at the beginning of this chapter, we perform the coordinate transformation from the global AdS space to the Poincaré AdS space to compare with the gauge theory side.

In the Euclidean signature, the global coordinates and the Poincaré coordinates are related through the following coordinate redefinition:

$$Y = \frac{re^{t_{\rm E}}}{\cosh\rho}, \qquad R = re^{t_{\rm E}}\tanh\rho = Y\sinh\rho, \tag{4.38}$$

where r is a constant parameter. Because of the relation

$$\frac{dY^2 + dR^2}{Y^2} = \cosh^2\rho dt_{\rm E}^2 + d\rho^2,$$

the AdS_5 part of the (Euclidean) line element (4.7) is changed to

$$\begin{aligned} ds^2 &= L^2\frac{dY^2 + dR^2 + R^2(d\varphi_1^2 + \sin^2\varphi_1^2 d\varphi_2^2 + \cos^2\varphi_1^2 d\varphi_3^2)}{Y^2} \\ &= L^2\frac{dY^2 + d\boldsymbol{X}^2}{Y^2}. \end{aligned} \tag{4.39}$$

Here we introduce the flat four-dimensional vector defined by

$$\boldsymbol{X} = R(\sin\varphi_1\cos\varphi_2, \sin\varphi_1\sin\varphi_2, \cos\varphi_1\cos\varphi_3, \cos\varphi_1\sin\varphi_3). \tag{4.40}$$

Substituting the AdS_5 part of the solution (4.27) and (4.29) into (4.38), the solution in the Poincaré coordinate is given by[3]

$$Y_0(\tau_{\rm E}) = re^{j\tau_{\rm E}}[\sqrt{j^2+1}\tanh(\sqrt{j^2+1}\tau_{\rm E} + \xi) - j], \tag{4.41}$$

$$\boldsymbol{X}_0 = R_0(\cos\sigma, \sin\sigma, 0, 0) \quad \left(R_0(\tau_{\rm E}) = re^{j\tau_{\rm E}}\frac{\sqrt{j^2+1}}{\cosh(\sqrt{j^2+1}\tau_{\rm E} + \xi)}\right). \tag{4.42}$$

Behavior of the solution at its boundaries are shown in Table 4.2.

Table 4.2: The behavior of Y_0 and R_0

	$\tau_{\rm E} \to 0$	$\tau_{\rm E} \to \infty$
Y_0	0	∞
R_0	r	0

[3] This form of the solution was derived in [23] to study the case of the 1/2 BPS Wilson loop operator.

Although we have transferred the solution from the global to the Poincaré coordinates, the position of O_J is still at infinity, $Y_0 = \infty$. Then using the isometry of AdS_5, we bring it to a finite distance from the position of W_C. The isometry transformation is given by

$$\begin{aligned} \boldsymbol{X}_{\mathrm{I}} &= \frac{\boldsymbol{X} + \boldsymbol{c}(\boldsymbol{X}^2 + Y^2)}{1 + 2\boldsymbol{c} \cdot \boldsymbol{X} + \boldsymbol{c}^2(\boldsymbol{X}^2 + Y^2)}, \\ Y_{\mathrm{I}} &= \frac{1}{1 + 2\boldsymbol{c} \cdot \boldsymbol{X} + \boldsymbol{c}^2(\boldsymbol{X}^2 + Y^2)}, \end{aligned} \tag{4.43}$$

where $\boldsymbol{c}$ is a constant vector. Now we consider the case in which O_J is on a line which passes the centre of W_C. For this purpose, we take $\boldsymbol{c} = (0, 0, 0, 1/\ell)$. After the transformation (4.43), we further perform the translation into the X^4_{I} direction,

$$\boldsymbol{X}_{\mathrm{T}} = \boldsymbol{X}_{\mathrm{I}} + \left(0, 0, 0, -\frac{\ell r^2}{\ell^2 + r^2}\right) \tag{4.44}$$

and the scale transformation,

$$\tilde{\boldsymbol{X}} = \frac{\ell^2 + r^2}{\ell^2}\boldsymbol{X}_{\mathrm{T}}, \quad \tilde{Y} = \frac{\ell^2 + r^2}{\ell^2} Y_{\mathrm{I}}, \tag{4.45}$$

so that the centre of W_C is located on the origin and its radius becomes r. Applying these transformations to $\boldsymbol{X}$ and Y, they become

$$\begin{aligned} \tilde{X}^i &= \frac{(\ell^2 + r^2)X^i}{(\boldsymbol{X} + \boldsymbol{x})^2 + Y^2} \qquad (i = 1, 2, 3), \\ \tilde{X}^4 &= -\frac{(\ell^2 + r^2)X^4}{(\boldsymbol{X} + \boldsymbol{x})^2 + Y^2} + \ell, \\ \tilde{Y} &= \frac{(\ell^2 + r^2)Y}{(\boldsymbol{X} + \boldsymbol{x})^2 + Y^2}, \end{aligned}$$

where $\boldsymbol{x} = (0, 0, 0, \ell)$. Using these relations, the solution (4.41) and (4.42) is mapped to

$$\tilde{\boldsymbol{X}}_0 := \frac{(\ell^2 + r^2)}{\ell^2 + R_0^2 + Y_0^2}(R_0 \cos\sigma, R_0 \sin\sigma, 0, -\ell) + (0, 0, 0, \ell), \tag{4.46}$$

$$\tilde{Y}_0 := \frac{(\ell^2 + r^2)Y_0}{\ell^2 + R_0^2 + Y_0^2}. \tag{4.47}$$

Table 4.3 shows the behavior of the $\tilde{\boldsymbol{X}}_0$ and $\tilde{Y}_0$ and Fig.4.5 depicts the world-sheet in the Poincaré coordinates.

Table 4.3: The behavior of $\tilde{\boldsymbol{X}}_0$ and $\tilde{Y}_0$

	$\tau_{\mathrm{E}} \to 0$	$\tau_{\mathrm{E}} \to \infty$
$\tilde{\boldsymbol{X}}_0$	$(r\cos\sigma, r\sin\sigma, 0, 0)$	$(0, 0, 0, \ell)$
$\tilde{Y}_0$	0	0

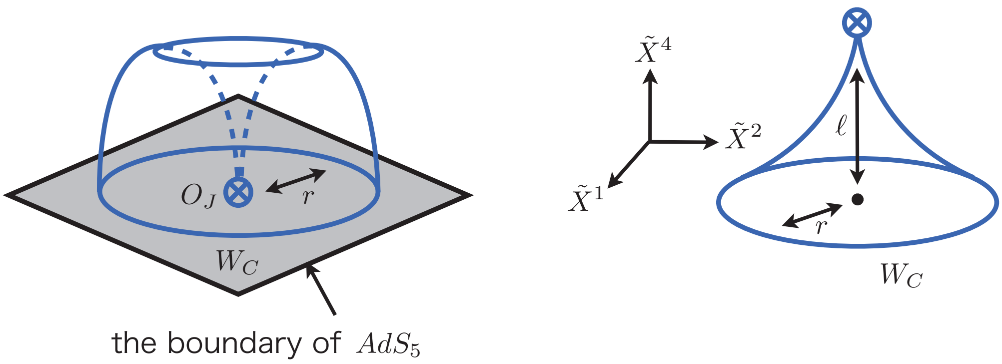

Figure 4.5: String world-sheet in the Poincaré coordinates

4.5 Evaluation of the path integral

We finally reach the step of evaluating the path integral. Since we consider supergravity approximation of strings, the path integral can be evaluated by the classical solution:

$$\int \mathcal{D}X e^{-S_{\text{string}}} \sim e^{-(S_{\text{bulk}}+S_{\text{boundary}})}\Big|_{\text{classical}} . \tag{4.48}$$

where S_{bulk} is the type IIB superstring action. S_{boundary} is a term which comes from the boundary conditions. We introduce cutoffs τ_- at $\tau_{\mathrm{E}} = 0$ and τ_+ at $\tau_{\mathrm{E}} = \infty$ because divergences emerge at these points.

First we calculate S_{bulk}. For convenience, we use the global coordinates here. Using

(4.10), (4.27) and (4.29),

$$
\begin{aligned}
S_{\text{bulk}} &= \sqrt{\lambda}\int_{\tau_-}^{\infty}(\sinh^2\rho+\sin^2\theta)d\tau_{\text{E}} \\
&= \sqrt{\lambda}\int_{\tau_-}^{\infty}\frac{d}{d\tau_{\text{E}}}\left(-\frac{1}{\sin\theta_0}\sinh\rho\sin\theta\right)d\tau_{\text{E}} \\
&= \sqrt{\lambda}\Bigg[\frac{1}{\sin\theta_0}\cdot\frac{\sqrt{j^2+1}}{\sqrt{j^2+1}\tau_-+\cdots}\times \\
&\qquad\qquad \times\frac{\sqrt{j^2+1}}{\cosh\sqrt{j^2+1}\tau_0(1+\cdots)+\sinh\sqrt{j^2+1}\tau_0(\sqrt{j^2+1}\tau_-+\cdots)}\Bigg] \\
&\sim \sqrt{\lambda}\cdot\frac{1}{\sin\theta_0}\cdot\sqrt{j^2+1}\left(\frac{1}{\cosh\sqrt{j^2+1}\tau_0}\cdot\frac{1}{\tau_-}-\frac{\sqrt{j^2+1}\sinh\sqrt{j^2+1}\tau_0}{\cosh^2\sqrt{j^2+1}\tau_0}\right) \\
&= \sqrt{\lambda}\left[\frac{1}{\tau_-}-\sqrt{j^2+\cos^2\theta_0}\right]. \qquad (4.49)
\end{aligned}
$$

Here we used the equation

$$
\frac{d}{d\tau_{\text{E}}}(\sinh\rho\sin\theta) = -\sin\theta_0(\sinh^2\rho+\sin^2\theta)
$$

in the second line. Its proof is given in Appendix C.

From the references [8, 23–25], we find that the appropriate boundary term is given by

$$
S_{\text{boundary}} = \frac{\partial L}{\partial\dot{u}}u\bigg|_{\tau_{\text{E}}=\tau_-} - J\left[\ln\frac{\tilde{Y}}{\tilde{Y}^2+(\tilde{\boldsymbol{X}}-\boldsymbol{x})^2}+\ln\cos\theta\sin\chi_1e^{-i\chi_2}\right]_{\tau_{\text{E}}=\tau_+}, \qquad (4.50)
$$

where L is the string Lagrangian

$$
L = \frac{\sqrt{\lambda}}{2}\left(\frac{\dot{\tilde{Y}}^2+\dot{\tilde{\boldsymbol{X}}}^2}{\tilde{Y}^2}+(S^5\text{part})\right),
$$

and $u = 1/\tilde{Y}$. The first term in (4.50) originates from the initial boundary condition coming from the Wilson loop operator W_C, while the second term originates from the final boundary condition coming from the local operator O_J.

The calculation of the first term in $S_{\rm boundary}$ is

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{u}}u\bigg|_{\tau_{\rm E}=\tau_-} &= -\sqrt{\lambda}\frac{\dot{\tilde{Y}}}{\tilde{Y}}\bigg|_{\tau_{\rm E}=\tau_-} \\
&= -\sqrt{\lambda}\left(\frac{\dot{Y}}{Y} - \frac{1}{\ell^2+R^2+Y^2}\cdot\frac{d}{d\tau_{\rm E}}(R^2+Y^2)\right)\bigg|_{\tau_{\rm E}=\tau_-} \\
&\sim -\sqrt{\lambda}\frac{\dot{Y}}{Y}\bigg|_{\tau_{\rm E}=\tau_-} \\
&= -\sqrt{\lambda}\left(j + \frac{j^2+1}{[\sqrt{j^2+1}\tanh(\sqrt{j^2+1}\tau_{\rm E}+\xi)-j]\cosh^2(\sqrt{j^2+1}\tau_{\rm E}+\xi)}\right)\bigg|_{\tau_{\rm E}=\tau_-} \\
&\sim -\frac{\sqrt{\lambda}}{\tau_-}.
\end{aligned}
\tag{4.51}
$$

The one of the second term is

$$
\begin{aligned}
&-J\left[\ln\frac{\tilde{Y}}{\tilde{Y}^2+(\tilde{\boldsymbol{X}}-\boldsymbol{x})^2} + \ln\cos\theta\sin\chi_1 e^{-i\chi_2}\right]_{\tau_{\rm E}=\tau_+} \\
&\quad= -J\left[\ln\frac{\tilde{Y}}{\tilde{Y}^2+(\tilde{\boldsymbol{X}}-\boldsymbol{x})^2} + \ln\cos\theta e^{-i\chi_2}\right]_{\tau_{\rm E}=\tau_+} \\
&\quad= -J\left[\ln\frac{Y}{\ell^2+r^2} + \ln\left(\left(1-\frac{j^2+1}{\cosh^2(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0))}\right)^{\frac{1}{2}} \times e^{-j\tau_{\rm E}}\right.\right. \\
&\qquad\left.\left.\times\left(\frac{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)+\xi)\sinh(\sqrt{j^2+1}\tau_0-\xi)}{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)-\xi)\sinh(\sqrt{j^2+1}\tau_0+\xi)}\right)^{\frac{1}{2}}\right)\right]_{\tau_{\rm E}=\tau_+} \\
&\quad\sim -J\left[\ln\frac{r}{\ell^2+r^2} + \ln\left(\sqrt{\frac{j^2}{\cos^2\theta_0}+1} - \frac{j^2}{\cos\theta_0}\right)\right].
\end{aligned}
\tag{4.52}
$$

Combinig (4.49), (4.51), and (4.52), the evaluation of the path integral is

$$
e^{-(S_{\rm bulk}+S_{\rm boundary})}\bigg|_{\rm classical} = \left(\frac{r}{\ell^2+r^2}\right)^J \exp[\sqrt{\lambda'}(\sqrt{j'^2+1}+j'\ln(\sqrt{j'^2+1}-j'))], \tag{4.53}
$$

where $\lambda' = \lambda\cos^2\theta_0$ and $j' = j/\cos\theta_0$. This result reproduces the behavior of the correlation function (4.6) in the limit $\lambda\to\infty$ and $J\to\infty$ with j' fixed.[4]

[4](4.53) was also derived in [19] to be compared with the exact calculation of $\langle W_C O_J\rangle$ applying the localization theorem.

4.6 A second solution

There exist a second solution, for which $\sin\theta$ and χ_2 are given by

$$\begin{aligned}\sin\theta &= \frac{\sqrt{j^2+1}}{\cosh\sqrt{j^2+1}(-\tau_{\rm E}+\tau_0)},\\ \chi_2 &= -ij\tau_{\rm E} + \frac{i}{2}\ln\left(\frac{\sinh(\sqrt{j^2+1}(-\tau_{\rm E}+\tau_0)-\xi)\sinh(\sqrt{j^2+1}\tau_0+\xi)}{\sinh(\sqrt{j^2+1}(-\tau_{\rm E}+\tau_0)+\xi)\sinh(\sqrt{j^2+1}\tau_0-\xi)}\right).\end{aligned} \tag{4.54}$$

We evaluate the action with this solution:

$$S_{\rm bulk} = \sqrt{\lambda}\int_{\tau_-}^{\infty}(\sinh^2\rho + \sin^2\theta)d\tau_{\rm E} \sim \sqrt{\lambda}\left[\frac{1}{\tau_-} + \sqrt{j^2+\cos\theta_0^2}\right]. \tag{4.55}$$

The boundary terms are evaluated as

$$\begin{aligned}&\frac{\partial L}{\partial \dot{u}}u\bigg|_{\tau_{\rm E}=\tau_-} \sim -\frac{\sqrt{\lambda}}{\tau_-},\\ &-J\left[\ln\frac{Y'}{Y'^2+(\vec{X}'-\vec{x})^2} + \ln\cos\theta e^{-i\chi_2}\right]_{\tau_{\rm E}=\tau_+}\\ &\quad\sim -J\left[\ln\frac{r}{\ell^2+r^2} - \ln\left(\sqrt{\frac{j^2}{\cos^2\theta_0}+1} - \frac{j^2}{\cos\theta_0}\right) + \pi i\right].\end{aligned} \tag{4.56}$$

Then combing these terms, we obtain

$$e^{-(S_{\rm bulk}+S_{\rm boundary})} = \left(\frac{r}{\ell^2+r^2}\right)^J(-1)^J\exp[-\sqrt{\lambda'}(\sqrt{j'^2+1} - j'\ln(\sqrt{j'^2+1}-j'))]. \tag{4.57}$$

To interpret this result, we recall that the behavior of the correlation function $\langle W_C O_J\rangle$ in the gauge theory side is determined by the method of steepest descent. The saddle points of (4.4) are classified into two categories depending on whether it is on the steepest descent path P or not, see Fig.4.6.[5] The evaluation of the action including the boundary terms for the second solution reproduces the saddle point value which is not on P, so it does not contribute to $I_J(\sqrt{\lambda'})$ in contrast to (4.53).

The semi-classical evaluation of the string path integral would correspond to applying the method of steepest descent to the functional integral. If the AdS/CFT correspondence is correct, the complete calculation of the string path integral would reproduce the

[5] The saddle points which belong to each category are located at $z = -\ln(\sqrt{j'^2+1}+j') + 2(n+1)\pi i$ and $z = \ln(\sqrt{j'^2+1}+j') + 2n\pi i\ (n\in\mathbb{Z})$, respectively and there are $2\pi i$ periodicity.

modified Bessel function in the present case. Hence our result seems to be natural, i.e. we found that both of the saddle points for the integrand of the modified Bessel function are reproduced from the semi-classical analysis of the string path integral.

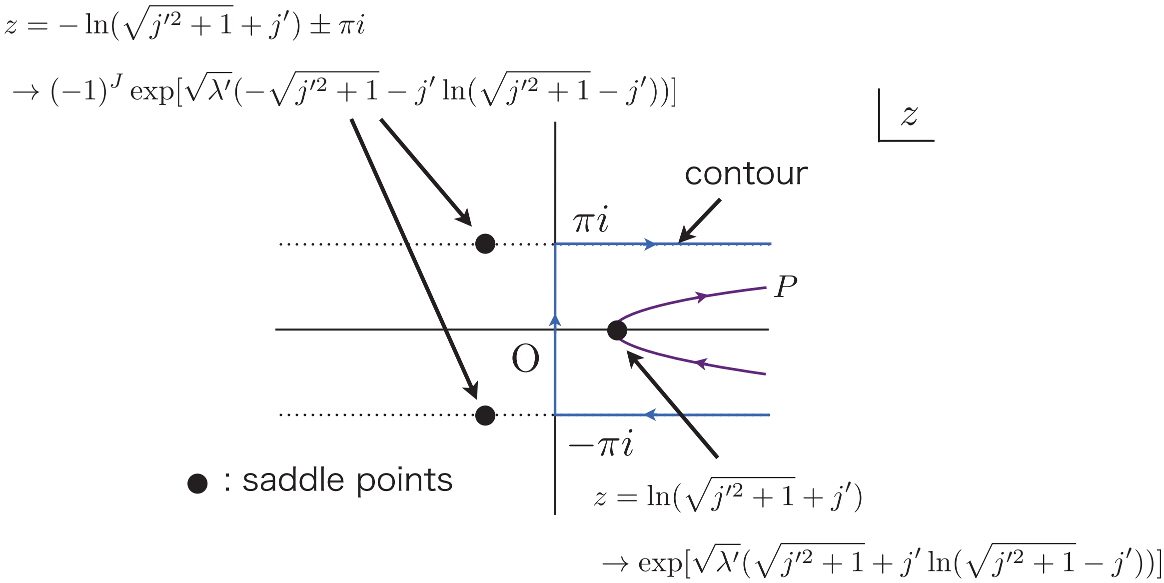

Figure 4.6: The saddle points of $I_J(\sqrt{\lambda'})$

Chapter 5

Summary and future directions

The main theme of this thesis is giving a support for the AdS/CFT correspondence. In chapter 2, we reviewed the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. It was led by considering the two different pictures of the coincident N D3-branes at low-energy; one is $\mathcal{N} = 4$ SYM theory as the open string picture and the other is type IIB supergravity theory on the $AdS_5 \times S^5$ as the closed string picture. We found that SYM theory in the strong 't Hooft coupling regime corresponds to type IIB supergravity theory in the parameter range which is valid for the low-energy approximation. In chapter 3, we discussed how to introduce the Wilson loop operator in $\mathcal{N} = 4$ SYM theory and its interpretation in the AdS/CFT correspondence. We also calculated the expectation value of a single 1/2 BPS Wilson loop operator, evaluated the string path integral semi-classically, and shown that they agree. In chapter 4, we first presented that the correlation function of a 1/4 BPS Wilson loop and a local operator is given by the modified Bessel function and studied its behavior in the large λ and J limit with keeping $j' = J/\sqrt{\lambda'}$ fixed. Next, we found the classical string solution which reflects the information of the gauge theory operators and studied its preserved supersymmetry. Finally we evaluated the string path integral semi-classically and shown that the result reproduces the large λ and J behavior of the correlation function. We also studied the second string solution and the result reproduced the saddle point value which does not contribute the modified Bessel function.

As for future directions, one possibility is to obtain more precise supports of the correspondence. In our analysis in the gravity theory side, we do not consider fluctuation around the classical solution. The supergravity approximation corresponds to taking the leading order of α' expansion of superstring theory. If we calculate the higher order terms, involving stringy corrections, we would obtain a more precise support. However, it is difficult to perform calculation, so we would need some tools, such as integrability. The other possibility is to change the gauge theory operators. For example, since we consider Wilson loop operators in the fundamental representation in this thesis, we would change them to ones in larger representations. In this situation, we deal with a bundle of heavy open strings in the gravity theory side and it is known that such an object is described by a D-brane [26, 27]. Applying such studies, we may obtain new supports for the correspondence or new knowledge about D-brane dynamics.

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Appendix A

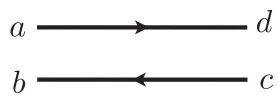
Some ingredients for the AdS/CFT correspondence

A.1 The ’t Hooft limit

In order to explain the concept of the ’t Hooft limit, we deal with *the matrix model* as a toy model of gauge theories in this section [28–30]. We consider the system whose action is given by

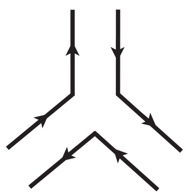
$$S = \frac{1}{g_{\mathrm{YM}}^2} \mathrm{Tr}\left[\frac{1}{2}M^2 + M^3\right], \tag{A.1}$$

where M is an $N \times N$ hermitian matrix. $\mathrm{Tr}M^2$ and $\mathrm{Tr}M^3$ corresponds to the kinetic and the interaction term in usual quantum field theory, respectively. Introducing “fatted” Feynman diagram notation, a free propagator is given by

$\langle M_{ab} M_{cd} \rangle =$ 

$= g_{\mathrm{YM}}^2 \delta_{ad} \delta_{bc},$

and it contributes a factor g_{YM}^2. On the other hand, the term $\mathrm{Tr}M^3$ is represented as

$\mathrm{Tr}M^3 =$ 

and contributes a factor $1/g_{\mathrm{YM}}^2$.

When we connect vertices by propagators and make a Feynman diagram in perturbative calculation, some closed lines appear which are called *closed index lines*, see examples shown below. They represent taking trace of products of several Kronecker deltas. Then each closed index line contributes a factor N.

Fig.A.1 depicts two examples of diagrams appearing in the perturbative expansion of the partition function which are called *planar diagrams*. The left one has three propagators, two vertices, and three closed index lines, then it contributes a factor

$$(g_{\mathrm{YM}}^2)^3 \cdot (g_{\mathrm{YM}}^{-2})^2 \cdot N^3 = g_{\mathrm{YM}}^2 N^3 = \lambda N^2.$$

The right one has six propagators, four vertices, and four closed index lines, and then it contributes a factor

$$(g_{\mathrm{YM}}^2)^6 \cdot (g_{\mathrm{YM}}^{-2})^4 \cdot N^4 = g_{\mathrm{YM}}^4 N^4 = \lambda^2 N^2.$$

$\langle \mathrm{Tr} M^3 \mathrm{Tr} M^3 \rangle$ $\qquad$ $\langle \mathrm{Tr} M^3 \mathrm{Tr} M^3 \mathrm{Tr} M^3 \mathrm{Tr} M^3 \rangle$

$g_{\mathrm{YM}}^2 N^3 = \lambda N^2$ $\qquad$ $g_{\mathrm{YM}}^4 N^4 = \lambda^2 N^2$

Figure A.1: Planar diagrams

On the other hand, Fig.A.2 depicts an example of the diagrams, which are called *non-planar diagrams*. This diagram has six propagators, four vertices and two closed index lines, and then it contributes a factor

$$(g_{\mathrm{YM}}^2)^6 \cdot (g_{\mathrm{YM}}^{-2})^4 \cdot N^2 = g_{\mathrm{YM}}^4 N^2 = \lambda^2 N^0.$$

$\langle \mathrm{Tr} M^3 \mathrm{Tr} M^3 \mathrm{Tr} M^3 \mathrm{Tr} M^3 \rangle$

$g_{\mathrm{YM}}^4 N^2 = \lambda^2 N^0$

Figure A.2: A non-planar diagram

Generally, a diagram which has p propagators, q vertices, and r closed index lines contributes a factor

$$(g_{\rm YM}^2)^p \cdot (g_{\rm YM}^{-2})^q \cdot N^r = \lambda^{p-q} N^{r-p+q} = \lambda^{r-2+2g} N^{2-2g},$$

where g denotes a genus of a surface corresponding to the diagram. The sum of all the diagrams are given by a power series of N^{-2}:

$$Z = \sum_{g=0}^{\infty} F_g(\lambda) N^{2-2g},$$

where $F_g(\lambda)$ are functions of λ which are given by calculations of diagrams. In the 't Hooft limit (2.12), a summation of all planar diagrams gives the leading contribution while the ones with higher genera are suppressed.

A.2 Coordinate systems for anti-de Sitter (AdS) space

The five-dimensional anti-de Sitter space, AdS_5 can be defined as a hyperboloid embedded in flat six-dimensional space with a signature $(-,-,+,+,+,+)$,

$$-\mathsf{X}_1^2 - \mathsf{X}_2^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -L^2, \tag{A.2}$$

where L is a constant.

In this thesis we use mainly two coordinate systems. The first one is given by setting each coordinate as

$$\begin{aligned} \mathsf{X}_1 &= \frac{1}{2y}(1 + y^2(L^2 + x_i^2 - x_0^2)), \quad \mathsf{X}_2 = Lyx_0, \\ X_4 &= \frac{1}{2y}(1 - y^2(L^2 - x_i^2 + x_0^2)), \quad X_i = Lyx_i \, (i = 1, 2, 3), \end{aligned}$$

where $y > 0$. Hence the line element of the hyperboloid becomes

$$\begin{aligned} ds^2 &= -d\mathsf{X}_1^2 - d\mathsf{X}_2^2 + \sum_{I=1}^{4} dX_I^2 \\ &= L^2 \left(\frac{dy^2}{y^2} + y^2(-dx_0^2 + dx_i^2) \right) \\ &= L^2 \left(\frac{du^2 + (-dx_0^2 + dx_i^2)}{u^2} \right) \end{aligned} \tag{A.3}$$

We performed the coordinate transformation $y = 1/u$ in the last line. This coordinate system is called *the Poincaré coordinates*.

The second coordinate system for AdS_5 is given by

$$\mathsf{X}_1 = R\cosh\rho\cos t, \quad \mathsf{X}_2 = R\cosh\rho\sin t,$$
$$X_I = R\sinh\rho\Omega_I \quad \left(\sum_{I=1}^{4}\Omega_I^2 = 1\right).$$

In this case, the line element becomes

$$\begin{aligned} ds^2 &= -\mathsf{dX}_1^2 - \mathsf{dX}_2^2 + \sum_{I=1}^{4} dX_I^2 \\ &= R^2\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\sum_{I=1}^{4} d\Omega_I^2\right). \end{aligned} \tag{A.4}$$

This coordinate system is called *the global coordinates*.

Appendix B

The solution of the equation of motion

In this appendix, we show calculation to obtain the solution of the equations of motion for the string.

B.1 ρ and θ

The equation of motion for ρ is given by (4.25)

$$\left(\frac{d\rho}{d\tau_{\mathrm{E}}}\right)^2 = \sinh^2\rho + j^2\tanh^2\rho. \tag{B.1}$$

The solution is given formally by

$$\int \frac{d\rho}{\sqrt{\sinh^2\rho + j^2\tanh^2\rho}} = \pm\tau_{\mathrm{E}} + A \quad (A\text{: arbitrary constant}). \tag{B.2}$$

The integral on the left hand side is calculated as follows:

$$\begin{aligned}
\int \frac{d\rho}{\sqrt{\sinh^2\rho + j^2\tanh^2\rho}} &= \int \frac{\cosh\rho d\rho}{\sinh\rho\sqrt{\cosh^2\rho + j^2}} \\
&= \int \frac{udu}{u^2\sqrt{u^2+(j^2+1)}} \quad (\sinh\rho = u) \\
&= \int \frac{dr}{r^2-(j^2+1)} \quad \left(\, r = \sqrt{u^2+(j^2+1)}\,\right) \\
&= \frac{1}{2\sqrt{j^2+1}} \ln\left|\frac{r-\sqrt{j^2+1}}{r+\sqrt{j^2+1}}\right| \\
&= \frac{1}{\sqrt{j^2+1}} \ln\frac{\sqrt{\sinh^2\rho+(j^2+1)}-\sqrt{j^2+1}}{\sinh\rho}.
\end{aligned}$$

Hence the solution which satisfies $\rho \to 0$ at $\tau_{\rm E} \to \infty$ is

$$\frac{\sqrt{j^2+1}}{\sinh \rho} = \frac{1}{2}(\tilde{A}^{-1} e^{\sqrt{j^2+1}\tau_{\rm E}} - \tilde{A} e^{-\sqrt{j^2+1}\tau_{\rm E}}) \quad (\tilde{A} = e^{\sqrt{j^2+1}A}).$$

Moreover, we set $\tilde{A} = 1$ so that the solution satisfies $\rho \to \infty$ at $\tau_{\rm E} \to 0$. Then we obtain

$$\sinh \rho = \frac{\sqrt{j^2+1}}{\sinh \sqrt{j^2+1}\tau_{\rm E}}. \tag{B.3}$$

Since the equation of motion for θ, (4.26)

$$\left(\frac{d\theta}{d\tau_{\rm E}}\right)^2 = \sin^2\theta + j^2 \tan^2\theta, \tag{B.4}$$

is analogous to (4.25), its calculation is done by a similar way as in the case of ρ. The solution which satisfies the final state condition $\sin\theta \to 0\,(\tau_{\rm E} \to \infty)$ is

$$\frac{\sqrt{j^2+1}}{\sin\theta} = \frac{1}{2}(B e^{\sqrt{j^2+1}\tau_{\rm E}} + B^{-1} e^{-\sqrt{j^2+1}\tau_{\rm E}}). \quad (B\text{: arbitrary constant})$$

We introduce a non negative constant τ_0 which is related to the parameter θ_0 by

$$\sin\theta_0 = \frac{\sqrt{j^2+1}}{\cosh \sqrt{j^2+1}\tau_0}. \tag{B.5}$$

Here θ_0 is the parameter in the scalar field coefficients Θ_I of the Wilson loop and it determines the boundary condition in S^5, which is $\sin\theta|_{\tau_{\rm E}=0} = \sin\theta_0$. Finally we obtain the solution

$$\sin\theta = \frac{\sqrt{j^2+1}}{\cosh \sqrt{j^2+1}(\tau_{\rm E}+\tau_0)}. \tag{B.6}$$

B.2 $t_{\rm E}$

The Wick rotated version of (4.21) is given by

$$\cosh^2\rho \frac{dt_{\rm E}}{d\tau_{\rm E}} = j. \tag{B.7}$$

The solution is given formally by

$$\begin{aligned} t_{\rm E} &= j \int \frac{d\tau_{\rm E}}{\cosh^2\rho} + C \\ &= j\tau_{\rm E} - j\int \tanh^2\rho\, d\tau_{\rm E} + C. \quad (C\text{: arbitrary constant}) \end{aligned} \tag{B.8}$$

Using (B.3), the integral on the right hand side can be calculated as follows (The factor $-j$ is omitted.):

$$
\begin{aligned}
\int \tanh^2 \rho d\tau_{\mathrm{E}} &= \int \frac{j^2+1}{(j^2+1)+\sinh^2 \sqrt{j^2+1}\tau_{\mathrm{E}}} d\tau_{\mathrm{E}} \\
&= \int \frac{dv}{(1+v^2)\sqrt{1+a^2v^2}} \quad \left(v = \frac{\sinh \sqrt{j^2+1}\tau_{\mathrm{E}}}{\sqrt{j^2+1}},\ a = \sqrt{j^2+1} \right) \\
&= \frac{1}{2\sqrt{a^2-1}} \ln \left| \frac{v\sqrt{a^2-1}+\sqrt{a^2v^2+1}}{v\sqrt{a^2-1}-\sqrt{a^2v^2+1}} \right| \\
&= \frac{1}{2j} \ln \left| \frac{j \sinh \sqrt{j^2+1}\tau_{\mathrm{E}} + \sqrt{j^2+1} \cosh \sqrt{j^2+1}\tau_{\mathrm{E}}}{j \sinh \sqrt{j^2+1}\tau_{\mathrm{E}} - \sqrt{j^2+1} \cosh \sqrt{j^2+1}\tau_{\mathrm{E}}} \right| \\
&= \frac{1}{2j} \ln \left(\frac{\cosh(\sqrt{j^2+1}\tau_{\mathrm{E}} + \xi)}{\cosh(\sqrt{j^2+1}\tau_{\mathrm{E}} - \xi)} \right). \quad (\, \xi = \ln(j + \sqrt{j^2+1})\,)
\end{aligned}
$$

Finally, we obtain

$$
t_{\mathrm{E}} = j\tau_{\mathrm{E}} - \frac{1}{2} \ln \left(\frac{\cosh(\sqrt{j^2+1}\tau_{\mathrm{E}} + \xi)}{\cosh(\sqrt{j^2+1}\tau_{\mathrm{E}} - \xi)} \right). \tag{B.9}
$$

Here we choose the arbitrary constant C so that the boundary condition $t_{\mathrm{E}}|_{\tau_{\mathrm{E}}=0} = 0$ is satisfied.

B.3 χ_2

The Wick rotated version of (4.22) is given by

$$
\cos^2 \theta \frac{d\chi_2}{d\tau_{\mathrm{E}}} = -ij. \tag{B.10}
$$

The solution is given formally by

$$
\begin{aligned}
\chi_2 &= -ij \int \frac{d\tau_{\mathrm{E}}}{\cos^2 \theta} + D \\
&= -ij\tau_{\mathrm{E}} - ij \int \tan^2 \theta d\tau_{\mathrm{E}} + D. \quad (D\text{: arbitrary constant})
\end{aligned} \tag{B.11}
$$

Using (B.6), the integral on the right hand side is calculated as follows (The factor $-ij$ is omitted.):

$$\begin{aligned}
\int \tan^2\theta d\tau_{\mathrm{E}} &= \int \frac{j^2+1}{\cosh^2\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)-(j^2+1)}d\tau_{\mathrm{E}} \\
&= \frac{1}{2}\int\left(\frac{1}{\cosh k - a}-\frac{1}{\cosh k + a}\right)dk \quad (k=\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0),\ a=\sqrt{j^2+1}) \\
&= -\frac{1}{2}\frac{1}{\sqrt{a^2-1}}\ln\left|\frac{\tanh\frac{k}{2}+\sqrt{\frac{a-1}{a+1}}}{\tanh\frac{k}{2}-\sqrt{\frac{a-1}{a+1}}}\cdot\frac{\tanh\frac{k}{2}+\sqrt{\frac{a+1}{a-1}}}{\tanh\frac{k}{2}-\sqrt{\frac{a+1}{a-1}}}\right| \\
&= -\frac{1}{2j}\ln\left|\frac{j\cosh k+\sqrt{j^2+1}\sinh k}{j\cosh k-\sqrt{j^2+1}\sinh k}\right| \\
&= -\frac{1}{2j}\ln\left(\frac{\sinh(\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)+\xi)}{\sinh(\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)-\xi)}\right). \quad (\,\xi=\ln(j+\sqrt{j^2+1})\,)
\end{aligned}$$

Therefore, the solution is

$$\chi_2 = -ij\tau_{\mathrm{E}}+\frac{i}{2}\ln\left(\frac{\sinh(\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)+\xi)}{\sinh(\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)-\xi)}\right)+D.$$

Here we choose the arbitrary constant D so that the boundary condition $\chi_2|_{\tau_{\mathrm{E}}=0}=0$ is satisfied, which is determined by the scalar field coefficient Θ_I. Finally, we obtain the solution

$$\chi_2 = -ij\tau_{\mathrm{E}}+\frac{i}{2}\ln\left(\frac{\sinh(\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)+\xi)\sinh(\sqrt{j^2+1}\tau_0-\xi)}{\sinh(\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)-\xi)\sinh(\sqrt{j^2+1}\tau_0+\xi)}\right). \qquad \text{(B.12)}$$

Appendix C

The BPS condition

In this appendix, we show calculation to obtain the conditions (4.35), (4.36) and (4.37).

We show again the BPS condition (4.34) here:

$$\frac{i}{\sinh^2\rho+\sin^2\theta}(\dot{t}_{\mathrm{E}}\cosh\rho\Gamma_{\mathrm{E}}+\dot{\rho}\Gamma_1+\dot{\theta}\Gamma_5+\dot{\chi}_2\cos\theta\Gamma_8)(\sinh\rho\Gamma_3+\sin\theta\Gamma_6)\sigma_3\epsilon=\epsilon. \quad \text{(C.1)}$$

After the Wick rotation and some calculation, (4.33) are rewritten as

$$\epsilon=e^{\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}e^{\frac{t_{\mathrm{E}}}{2}\varepsilon\Gamma_\star\Gamma_{\mathrm{E}}}e^{\frac{\theta}{2}\varepsilon\Gamma_\star\Gamma_5}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}e^{\frac{\sigma}{2}(\Gamma_{13}+\Gamma_{56})}\begin{pmatrix}\bar{\epsilon}_1\\ \bar{\epsilon}_2\end{pmatrix}. \quad \text{(C.2)}$$

Since no σ-dependence appears in (C.1), we impose the first condition

$$(\Gamma_{13}+\Gamma_{56})\begin{pmatrix}\bar{\epsilon}_1\\ \bar{\epsilon}_2\end{pmatrix}=0, \quad \text{(C.3)}$$

so we obtain (4.35). Then the Killing spinor (C.2) is reduced to

$$\epsilon=e^{\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}e^{\frac{t_{\mathrm{E}}}{2}\varepsilon\Gamma_\star\Gamma_{\mathrm{E}}}e^{\frac{\theta}{2}\varepsilon\Gamma_\star\Gamma_5}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar{\epsilon}_1\\ \bar{\epsilon}_2\end{pmatrix}. \quad \text{(C.4)}$$

Since our purpose is to obtain constant projection conditions imposed on the spinors $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$, we multiply the inverse of the factor of (C.4) on each side of (C.1).

First we multiply $e^{-\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}e^{-\frac{\theta}{2}\varepsilon\Gamma_\star\Gamma_5}$. The calculation of the left hand side of (C.1) is as follows (The factor $i/(\sinh^2\rho+\sin^2\theta)$ is omitted):

$$e^{-\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}e^{-\frac{\theta}{2}\varepsilon\Gamma_\star\Gamma_5}(\dot{t}_{\rm E}\cosh\rho\Gamma_{\rm E}+\dot\rho\Gamma_1+\dot\theta\Gamma_5+\dot\chi_2\cos\theta\Gamma_8)(\sinh\rho\Gamma_3+\sin\theta\Gamma_6)\sigma_3\epsilon$$

$$\begin{aligned}&=e^{-\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}(\dot{t}_{\rm E}\cosh\rho\sinh\rho\Gamma_{\rm E3}+\dot\rho\sinh\rho\Gamma_{13}+\dot\theta\sinh\rho\Gamma_{53}+\dot\chi_2\cos\theta\sin\theta\Gamma_{86})\sigma_3e^{\frac{\theta}{2}\varepsilon\Gamma_\star\Gamma_5}\epsilon+\\&\quad+e^{-\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}(\dot{t}_{\rm E}\cosh\rho\sin\theta\Gamma_{\rm E6}+\dot\rho\sin\theta\Gamma_{16}+\dot\theta\sin\theta\Gamma_{56}+\dot\chi_2\cos\theta\sinh\rho\Gamma_{83})\sigma_3e^{-\frac{\theta}{2}\varepsilon\Gamma_\star\Gamma_5}\epsilon\end{aligned}$$

$$\begin{aligned}&=e^{-\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}(\dot{t}_{\rm E}\cosh\rho\sinh\rho\Gamma_{\rm E3}+\dot\rho\sinh\rho\Gamma_{13}+\dot\theta\sinh\rho\Gamma_{53}+\dot\chi_2\cos\theta\sin\theta\Gamma_{86})\sigma_3\times\\&\quad\times e^{\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}}e^{\theta\varepsilon\Gamma_\star\Gamma_5}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}+\\&\quad+e^{-\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}(\dot{t}_{\rm E}\cosh\rho\sin\theta\Gamma_{\rm E6}+\dot\rho\sin\theta\Gamma_{16}+\dot\theta\sin\theta\Gamma_{56}+\dot\chi_2\cos\theta\sinh\rho\Gamma_{83})\times\\&\quad\times\sigma_3e^{\frac{\rho}{2}\varepsilon\Gamma_\star\Gamma_1}e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}\end{aligned}$$

$$\begin{aligned}&=(\dot{t}_{\rm E}\cosh\rho\sinh\rho\Gamma_{\rm E3}+\dot\chi_2\cos\theta\sin\theta\Gamma_{86})\sigma_3e^{\rho\varepsilon\Gamma_\star\Gamma_1}e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}}e^{\theta\varepsilon\Gamma_\star\Gamma_5}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}+\\&\quad+(\dot\rho\sinh\rho\Gamma_{13}+\dot\theta\sinh\rho\Gamma_{53})\sigma_3e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}}e^{\theta\varepsilon\Gamma_\star\Gamma_5}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}+\\&\quad+(\dot\rho\sin\theta\Gamma_{16}+\dot\theta\sin\theta\Gamma_{56})\sigma_3e^{\rho\varepsilon\Gamma_\star\Gamma_1}e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}+\\&\quad+(\dot{t}_{\rm E}\cosh\rho\sin\theta\Gamma_{\rm E6}+\dot\chi_2\cos\theta\sinh\rho\Gamma_{83})\sigma_3e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}.\end{aligned}$$

Then (C.1) becomes

$$\begin{aligned}\frac{i}{\sinh^2\rho+\sin^2\theta}&[(\dot{t}_{\rm E}\cosh\rho\sin\theta\Gamma_{\rm E6}+\dot\chi_2\cos\theta\sinh\rho\Gamma_{83})+\\&+e^{-\rho\varepsilon\Gamma_\star\Gamma_1}\sin\theta(\dot\rho\Gamma_{16}+\dot\theta\Gamma_{56})+\\&+e^{-\theta\varepsilon\Gamma_\star\Gamma_5}\sinh\rho(\dot\rho\Gamma_{13}+\dot\theta\Gamma_{53})+\\&+e^{-\rho\varepsilon\Gamma_\star\Gamma_1-\theta\varepsilon\Gamma_\star\Gamma_5}(\dot{t}_{\rm E}\cosh\rho\sinh\rho\Gamma_{\rm E3}+\dot\chi_2\cos\theta\sin\theta\Gamma_{86})]\sigma_3\tilde\epsilon=\tilde\epsilon,\end{aligned}\tag{C.5}$$

where $\tilde\epsilon$ is defined by

$$\tilde\epsilon=e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}}e^{\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}.$$

Using (C.3), the second and the third terms in the square bracket of (C.5) are calculated as

$$
\begin{aligned}
&e^{-\rho\varepsilon\Gamma_\star\Gamma_1}\sin\theta(\dot\rho\Gamma_{16}+\dot\theta\Gamma_{56})+e^{-\theta\varepsilon\Gamma_\star\Gamma_5}\sinh\rho(\dot\rho\Gamma_{13}+\dot\theta\Gamma_{53})\\
&\quad=\sin\theta\cosh\rho(\dot\rho\Gamma_{16}+\dot\theta\Gamma_{56})+\cos\theta\sinh\rho(\dot\rho\Gamma_{13}+\dot\theta\Gamma_{53})\\
&\quad=\frac{d}{d\tau_{\mathrm{E}}}(\sinh\rho\sin\theta\Gamma_{16}+\cosh\rho\cos\theta\Gamma_{13})+\cosh\rho\sin\theta\cdot\dot\theta(\Gamma_{13}+\Gamma_{56})+\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\sinh\rho\cos\theta\cdot\dot\theta\Gamma_{51}(\Gamma_{13}+\Gamma_{56})\\
&\quad=\frac{d}{d\tau_{\mathrm{E}}}(\sinh\rho\sin\theta\Gamma_{16}+\cosh\rho\cos\theta\Gamma_{13}).
\end{aligned}
$$

On the other hand, the first and the fourth terms are calculated as

$$
\begin{aligned}
&(\dot t_{\mathrm{E}}\cosh\rho\sin\theta\Gamma_{\mathrm{E}6}+\dot\chi_2\cos\theta\sinh\rho\Gamma_{83})+\\
&\qquad\qquad\qquad+e^{-\rho\varepsilon\Gamma_\star\Gamma_1-\theta\varepsilon\Gamma_\star\Gamma_5}(\dot t_{\mathrm{E}}\cosh\rho\sinh\rho\Gamma_{\mathrm{E}3}+\dot\chi_2\cos\theta\sin\theta\Gamma_{86})\\
&\quad=\dot t_{\mathrm{E}}\cosh\rho(e^{-\theta\varepsilon\Gamma_\star\Gamma_5}-\cos\theta)\varepsilon\Gamma_\star\Gamma_5\Gamma_{\mathrm{E}6}+\dot\chi_2\cos\theta(\cosh\rho-e^{-\rho\varepsilon\Gamma_\star\Gamma_1})\varepsilon\Gamma_\star\Gamma_1\Gamma_{83}\\
&\qquad+e^{-\rho\varepsilon\Gamma_\star\Gamma_1-\theta\varepsilon\Gamma_\star\Gamma_5}[\dot t_{\mathrm{E}}\cosh\rho(e^{\rho\varepsilon\Gamma_\star\Gamma_1}-\cosh\rho)\varepsilon\Gamma_\star\Gamma_1\Gamma_{\mathrm{E}3}+\\
&\qquad\qquad\qquad+\dot\chi_2\cos\theta(\cos\theta-e^{\theta\varepsilon\Gamma_\star\Gamma_5})\varepsilon\Gamma_\star\Gamma_5\Gamma_{86}]\\
&\quad=-\cosh\rho\cos\theta\Gamma_{13}(\dot t_{\mathrm{E}}\varepsilon\Gamma_\star\Gamma_{\mathrm{E}}+\dot\chi_2\varepsilon\Gamma_\star\Gamma_8)+\Gamma_{13}e^{\rho\varepsilon\Gamma_\star\Gamma_1-\theta\varepsilon\Gamma_\star\Gamma_5}\times\\
&\qquad\qquad\qquad\times(\dot t_{\mathrm{E}}\cosh^2\rho\varepsilon\Gamma_\star\Gamma_{\mathrm{E}}+\dot\chi_2\cos^2\theta\varepsilon\Gamma_\star\Gamma_8).
\end{aligned}
$$

Then (C.5) becomes

$$
\begin{aligned}
\frac{i}{\sinh^2\rho+\sin^2\theta}\Big[&\Gamma_{13}e^{\rho\varepsilon\Gamma_\star\Gamma_1-\theta\varepsilon\Gamma_\star\Gamma_5}(\dot t_{\mathrm{E}}\cosh^2\rho\varepsilon\Gamma_\star\Gamma_{\mathrm{E}}+\dot\chi_2\cos^2\theta\varepsilon\Gamma_\star\Gamma_8)+\\
&+\frac{d}{d\tau_{\mathrm{E}}}(\sinh\rho\sin\theta\Gamma_{16}+\cosh\rho\cos\theta\Gamma_{13})-\\
&-\cosh\rho\cos\theta\Gamma_{13}(\dot t_{\mathrm{E}}\varepsilon\Gamma_\star\Gamma_{\mathrm{E}}+\dot\chi_2\varepsilon\Gamma_\star\Gamma_8)\Big]\sigma_3\tilde\epsilon=\tilde\epsilon.
\end{aligned}
\tag{C.6}
$$

Using the equations (4.21) and (4.22), the first term in the square bracket of (C.6) can be rewritten as

$$
\Gamma_{13}e^{\rho\varepsilon\Gamma_\star\Gamma_1-\theta\varepsilon\Gamma_\star\Gamma_5}j\varepsilon\Gamma_\star(\Gamma_{\mathrm{E}}-i\Gamma_8).
$$

Here we impose the second condition (4.36),

$$
(\Gamma_{\mathrm{E}}-i\Gamma_8)\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}=0,
\tag{C.7}
$$

which corresponds to the BPS condition for the counterpart of O_J. Hence (C.6) is reduced to

$$\begin{aligned}\frac{i}{\sinh^2\rho+\sin^2\theta}\Bigg[&\frac{d}{d\tau_{\rm E}}(\sinh\rho\sin\theta\Gamma_{16}+\cosh\rho\cos\theta\Gamma_{13})\\ &-\cosh\rho\cos\theta\Gamma_{13}(\dot t_{\rm E}\varepsilon\Gamma_\star\Gamma_{\rm E}+\dot\chi_2\varepsilon\Gamma_\star\Gamma_8)\Bigg]\sigma_3\tilde\epsilon=\tilde\epsilon.\end{aligned}\tag{C.8}$$

Next we multiply $e^{-\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}-\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}$ on each side of (C.8). The left hand side is calculated as

$$\begin{aligned}&\frac{i}{\sinh^2\rho+\sin^2\theta}\Bigg[e^{-\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}-\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\frac{d}{d\tau_{\rm E}}(\sinh\rho\sin\theta\Gamma_{16}+\cosh\rho\cos\theta\Gamma_{13})-\\ &\quad-e^{-\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}-\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\cosh\rho\cos\theta\Gamma_{13}(\dot t_{\rm E}\varepsilon\Gamma_\star\Gamma_{\rm E}+\dot\chi_2\varepsilon\Gamma_\star\Gamma_8)\Bigg]\sigma_3\tilde\epsilon\\ =&\frac{i}{\sinh^2\rho+\sin^2\theta}\Bigg[\frac{d}{d\tau_{\rm E}}(\sinh\rho\sin\theta)\Gamma_{16}e^{\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}+\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\\ &\quad+\frac{d}{d\tau_{\rm E}}(\cosh\rho\cos\theta)\Gamma_{13}e^{-\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}-\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\\ &\quad-\cosh\rho\cos\theta\Gamma_{13}(\dot t_{\rm E}\varepsilon\Gamma_\star\Gamma_{\rm E}+\dot\chi_2\varepsilon\Gamma_\star\Gamma_8)e^{-\frac{t_{\rm E}}{2}\varepsilon\Gamma_\star\Gamma_{\rm E}-\frac{\chi_2}{2}\varepsilon\Gamma_\star\Gamma_8}\Bigg]\sigma_3\tilde\epsilon\\ =&\frac{i}{\sinh^2\rho+\sin^2\theta}\left[\frac{d}{d\tau_{\rm E}}(\sinh\rho\sin\theta\Gamma_{16}+\cosh\rho\cos\theta\Gamma_{13}e^{-(t_{\rm E}-i\chi_2)\varepsilon\Gamma_\star\Gamma_{\rm E}})\right]\sigma_3\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}.\end{aligned}\tag{C.9}$$

In (C.9), we further use the following two equations which we prove shortly:

$$\cosh\rho\cos\theta e^{-(t_{\rm E}-i\chi_2)\varepsilon\Gamma_\star\Gamma_{\rm E}}=\frac{\cos\theta_0}{\sin\theta_0}\sinh\rho\sin\theta+\sqrt{1+\frac{j^2}{\cos^2\theta_0}}+\frac{j}{\cos\theta_0}\varepsilon\Gamma_\star\Gamma_{\rm E},\tag{C.10}$$

$$\frac{d}{d\tau_{\rm E}}(\sinh\rho\sin\theta)=-\sin\theta_0(\sinh^2\rho+\sin^2\theta).\tag{C.11}$$

Then we obtain the third condition (4.37),

$$-i(\sin\theta_0\Gamma_{16}+\cos\theta_0\Gamma_{13})\sigma_3\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}=\begin{pmatrix}\bar\epsilon_1\\ \bar\epsilon_2\end{pmatrix}.\tag{C.12}$$

To show (C.10), we first calculate the exponential part of the left hand side:

$$e^{-(t_{\rm E}-i\chi_2)\varepsilon\Gamma_\star\Gamma_{\rm E}}=[\cosh(t_{\rm E}-i\chi_2)-\sinh(t_{\rm E}-i\chi_2)\varepsilon\Gamma_\star\Gamma_{\rm E}].$$

The calculation of $\cosh(t_\mathrm{E} - i\chi_2)$ is done by the following steps:

$$
\begin{aligned}
&\cosh(t_\mathrm{E} - i\chi_2)\\
&= \frac{1}{2}\Bigg[\left(\frac{\cosh(\sqrt{j^2+1}\tau_\mathrm{E} - \xi)}{\cosh(\sqrt{j^2+1}\tau_\mathrm{E} + \xi)} \cdot \frac{\sinh(\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) + \xi)\sinh(\sqrt{j^2+1}\tau_0 - \xi)}{\sinh(\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) - \xi)\sinh(\sqrt{j^2+1}\tau_0 + \xi)}\right)^{\frac{1}{2}} +\\
&\qquad + \left(\frac{\cosh(\sqrt{j^2+1}\tau_\mathrm{E} + \xi)}{\cosh(\sqrt{j^2+1}\tau_\mathrm{E} - \xi)} \cdot \frac{\sinh(\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) - \xi)\sinh(\sqrt{j^2+1}\tau_0 + \xi)}{\sinh(\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) + \xi)\sinh(\sqrt{j^2+1}\tau_0 - \xi)}\right)^{\frac{1}{2}}\Bigg]\\
&= \frac{1}{2}\cdot\frac{1}{D}[\cosh(\sqrt{j^2+1}\tau_\mathrm{E} - \xi)\cdot\sinh(\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) + \xi)\sinh(\sqrt{j^2+1}\tau_0 - \xi) +\\
&\qquad\qquad + \cosh(\sqrt{j^2+1}\tau_\mathrm{E} + \xi)\cdot\sinh(\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) - \xi)\sinh(\sqrt{j^2+1}\tau_0 + \xi)]\\
&= \frac{1}{2}\cdot\frac{1}{D}\cdot\frac{1}{2}[(\sinh\sqrt{j^2+1}(2\tau_\mathrm{E}+\tau_0) + \sinh(\sqrt{j^2+1}\tau_0 + 2\xi))\cdot\sinh(\sqrt{j^2+1}\tau_0 - \xi) +\\
&\qquad\qquad + (\sinh\sqrt{j^2+1}(2\tau_\mathrm{E}+\tau_0) + \sinh(\sqrt{j^2+1}\tau_0 - 2\xi))\cdot\sinh(\sqrt{j^2+1}\tau_0 + \xi)]\\
&= \frac{1}{2}\cdot\frac{1}{D}[\sqrt{j^2+1}\sinh\sqrt{j^2+1}(2\tau_\mathrm{E}+\tau_0)\sinh\sqrt{j^2+1}\tau_0\\
&\qquad\qquad\qquad\qquad + \sqrt{j^2+1}\sinh^2\sqrt{j^2+1}\tau_0 - 2j^2\sqrt{j^2+1}]\\
&= \frac{1}{D}\left[\frac{(j^2+1)^{\frac{3}{2}}}{\sinh\rho\sin\theta}\sinh\sqrt{j^2+1}\tau_0 + (j^2+1)^{\frac{3}{2}}\frac{\cos^2\theta_0}{\sin^2\theta_0}\right].
\end{aligned}
$$

Here we used the relation

$$
\frac{1}{\sinh\rho\sin\theta} = \frac{1}{j^2+1}\cdot\frac{1}{2}(\sinh\sqrt{j^2+1}(2\tau_\mathrm{E}+\tau_0) - \sinh\sqrt{j^2+1}\tau_0) \tag{C.13}
$$

and

$$
\frac{\cos^2\theta}{\sin^2\theta}\cdot(j^2+1) = \sinh^2\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) - j^2 \tag{C.14}
$$

to obtain the last line. Using the equations (C.13), (C.14) and

$$
\frac{\cosh^2\rho}{\sinh^2\rho}\cdot(j^2+1) = \cosh^2\sqrt{j^2+1}\tau_\mathrm{E} + j^2, \tag{C.15}
$$

the calculation of the common denominator D is

$$
\begin{aligned}
D &= [\cosh(\sqrt{j^2+1}\tau_\mathrm{E} + \xi)\cosh(\sqrt{j^2+1}\tau_\mathrm{E} - \xi)\sinh((\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) - \xi)\times\\
&\qquad \times\sinh((\sqrt{j^2+1}(\tau_\mathrm{E}+\tau_0) + \xi)\sinh(\sqrt{j^2+1}\tau_0 + \xi)\sinh(\sqrt{j^2+1}\tau_0 - \xi)]^{\frac{1}{2}}\\
&= [(\cosh^2\sqrt{j^2+1}\tau_\mathrm{E}\cdot(j^2+1) - \sinh^2\sqrt{j^2+1}\tau_\mathrm{E}\cdot j^2)\times
\end{aligned}
$$

$$\times (\sinh^2 \sqrt{j^2+1}(\tau_{\rm E}+\tau_0)\cdot(j^2+1) - \cosh^2 \sqrt{j^2+1}(\tau_{\rm E}+\tau_0)\cdot j^2)\times$$
$$\times (\sinh^2 \sqrt{j^2+1}\tau_0\cdot(j^2+1) - \cosh^2 \sqrt{j^2+1}\tau_0\cdot j^2)]^{\frac{1}{2}}$$
$$= [(\cosh^2 \sqrt{j^2+1}\tau_{\rm E}+j^2)(\sinh^2 \sqrt{j^2+1}(\tau_{\rm E}+\tau_0)-j^2)(\sinh^2 \sqrt{j^2+1}\tau_0-j^2)]^{\frac{1}{2}}$$
$$= (j^2+1)^{\frac{3}{2}}\frac{\cosh\rho}{\sinh\rho}\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta_0}{\sin\theta_0}.$$

Therefore, $\cosh(t_{\rm E}-i\chi_2)$ becomes

$$\begin{aligned}\cosh(t_{\rm E}-i\chi_2) &= \frac{1}{\cosh\rho\cos\theta}\left[\frac{\cos\theta_0}{\sin\theta_0}\sinh\rho\sin\theta + \frac{\sin\theta_0}{\cos\theta_0}\sinh\sqrt{j^2+1}\tau_0\right] \\ &= \frac{1}{\cosh\rho\cos\theta}\left[\frac{\cos\theta_0}{\sin\theta_0}\sinh\rho\sin\theta + \sqrt{1+\frac{j^2}{\cos^2\theta_0}}\right], \end{aligned} \tag{C.16}$$

because $\sinh^2\sqrt{j^2+1}\tau_0$ is rewritten as

$$\sinh^2\sqrt{j^2+1}\tau_0 = \frac{j^2+\cos^2\theta_0}{\sin^2\theta_0}.$$

Similarly, the calculation of $\sinh(t_{\rm E}-i\chi_2)$ is given by

$$\begin{aligned}&\sinh(t_{\rm E}-i\chi_2) \\ &= \frac{1}{2}\Bigg[\left(\frac{\cosh(\sqrt{j^2+1}\tau_{\rm E}-\xi)}{\cosh(\sqrt{j^2+1}\tau_{\rm E}+\xi)}\cdot\frac{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)+\xi)\sinh(\sqrt{j^2+1}\tau_0-\xi)}{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)-\xi)\sinh(\sqrt{j^2+1}\tau_0+\xi)}\right)^{\frac{1}{2}} - \\ &\quad - \left(\frac{\cosh(\sqrt{j^2+1}\tau_{\rm E}+\xi)}{\cosh(\sqrt{j^2+1}\tau_{\rm E}-\xi)}\cdot\frac{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)-\xi)\sinh(\sqrt{j^2+1}\tau_0+\xi)}{\sinh(\sqrt{j^2+1}(\tau_{\rm E}+\tau_0)+\xi)\sinh(\sqrt{j^2+1}\tau_0-\xi)}\right)^{\frac{1}{2}}\Bigg] \\ &= \frac{1}{2}\cdot\frac{1}{D}[-j\cosh\sqrt{j^2+1}\tau_0(\sinh\sqrt{j^2+1}(2\tau_{\rm E}+\tau_0)-\sinh\sqrt{j^2+1}\tau_0)] \\ &= -\frac{1}{D}\cdot j(j^2+1)\cosh\sqrt{j^2+1}\tau_0\cdot\frac{1}{\sinh\rho\sin\theta} \\ &= -\frac{1}{\cosh\rho\cos\theta}\cdot\frac{j}{\cos\theta_0}, \end{aligned} \tag{C.17}$$

where D is the same common denominator in the case of $\cosh(t_{\rm E}-i\chi_2)$. Therefore, substituting (C.16) and (C.17) into the left hand side, (C.10) is shown.

Next we show the relation (C.11). It is done by straightforward calculation:

$$
\begin{aligned}
\frac{d}{d\tau_{\mathrm{E}}}(\sinh\rho\sin\theta) &= -\sinh^2\rho\sin\theta\cosh\sqrt{j^2+1}\tau_{\mathrm{E}} - \sin^2\theta\sinh\rho\sinh\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0) \\
&= -\sinh^2\rho\sin\theta_0\frac{\cosh\sqrt{j^2+1}\tau_0}{\cosh\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)}\cosh\sqrt{j^2+1}\tau_{\mathrm{E}} - \\
&\quad -\sin^2\theta\sin\theta_0\frac{\cosh\sqrt{j^2+1}\tau_0}{\sinh\sqrt{j^2+1}\tau_{\mathrm{E}}}\sinh\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0) \\
&= -\sin\theta_0(\sinh^2\rho+\sin^2\theta)+ \\
&\quad +\sin\theta_0\sinh^2\rho\left(1-\frac{\cosh\sqrt{j^2+1}\tau_0\cosh\sqrt{j^2+1}\tau_{\mathrm{E}}}{\cosh\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)}\right)+ \\
&\quad +\sin\theta_0\sin^2\theta\left(1-\frac{\cosh\sqrt{j^2+1}\tau_0\sinh\sqrt{j^2+1}(\tau_{\mathrm{E}}+\tau_0)}{\sinh\sqrt{j^2+1}\tau_{\mathrm{E}}}\right) \\
&= -\sin\theta_0(\sinh^2\rho+\sin^2\theta).
\end{aligned}
$$

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