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CLASSIFICATION OF QUASIHOMOGENEOUS POLYNOMIALS WITH INNER MODALITY=ARITHMETIC INNER MODALITY

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This article is dedicated to the memory of Professor E. Yoshinaga, 1945–1994.

Abstract. For quasihomogeneous polynomials with isolated singularity, V.I.Arnold introduced the notion of inner modality and classified them with inner modality $= 0, 1$ in [1]. After that, E.Yoshinaga, M.Suzuki, J. Estrada Sarlabous, J. Arocha and A. Fuentes classified them with inner modality ≤ 9 (see [18], [11], [7], [15]). The author introduced a concept of *arithmetic inner modality* for quasihomogeneous polynomials with isolated singularity in [15], and he observed that these two invariants match each other for quasihomogeneous polynomials with inner modality ≤ 9, and also he found examples with inner modality $= 10$ for which two invariants don't match (see [14]). We are interested in how many quasihomogeneous polynomials with the same inner modality as arithmetic inner modality.

The purpose of this paper is to give the complete classification of quasihomogeneous polynomials with the same inner modality as arithmetic inner modality.

1. Preliminaries

In this section, we explain the terms and the results used in this article.

A local analytic function $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$, that is $f \in \mathfrak{M} \subset \mathbb{C}\{x_1, \cdots, x_n\}$, has *an isolated singularity* if

$$\left\{x \;\middle|\; \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0\right\} = \{\mathbf{0}\}$$

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locally, where $\mathfrak{M}$ is the maximal ideal of the local ring $\mathbb{C}\{x_1, \cdots, x_n\}$. We denote the ideal $\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ of the ring $\mathbb{C}\{x_1, \ldots, x_n\}$ by $\Delta(f)$ and denote the quotient ring $\mathbb{C}\{x_1, \ldots, x_n\}/\Delta(f)$ by $\mathcal{R}_f$. Note that by Hilbert's Nullstellensatz, if f has an isolated singularity at the origin then $\mathfrak{M}^p \subset \Delta(f) \subset \mathbb{C}\{x_1, \ldots, x_n\}$ for some $p \in \mathbb{N}$, thus the dimension of $\mathcal{R}_f$ over $\mathbb{C}$ is finite. We call it *the Milnor number* of f at the origin and it is denoted by the notation $\mu(f)$. It is well known that a local analytic function with isolated singularity is a polynomial up to a suitable local coordinate transformation (see [5], [17]) and hence we will study "quasihomogeneous" polynomials with isolated singularity at the origin.

For positive rational numbers $r_1, \ldots, r_n \in \mathbb{Q}^+$, a monomial

$$\mathfrak{m} = x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}[x_1, \ldots, x_n] \quad (i_1, \ldots, i_n \in \mathbb{N} \cup \{0\})$$

has *generalized degree* d if $r_1 i_1 + \cdots + r_n i_n = d$ and we denote the generalized degree of $\mathfrak{m}$ by $\mathrm{gdeg}(\mathfrak{m})$. A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is *quasihomogeneous of type* $(d; r_1, \ldots, r_n)$ if each monomial term of f with non-zero coefficient has generalized degree d. Then we call the number d the generalized degree of f and call r_i's the weights of f. Also a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is *semi-quasihomogeneous of type* $(d; r_1, \ldots, r_n)$ if it is of the form $f = f_0 + g$, where f_0 is a quasihomogeneous polynomial of type $(d; r_1, \ldots, r_n)$ with isolated singularity at the origin and any term of g has a generalized degree greater than d for weights $(r_1, \ldots, r_n)$. Note that it is shown that $\mu(f) = \mu(f_0)$ (see 3.1 Theorem in [1]).

A local analytic function $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ has *a quasihomogeneous singularity* at the origin if f becomes a quasihomogeneous polynomial after a suitable local coordinate transformation.

Theorem 1.1 ([8], Satz 1.3). *Suppose that $f \in \mathfrak{M} \subset \mathbb{C}\{x_1, \ldots, x_n\}$ has a quasihomogeneous isolated singularity. Then there exist a coordinate system $(y_1, \ldots, y_n)$ in which f has the form*

$$f = h(y_1, \ldots, y_k) + y_{k+1}^2 + \cdots + y_n^2$$

with a quasihomogeneous polynomial $h \in \mathbb{C}[y_1, \ldots, y_k]$ of type $(1; s_1, \ldots, s_k)$ $(0 < s_i < \frac{1}{2}, i = 1, \ldots, k)$. The natural number k and $(s_1, \ldots, s_k)$ are uniquely determined up to permutations of components.

We call the number k *the corank of* f and call the polynomial h *the residual part* of f. We denote the corank of f by $\mathrm{corank}(f)$. In order to classify them, it is sufficient to classify their residual parts.

The following proposition and theorem are frequently used in this article.

Proposition 1.2 ([8], Korollar 1.6). *Let f be the residual part of a quasihomogeneous polynomial with isolated singularity at the origin. Then for any i ($i = 1, \cdots, k$), there are integers m_i (≥ 2) and j_i ($j_i = 1, \ldots, k$) such that f contains the monomial $x_i^{m_i} x_{j_i}$ as a term.*

Note that we have $\frac{1}{2} < m_i r_i < 1$ since $\frac{1}{2} < 1 - r_{j_i} < 1$.

Theorem 1.3 ([9], 2.12. Proposition). *Let f be the residual part of a quasihomogeneous polynomial with isolated singularity at the origin and suppose that f have the type $(1; r_1, \ldots, r_k)$ ($0 < r_i < \frac{1}{2}$, $i = 1, \ldots, k$). Then*

$$\sum_{i=1}^{k} r_i \leq \frac{k}{3}.$$

The inequality in the above theorem is derived from Proposition 2.12 in K. Saito [9]. From this theorem it follows easily that the minimum of r_i is less than or equal to $\frac{1}{3}$.

From now on let f be a quasihomogeneous polynomial of type $(1; r_1, \ldots, r_n)$ with isolated singularity at the origin. Then the number of basis monomials of $\mathcal{R}_f$ is the same for all (semi-)quasihomogeneous polynomials f of the same type as follows.

Theorem 1.4 ([1], 4.5. Theorem). *Let $r_1, \ldots, r_n$ be positive rational numbers for which $r_i = A_i/N$ ($i = 1, \ldots, n$), where N and A_i's are positive natural numbers. If f is a (semi-)quasihomogeneous polynomial of type $(1; r_1, \ldots, r_n)$, then*

$$\sum \mu_j z^j = \prod_{j=1}^{n} \frac{z^{N-A_j} - 1}{z^{A_j} - 1},$$

where μ_i is the number of basis monomials in $\mathcal{R}_f$ with generalized degree i/N.

We denote the right side of the equality in the above theorem by $\chi_f(z)$ and we call it *the characteristic function of f*, and when it becomes a polynomial, it is especially called *the characteristic polynomial of f*. The following result follows immediately from this theorem.

Corollary 1.5 ([1], 4.6. Corollary). *The Milnor number $\mu(f)$ is given by the formula*

$$\mu(f) = \chi_f(1) = \prod_{j=1}^{n} \left(\frac{1}{r_j} - 1 \right).$$

By the above theorem we can define the following.

Definition 1.1 ([1], 8.6. Definition)**.** The number of basis monomials of $\mathcal{R}_f$ with generalized degree ≥ 1 is called the *inner modality* of f and it is denoted by $m_0(f)$.

From Theorem 1.4 we see that the highest degree of generalized degrees of basis monomials of $\mathcal{R}_f$ is $n - 2\sum r_i$ and it is denoted by d_f. The coefficients of $\chi_f(z)$ are symmetric because it is the product of cyclotomic polynomials. Hence we have

$$m(f) = \sum_{j \geq N} \mu_j = \sum_{j \leq D-N} \mu_j,$$

where $D = nN - 2\sum A_j = N(n - 2\sum r_j) = Nd_f$.

For a (semi-)quasihomogeneous polynomial f of type $(1; r_1, \ldots, r_n)$, we define the following invariant expressed in terms of their weights.

Definition 1.2. We call the number of monomials in $\mathbb{C}[x_1, \ldots, x_n]$ with generalized degree $\leq d_f - 1$ *the arithmetic inner modality* of f and it is denoted it by $m_a(f)$.

By the definition, we have $m_0(f) \leq m_a(f)$ in general. If the images of monomials in $\mathcal{R}_f$ with gdeg$\leq d_f - 1$ are linearly independent, we have $m_0(f) = m_a(f)$.

2. Preparations

In this section, we give a necessary and sufficient condition for $m_0(f) = m_a(f)$, and we show that corank$(f) \leq 4$ for f with $m_0(f) = m_a(f)$ in 2.1. Further for each value of corank(f), we give a limit of values of exponents of f in 2.2.

2.1. **Corank limit.** In this subsection, we consider a quasihomogeneous polynomial f of type $(1; r_1, \ldots, r_k)$ $(0 < r_1, \ldots, r_k < \frac{1}{2})$ with isolated singularity at the origin. First we prove the following proposition.

Proposition 2.1. *We have* $m_0(f) = m_a(f)$ *if and only if*

$$\mathrm{gdeg}\left(\frac{\partial f}{\partial x_i}\right) > d_f - 1$$

for any i $(i = 1, \ldots, k)$.

Proof. First we prove that the condition is sufficient. Let $l := m_a(f)$ and let $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_l\}$ be the set of monomials in $\mathbb{C}[x_1, \ldots, x_k]$ with gdeg $\leq d_f - 1$. Since $m_0(f) \leq m_a(f)$ by the definition of $m_a(f)$, it is enough to show that $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_l\}$ is linearly independent over $\mathbb{C}$ in $\mathcal{R}_f$.

Let $\lambda_1, \ldots, \lambda_l$ be complex numbers such that $\lambda_1 \mathfrak{m}_1 + \ldots + \lambda_l \mathfrak{m}_l \in \Delta(f)$. Suppose that there exist some i $(i = 1, \ldots, k)$ such that $\lambda_i \neq 0$. Then we obtain

$$\min_{j=1,\ldots k} \operatorname{gdeg}\left(\frac{\partial f}{\partial x_j}\right) \leq \operatorname{gord}(\lambda_1 \mathfrak{m}_1 + \ldots + \lambda_l \mathfrak{m}_l) \leq \operatorname{gdeg}(\mathfrak{m}_i) \leq d_f - 1,$$

where gord( ) means the generalized order of a polynomial which is defined as the smallest generalized degree of each term of it. However it contradicts the hypothesis and thus $\lambda_1 = \ldots = \lambda_l = 0$.

Next we prove that the condition is necessary. Suppose that there exists some i such that $\operatorname{gdeg}\left(\frac{\partial f}{\partial x_i}\right) \leq d_f - 1$. Then $\frac{\partial f}{\partial x_i}$ is quasihomogeneous of type $(1 - r_i; r_1, \ldots, r_k)$ and by Proposition 1.2 it contains a monomial $x_i^{m_i - 1} x_{j_i}$ with non-zero coefficient for some m_i $(m_i = 2, 3, \ldots)$ and some j_i $(j_i = 1, \ldots k)$. On the other hand, $\frac{\partial f}{\partial x_i} \equiv 0$ in $\mathcal{R}_f$ is obious and this means that there exists a linear dependent relation in $\mathcal{R}_f$ among the images of monomials with gdeg $\leq d_f - 1$ and thus $m_0(f) \neq m_a(f)$. This completes the proof. □

The condition of the above proposition can be rewritten as follows.

Corollary 2.2. *Let* $r_{max} := \max\{r_1, \ldots, r_k\}$. *Then we have* $m_0(f) = m_a(f)$ *if and only if*

$$r_1 + r_2 + \cdots + r_{k-1} + r_k - \frac{1}{2} r_{max} > \frac{k}{2} - 1. \tag{2.1}$$

Proof. The inequality $\operatorname{gdeg}\left(\frac{\partial f}{\partial x_i}\right) > d_f - 1$ $(i = 1, \ldots, k)$ means

$$1 - r_i > k - 1 - 2(r_1 + \cdots r_k) \quad \text{for} \quad \forall i \in \{1, \ldots, k\}.$$

This condition is clearly equivalent to

$$1 - r_{max} > k - 1 - 2(r_1 + \cdots r_k).$$

Hence we have the conclusion. □

In what follows, we denote the left hand side of the inequality (2.1) by $\alpha(f)$.

The next proposition shows the upper limit of corank(f) necessary for our classification.

Proposition 2.3. *If* $m_0(f) = m_a(f)$, *then* $\operatorname{corank}(f) = k \leq 4$.

Proof. Assume that $r_1 \leq \cdots \leq r_k$. By Corollary 2.2, we have

$$r_1 + r_2 + \cdots + r_{k-1} + \frac{r_k}{2} > \frac{k}{2} - 1.$$

Hence

$$(k-1)r_k + \frac{1}{2}r_k > \frac{k}{2} - 1, \quad r_k > \frac{k-2}{2k-1}.$$

On the other hand, by Saito's inequality (see Theorem 1.3) we have

$$\frac{1}{3}k - \frac{1}{2}r_k > r_1 + \cdots + r_{k-1} + \frac{1}{2}r_k > \frac{k}{2} - 1, \quad r_k < 2 - \frac{k}{3}$$

From the above two inequality with r_k, we have

$$\frac{k-2}{2k-1} < r_k < 2 - \frac{k}{3}, \text{ hence } k < 5$$

This completes the proof. □

From the above proposition, wee see that it is enough to consider the case of corank ≤ 4 to classify quasihomogeneous polynomials with $m_a = m_0$.

2.2. **Estimation of exponents.** If f is quasihomogeneous of corank $= k$ with isolated singularity at the origin, then from Proposition 1.2 f contains k monomials

$$x_1^{m_1}x_{j_1}, \ldots, x_k^{m_k}x_{j_k} \quad (m_1, \ldots, m_k \geq 2, j_1, \ldots j_k \in \{1, 2, \ldots, k\})$$

with non-zero coefficients. Conversely the weights of a quasihomogeneous polynomial containing such monomials is uniquely determined. Hence in order to determine quasihomogeneous polynomials having a certain property, it is sufficient to determine all quasihomogeneous polynomials containing monomials $x_1^{m_1}x_{j_1}$, …, $x_k^{m_k}x_{j_k}$ $(m_1, \ldots, m_k \geq 2)$ with the same property and isolated singularity at the origin. Hence in order to determine quasihomogeneous polynomials with $m_0 = m_a$, we first need to determine bounds of exponent $m_1, \ldots, m_k$ of f that satisfiies the condition $m_0(f) = m_a(f)$. In what follow, we consider a quasihomogeneous polynomial f of type $(1; r_1, \ldots, r_k)$ with isolated singularity at the origin and assume that it contains monomials $x_1^{m_1}x_{j_1}$, …, $x_k^{m_k}x_{j_k}$ with non-zero coefficients.

Lemma 2.4. *Assume that* corank$(f) = 1$, 2. *Then we have always* $m_a(f) = m_0(f)$ *for all* f.

Proof. The consequence is immediate from the inequality (2.1) in *Corollary* 2.2. □

Lemma 2.5. *Assume that* $\operatorname{corank}(f) = 3$ *and* $r_1 \le r_2 \le r_3$. *If* $m_0(f) = m_a(f)$, *then we have*

$$(m_2, m_3) = \begin{cases} (i,2), & i = 2,\dots,7 \\ (i,3), & i = 2,\dots,5 \\ (i,4), & i = 2,\dots,5. \end{cases}$$

In the case of $(m_2, m_3) \ne (2,2), (2,3), (3,2)$, *we have*

$$m_1 = \begin{cases} 2,\dots,13 & \textit{if } (m_2,m_3) = (2,4),(4,2). \\ 2,\dots,9 & \textit{if } (m_2,m_3) = (3,3). \\ 2,\dots,19 & \textit{if } (m_2,m_3) = (5,2). \\ 2,\dots,11 & \textit{if } (m_2,m_3) = (6,2),(4,3). \\ 2,\dots,9 & \textit{if } (m_2,m_3) = (7,2). \\ 2,\dots,7 & \textit{if } (m_2,m_3) = (5,3),(4,4). \\ 2,\dots,23 & \textit{if } (m_2,m_3) = (3,4). \\ 2,\dots,5 & \textit{if } (m_2,m_3) = (5,4). \end{cases}$$

In the case of $(m_2, m_3) = (2,2)$, *we have*

$$m_1 = \begin{cases} 2,3,4 & \textit{if } (j_1,j_2,j_3) = (1,1,1),(2,1,1),(3,1,1), \\ 2,3,\dots & \textit{otherwise.} \end{cases}$$

In the case of $(m_2, m_3) = (2,3)$, *we have*

$$m_1 = \begin{cases} 2,3,\dots & \textit{if } (j_1,j_2,j_3) = (1,1,3),(1,2,1),(1,3,1),(2,1,3) \\ & \qquad (2,2,1),(2,3,1),(3,1,3),(3,2,1) \\ & \qquad (3,3,1), \\ 2,\dots,14 & \textit{otherwise.} \end{cases}$$

In the case of $(m_2, m_3) = (3,2)$, *we have*

$$m_1 = \begin{cases} 2,3,\dots & \textit{if } (j_1,j_2,j_3) = (1,1,2),(1,1,3),(1,2,1),(2,1,2) \\ & \qquad (2,1,3),(2,2,1),(3,1,2),(3,1,3) \\ & \qquad (3,2,1), \\ 2,\dots,14 & \textit{otherwise.} \end{cases}$$

Proof. Since $m(f) = m_0(f)$ and $r_1 \le r_2 \le r_3$, from Corollary 2.2, we have

$$r_1 + r_2 + \frac{1}{2} r_3 > \frac{1}{2}. \tag{2.2}$$

Using the previously introduced symbol $\alpha(f)$, the condition (2.2) can be expressed as $\alpha(f) > \frac{1}{2}$. We try to find the bounds of m_1, m_2, m_3 under this condition. Since $r_1 \leq r_2 \leq r_3$, it follows that

$$\frac{5}{2}\frac{1}{m_3} \geq \frac{5}{2}r_3 \geq \alpha(f) > \frac{1}{2}, \quad m_3 < 5, \text{ namely } m_3 = 2, 3, 4.$$

Since $r_1 \leq r_2$, it follows that

$$2\frac{1}{m_2} + \frac{1}{2}\frac{1}{m_3} \geq 2r_2 + \frac{1}{2}r_3 \geq \alpha(f) > \frac{1}{2}, \quad m_2 < 4\left(1 - \frac{1}{m_3}\right)^{-1}.$$

Then

$$m_2 < \begin{cases} 8 & (m_3 = 2). \\ 6 & (m_3 = 3). \\ 5 + \dfrac{1}{3} & (m_3 = 4). \end{cases}$$

Hence we have

$$\begin{aligned} (m_2, m_3) =& (2,2), (3,2), (4,2), (5,2), (6,2), (7,2), \\ & (2,3), (3,3), (4,3), (5,3), \\ & (2,4), (3,4), (4,4), (5,4). \end{aligned}$$

Since $r_i \leq \dfrac{1}{m_i}$ $(i = 1, 2, 3)$,

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{2}\frac{1}{m_3} \geq \alpha(f) > \frac{1}{2},$$

and

$$\frac{1}{m_1} > \frac{1}{2} - \frac{1}{m_2} - \frac{1}{2}\frac{1}{m_3}.$$

Note that

$$\begin{aligned} \frac{1}{2} - \frac{1}{m_2} - \frac{1}{2}\frac{1}{m_3} \leq 0 \iff& (m_2 - 2)(m_3 - 1) \leq 2 \\ \iff& (m_2, m_3) = (2,2), (2,3), (2,4), (3,2), \\ & \qquad (3,3), (4,2). \end{aligned}$$

Then we have

$$m_1 < \left(\frac{1}{2} - \frac{1}{m_2} - \frac{1}{2}\frac{1}{m_3}\right)^{-1},$$

whenever $(m_2, m_3) \neq (2,2), (2,3), (2,4), (3,2), (3,3), (4,2)$, namely $(m_2, m_3) = (5,2), (6,2), (7,2), (4,3), (5,3), (3,4), (4,4), (5,4)$. Hence we have

$$m_1 = \begin{cases} 2, \dots, 19 & \text{if } (m_2, m_3) = (5,2). \\ 2, \dots, 11 & \text{if } (m_2, m_3) = (6,2), (4,3). \\ 2, \dots, 9 & \text{if } (m_2, m_3) = (7,2). \\ 2, \dots, 7 & \text{if } (m_2, m_3) = (5,3), (4,4). \\ 2, \dots, 23 & \text{if } (m_2, m_3) = (3,4). \\ 2, \dots, 5 & \text{if } (m_2, m_3) = (5,4). \end{cases}$$

Next we consider the cases of $(m_2, m_3) = (2,2), (2,3), (2,4), (3,2), (3,3), (4,2)$. Since f has an isolated singularity at the origin, it follows from Cororally 1.5 that for each case, the Milnor number must be a natural number, namely

$$\mu(f) = \left(\frac{1}{r_1} - 1\right)\left(\frac{1}{r_2} - 1\right)\left(\frac{1}{r_3} - 1\right) \in \mathbb{N}. \tag{2.3}$$

From here we try to find the bound of m_1 by adding the condition (2.3) in addition to (2.2). To do this, for all permutations (j_1, j_2, j_3) of the set $\{1,2,3\}$ with repetition, we try to calculate (r_1, r_2, r_3), $\alpha(f)$ and $\mu(f)$ for the quasihomogeneous polynomial $f(x_1, x_2, x_3)$ contains monomials $x_1^{m_1} x_{j_1}$, $x_2^{m_2} x_{j_2}$, $x_3^{m_3} x_{j_3}$ with non-zero coefficients, and try to find the bound of m_1. We show such a calculation by an example.

As an example, in the case of $(m_2, m_3) = (3,2)$, if $(j_1, j_2, j_3) = (1,2,3)$, then we have

$$\begin{aligned} (r_1, r_2, r_3) &= \left(\frac{1}{m_1+1}, \frac{1}{4}, \frac{1}{3}\right), \\ r_{max} &= r_3 \\ \alpha(f) - \frac{1}{2} &= r_1 + r_2 + r_3 - \frac{1}{2} r_{max} - \frac{1}{2} \\ &= r_1 + r_2 + \frac{1}{2} r_3 - \frac{1}{2} = \frac{11 - m_1}{12(m_1+1)}, \\ \mu(f) &= 6m_1 \in \mathbb{N}. \end{aligned}$$

Since $\alpha(f)-\frac{1}{2}>0$ by Corollary 2.2, hence we have $m_1<11$. If $(j_1,j_2,j_3)=(1,1,1)$, then we have

$$\begin{aligned}(r_1,r_2,r_3)&=\left(\frac{1}{m_1+1},\frac{m_1}{3(m_1+1)},\frac{m_1}{2(m_1+1)}\right),\\ r_{max}&=r_3\\ \alpha(f)-\frac{1}{2}&=r_1+r_2+\frac{1}{2}r_3-\frac{1}{2}=\frac{m_1+6}{12(m_1+1)}>0,\\ \mu(f)&=\frac{(m_1+2)^2}{m_1}=2m_1+7+\frac{6}{m_1}.\end{aligned}$$

Since $\mu\in\mathbb{N}$, we have $m_1=2,3,6$. If $(j_1,j_2,j_3)=(1,1,2)$, then

$$\begin{aligned}(r_1,r_2,r_3)&=\left(\frac{1}{m_1+1},\frac{m_1}{3(m_1+1)},\frac{2m_1+3}{6(m_1+1)}\right)\\ r_{max}&=r_3\\ \alpha(f)-\frac{1}{2}&=r_1+r_2+\frac{1}{2}r_3-\frac{1}{2}=\frac{3}{4(m_1+1)}>0\\ \mu(f)&=4m_1+3\in\mathbb{N}\end{aligned}$$

Since $\alpha(f)-\frac{1}{2}>0$ and $\mu(f)\in\mathbb{N}$ for any $m_1(m_1=2,3,\dots)$, m_1 is free. By doing same calculations for the other permutations (j_1,j_2,j_3), it follows that for $(m_2,m_3)=(3,2)$

$$m_1=\begin{cases}2,3,\dots & \text{if } (j_1,j_2,j_3)=(1,1,2),(1,1,3),(1,2,1),(2,1,2)\\ & \qquad (2,1,3),(2,2,1),(3,1,2),(3,1,3)\\ & \qquad (3,2,1),\\ 2,\dots,14 & \text{otherwise.}\end{cases}$$

We performe this calculation by a computer (see a sample program in Appendix A).

We perform similar calculations for the other values of (m_2,m_3) and each purmutation (j_1,j_2,j_3) of $\{1,2,3\}$ with repetition by a computer. As a result, if follows that for $(m_2,m_3)=(2,3)$

$$m_1=\begin{cases}2,3,\dots & \text{if } (j_1,j_2,j_3)=(1,1,3),(1,2,1),(1,3,1),(2,1,3)\\ & \qquad (2,2,1),(2,3,1),(3,1,3),(3,2,1)\\ & \qquad (3,3,1),\\ 2,\dots,14 & \text{otherwise,}\end{cases}$$

and for $(m_2, m_3) = (2, 2)$,

$$m_1 = \begin{cases} 2, 3, 4 & \text{if } (j_1, j_2, j_3) = (1, 1, 1), (2, 1, 1), (3, 1, 1), \\ 2, 3, \dots & \text{otherwise.} \end{cases}$$

Similarly it follows that for $(m_2, m_3) = (2, 4), (3, 3), (4, 2)$

$$m_1 = \begin{cases} 2, \dots, 13 & \text{if } (m_2, m_3) = (2, 4), (4, 2), \\ 2, \dots, 9 & \text{if } (m_2, m_3) = (3, 3). \end{cases}$$

This completes the proof. □

Lemma 2.6. *Assume that* corank$(f) = 4$ *and* $r_1 \le r_2 \le r_3 \le r_4$. *If* $m_0(f) = m_a(f)$, *then we have*

$$(m_2, m_3, m_4) = \begin{cases} (i, 2, 2) & i = 2, 3, 4, 5, 6, 7. \\ (i, 3, 2) & i = 2, 3, 4. \\ (i, 2, 3) & i = 2, 3, 4, 5. \\ (i, 3, 3) & i = 2, 3 \end{cases}$$

$$m_1 = \begin{cases} 2, \dots, 8 & \textit{if } (m_2, m_3, m_4) = (2, 2, 2). \\ 2, \dots, 12 & \textit{if } (m_2, m_3, m_4) = (2, 2, 3), (2, 3, 2), (3, 2, 2). \\ 2, 3 & \textit{if } (m_2, m_3, m_4) = (2, 3, 3), (3, 2, 3), (4, 2, 2). \\ 2, \dots, 19 & \textit{if } (m_2, m_3, m_4) = (5, 2, 2), \\ 2, \dots, 11 & \textit{if } (m_2, m_3, m_4) = (6, 2, 2), (3, 3, 2), (4, 2, 3) \\ 2, \dots, 9 & \textit{if } (m_2, m_3, m_4) = (7, 2, 2). \\ 2, \dots, 5 & \textit{if } (m_2, m_3, m_4) = (4, 3, 2), (3, 3, 3). \\ 2, \dots, 7 & \textit{if } (m_2, m_3, m_4) = (5, 2, 3). \end{cases}$$

Proof. Since $m(f) = m_0(f)$ and $r_1 \le r_2 \le r_3 \le r_4$, from Corollary 2.2, we have

$$r_1 + r_2 + r_3 + \frac{1}{2} r_4 > 1. \tag{2.4}$$

Using the previously introduced symbol $\alpha(f)$, the condition (2.2) can be expressed as $\alpha(f) > 1$. We try to find the bounds of m_1, m_2, m_3, m_4 under this condition. Since $r_1 \le r_2 \le r_3 \le r_4$, it follows that

$$\frac{7}{2} \frac{1}{m_4} \ge \frac{7}{2} r_4 > \alpha(f) > 1, \quad m_4 < \frac{7}{2}, \text{ namely } m_4 = 2, 3.$$

Since $r_1 \le r_2 \le r_3$, it follows that

$$3 \frac{1}{m_3} + \frac{1}{2} \frac{1}{m_4} \ge 3 r_3 + \frac{1}{2} r_4 \ge \alpha(f) > 1, \quad m_3 < 3 \left(1 - \frac{1}{2m_4}\right)^{-1}.$$

Hence $m_3 = 2, 3$, and we have

$$(m_3, m_4) = (2,2), (3,2), (2,3), (3,3).$$

Since $r_1 \le r_2$, it follows that

$$2\frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{2}\frac{1}{m_4} \ge 2r_2 + r_3 + \frac{1}{2}r_4 \ge \alpha(f) > 1$$

Hence we have

$$m_2 < 2\left(1 - \frac{1}{m_3} - \frac{1}{2m_4}\right)^{-1} = \begin{cases} 8 & \text{if } (m_3, m_4) = (2,2), \\ 4 + \frac{4}{5} & \text{if } (m_3, m_4) = (3,2), \\ 6 & \text{if } (m_3, m_4) = (2,3), \\ 4 & \text{if } (m_3, m_4) = (3,3). \end{cases}$$

Since $r_i \le \dfrac{1}{m_i}$ $(i = 1,2,3,4)$, we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{2}\frac{1}{m_4} \ge \alpha(f) > 1,$$

and

$$m_1 < \left(1 - \frac{1}{m_2} - \frac{1}{m_3} - \frac{1}{2m_4}\right)^{-1},$$

whenever $(m_2, m_3, m_4) \ne (2,2,2), (3,2,2), (4,2,2), (2,3,2), (2,2,3), (2,3,3), (3,2,3)$, namely $(m_2, m_3, m_4) = (5,2,2), (6,2,2), (3,3,2), (4,2,3), (7,2,2), (4,3,2), (3,3,3), (5,2,3)$. Hence we have

$$m_1 = \begin{cases} 2, \dots, 19 & \text{if } (m_2, m_3, m_4) = (5,2,2), \\ 2, \dots, 11 & \text{if } (m_2, m_3, m_4) = (6,2,2), (3,3,2), (4,2,3) \\ 2, \dots, 9 & \text{if } (m_2, m_3, m_4) = (7,2,2). \\ 2, \dots, 5 & \text{if } (m_2, m_3, m_4) = (4,3,2), (3,3,3). \\ 2, \dots, 7 & \text{if } (m_2, m_3, m_4) = (5,2,3). \end{cases}$$

Next we consider the cases of $(m_2, m_3, m_4) = (2,2,2), (3,2,2), (4,2,2), (2,3,2), (2,2,3), (2,3,3), (3,2,3)$. Since f has an isolated singularity at the origin, it follows from Cororally 1.5 that for each case, the Milnor number must be a natural number, namely

$$\mu(f) = \left(\frac{1}{r_1} - 1\right)\left(\frac{1}{r_2} - 1\right)\left(\frac{1}{r_3} - 1\right)\left(\frac{1}{r_4} - 1\right) \in \mathbb{N}. \tag{2.5}$$

From here we try to find the bound of m_1 by considering the condition (2.5) in addition to the condition (2.4). To do this, for all permutations (j_1, j_2, j_3, j_4) of the set $\{1,2,3,4\}$ with repetition, we calculate (r_1, r_2, r_3, r_4), $\alpha(f)$ and $\mu(f)$ for the quasihomogeneous polynomial $f(x_1, x_2, x_3, x_4)$ contains monomials $x_1^{m_1}x_{j_1}$, $x_2^{m_2}x_{j_2}$, $x_3^{m_3}x_{j_3}$, $x_4^{m_4}x_{j_4}$ with non-zero coefficients, and try to find the bound of m_1. We show such a calculation by an example.

As an example, we consider the case of $(m_2, m_3, m_4) = (2, 3, 2)$. If $(j_1, j_2, j_3, j_4) = (3, 1, 3, 1)$, then we have

$$
\begin{aligned}
(r_1, r_2, r_3, r_4) &= \left(\frac{3}{4m_1}, \frac{4m_1 - 3}{8m_1}, \frac{1}{4}, \frac{4m_1 - 3}{8m_1}\right) \\
r_3 \le r_2 = r_4 \le r_1 &\quad r_{max} = r_1 \quad \text{if } m = 2 \\
r_1 \le r_2 \le r_3 \le r_4 &\quad r_{max} = r_4 \quad \text{if } m \ge 3 \\
\alpha(f) - 1 &= \begin{cases} \dfrac{10m_1 - 3}{8m_1} & > 0 \text{ for } m_1 = 2 \\ \dfrac{3}{16m_1} & > 0 \text{ for } m_1 \ge 3 \end{cases} \\
\mu(f) &= \frac{(4m_1 + 3)^2}{4m_1 - 3} = 4m_1 + 9 + \frac{36}{4m_1 - 3}.
\end{aligned}
$$

Since $\mu(f) \in \mathbb{N}$, we have $m_1 = 3$. If $(j_1, j_2, j_3) = (1, 1, 1, 4)$, then we have

$$
\begin{aligned}
(r_1, r_2, r_3, r_4) &= \left(\frac{1}{m_1 + 1}, \frac{m_1}{2(m_1 + 1)}, \frac{m_1}{3(m_1 + 1)}, \frac{1}{3}\right) \\
r_{max} &= r_2 \\
\alpha(f) - 1 &= \frac{4 - m_1}{12(m_1 + 1)} \\
\mu(f) &= \frac{2(m_1 + 2)(2m_1 + 3)}{m_1} = 4m_1 + 14 + \frac{12}{m_1}
\end{aligned}
$$

Since $\alpha(f) - 1 > 0$, we have $m_1 = 2, 3$. Since $\mu \in \mathbb{N}$, we have $m_1 = 2, 3, 4, 6, 12$. Hence we have $m_1 = 2, 3$. By doing same calculations for the other permutations (j_1, j_2, j_3, j_4) by a computer (see a sample program in Appendix B), we have

$$m_1 \le 12 \quad \text{for } (m_2, m_3, m_4) = (2, 3, 2).$$

Similarly we perform such a calculation for the other values of (m_2, m_3, m_4) and each permutaion (j_1, j_2, j_3, j_4) by a computer. As a result, we have the bound of m_1 as a necessary condition to satisfy $\alpha(f) - 1 > 0$ and $\mu(f) \in \mathbb{N}$. The bound of m_1 is

$$
m_1 = \begin{cases} 2, \ldots, 8 & \text{if } (m_2, m_3, m_4) = (2, 2, 2). \\ 2, \ldots, 12 & \text{if } (m_2, m_3, m_4) = (2, 2, 3), (2, 3, 2), (3, 2, 2). \\ 2, 3 & \text{if } (m_2, m_3, m_4) = (2, 3, 3), (3, 2, 3), (4, 2, 2). \end{cases}
$$

This completes the proof. □

3. Classification

In this section, we give the complete classification of quasihomogeneous polynomials with isolated singularity and with $m_0 = m_a$. Let f be a quasihomogeneous polynomial of corank= k with isolated singularity that contains k monomials

$$x_1^{m_1}x_{j_1}, \ldots, x_k^{m_k}x_{j_k} \quad (m_1, \ldots, m_k \geq k,\ j_1, \ldots, j_k \in \{1, 2, \ldots, k\})$$

with non-zero coefficients. In the previous section, for each corank k ($k = 3, 4$), we have the limit value of $m_1, \ldots, m_k$ and the values of $(j_1, \ldots, j_k)$ as neccessary conditions for $m_0(f) = m_a(f)$ (see Lemma 2.5, 2.6). Hence in order to classify quasihomogeneous polynomials with $m_0(f) = m_a(f)$, it is sufficient to select those which satisfy both conditions

(a) $m_0(f) = m_a(f)$, i.e. $\alpha(f) > \dfrac{k}{2} - 1$ (see Corollary 2.2),

(b) f has an isolated singularity at the origin,

from f with $(m_1, \ldots, m_k)$ and $(j_1, \ldots, j_k)$ in Lemma 2.5, 2.6.

Theorem 3.1.

(1) For all f of corank $= 1, 2$, *$m_0(f) = m_a(f)$ is always established.*
(2) For all f of corank $= 3$, *f with $m_0(f) = m_a(f)$ are exhausted by the list in Table A.*
(3) For all f of corank $= 4$, *f with $m_0(f) = m_a(f)$ are exhausted by the list in Table B.*

Proof.

(1) The result follows immediately from Lemm 2.4.

(2) According to $(m_1, \ldots, m_k)$, $(j_1, \ldots, j_k)$ in Lemma 2.5, we select all quasihomogeneous polynomials with isolated singularity that satisfy the both conditions

(a) $\alpha(f) - \dfrac{1}{2} = r_1 + r_2 + r_3 - \dfrac{1}{2} r_{max} - \dfrac{1}{2} > 0$,

(b) $\chi_f(z) = \dfrac{(z^{N-A_1} - 1)(z^{N-A_2} - 1)(z^{N-A_3} - 1)}{(z^{A_1} - 1)(z^{A_2} - 1)(z^{A_3} - 1)} \in \mathbb{C}[z]$,
where N, $A_i's$ are the natural number defined in Theorem 1.4,

by a computer (see Appendix C as a sample program). As a result, we have 215 types of quasihomogeneous polynomials, 17 of which are infinite sequences (see Table A). In the computer program, in order to determine whether f has an isolated singularity, we use the fact that $\chi_f(z)$ is a polynomial. However we should note that Theorem 1.4 doesn't guarantee that $\chi_f(z)$ is a polynomial "only if" f has an isolated singularity at the origin. Fortunately in case of corank$(f) = 3$, it is shown that there exists a quasihomogeneous polynomial of type $(1; r_1, r_2, r_3)$ with isolated singularity at the origin, if $\chi_f(z)$ is a polynomial (see [10], Theorem 3 and also see [1], *Remark* in Page 22). Hence for all the wights we get by calculation, we can determine quasihomogeneous polynomials with isolated singularity and we have the list in Table A at the end of this section.

(3) According to $(m_1, \ldots, m_k)$, $(j_1, \ldots, j_k)$ in Lemma 2.6, we select all quasihomogeneous polynomials with isolated singularity that satisfy the both conditions

(a) $\alpha(f) - 1 = r_1 + r_2 + r_3 + r_4 - \dfrac{1}{2} r_{max} - 1 > 0,$

(b) $\chi_f(z) \;=\; \dfrac{(z^{N-A_1}-1)(z^{N-A_2}-1)(z^{N-A_3}-1)(z^{N-A_4}-1)}{(z^{A_1}-1)(z^{A_2}-1)(z^{A_3}-1)(z^{A_4}-1)} \;\in\; \mathbb{C}[z],$
where N, $A_i's$ are the natural number defined in Theorem 1.4,

by a computer (see Appendix D as a sample program). As a result, we have 25 types of quasihomogeneous polynomials (see Table B). As with corank $= 3$, we should note that Theorem 1.4 doesn't guarantee that $\chi_f(z)$ is a polynomial "only if" f has an isolated singularity at the origin. In fact, in the case of corank $= 4$, there is a example which says that there is no quasihomogeneous polynomial of type $(1; \frac{1}{265}, \frac{24}{265}, \frac{33}{265}, \frac{58}{265})$ with isolated singularity at the origin even though $\chi_f(z)$ is a polynomial (see [1], *Remark* in Page 22). But we see that all the quasihomogeneous polynomials which we get as the result of calculations have an isolated singularity at the origin. Hence we have all the weights of quasihomogeneous polynomials of corank $= 4$ with $m_0 = m_a$ in the list in Table B at the end of this section.

This completes the proof. □

Table A : The list of quasihomogeneous polynomials of corank=3 with $m_0 = m_a$

The following table contains the list of 17 infinite sequences of weights of quasihomogeneous polynomials of corank = 3 with $m_0 = m_a$.

m_0	(r_1, r_2, r_3)		
$*$	$\left(\frac{1}{m+1}, \frac{m}{2(m+1)}, \frac{m+2}{4(m+1)}\right)$	$\left(\frac{1}{m+1}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{2m-1}, \frac{m-1}{2m-1}, \frac{m}{2(2m-1)}\right)$
$*$	$\left(\frac{2}{3m}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{3}{4m+1}, \frac{m+1}{4m+1}, \frac{2m-1}{4m+1}\right)$	$\left(\frac{1}{m+1}, \frac{1}{2(m+1)}, \frac{1}{3}\right)$
$*$	$\left(\frac{2}{3m}, \frac{1}{3}, \frac{3m-2}{6m}\right)$	$\left(\frac{1}{2m-1}, \frac{m-1}{2m-1}, \frac{1}{3}\right)$	$\left(\frac{1}{m+1}, \frac{m}{2(m+1)}, \frac{1}{4}\right)$
$*$	$\left(\frac{1}{m+1}, \frac{1}{3}, \frac{m}{3(m+1)}\right)$	$\left(\frac{1}{m+1}, \frac{2m+3}{6(m+1)}, \frac{m}{3(m+1)}\right)$	$\left(\frac{1}{2m-1}, \frac{m-1}{2m-1}, \frac{1}{4}\right)$
$*$	$\left(\frac{2}{3m}, \frac{1}{3}, \frac{3m-2}{9m}\right)$	$\left(\frac{3}{4m}, \frac{4m-3}{8m}, \frac{1}{4}\right)$	$\left(\frac{2}{3m-1}, \frac{1}{3}, \frac{m-1}{3m-1}\right)$
$*$	$\left(\frac{2}{3m-1}, \frac{m}{3m-1}, \frac{m-1}{3m-1}\right)$	$\left(\frac{4}{6m+1}, \frac{2m+1}{6m+1}, \frac{2m-1}{6m+1}\right)$	

where $m \in \mathbb{N}$ and $m \geq 2$.

The following table contains the list of 198 weights of quasihomogeneous polynomials of corank = 3 with $m_0 = m_a$.

m_0	(r_1, r_2, r_3)				
1	$\left(\frac{1}{4}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{4}, \frac{1}{3}, \frac{3}{8}\right)$	$\left(\frac{1}{4}, \frac{5}{16}, \frac{3}{8}\right)$	$\left(\frac{3}{13}, \frac{4}{13}, \frac{5}{13}\right)$	$\left(\frac{2}{9}, \frac{1}{3}, \frac{7}{18}\right)$
1	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{5}, \frac{1}{3}, \frac{2}{5}\right)$			
2	$\left(\frac{2}{9}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{1}{3}, \frac{5}{12}\right)$	$\left(\frac{2}{15}, \frac{1}{3}, \frac{13}{30}\right)$	$\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{7}, \frac{1}{3}, \frac{3}{7}\right)$
2	$\left(\frac{1}{8}, \frac{1}{3}, \frac{7}{16}\right)$	$\left(\frac{1}{5}, \frac{3}{10}, \frac{2}{5}\right)$	$\left(\frac{1}{6}, \frac{7}{24}, \frac{5}{12}\right)$	$\left(\frac{3}{17}, \frac{5}{17}, \frac{7}{17}\right)$	
3	$\left(\frac{3}{16}, \frac{1}{4}, \frac{13}{32}\right)$	$\left(\frac{1}{9}, \frac{1}{3}, \frac{4}{9}\right)$	$\left(\frac{2}{21}, \frac{1}{3}, \frac{19}{42}\right)$	$\left(\frac{1}{5}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{5}, \frac{1}{4}, \frac{2}{5}\right)$
3	$\left(\frac{1}{11}, \frac{1}{3}, \frac{5}{11}\right)$	$\left(\frac{1}{5}, \frac{4}{15}, \frac{11}{30}\right)$	$\left(\frac{5}{24}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{4}{19}, \frac{5}{19}, \frac{7}{19}\right)$
4	$\left(\frac{2}{9}, \frac{7}{27}, \frac{1}{3}\right)$	$\left(\frac{3}{20}, \frac{1}{4}, \frac{17}{40}\right)$	$\left(\frac{2}{9}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{1}{4}, \frac{5}{12}\right)$
4	$\left(\frac{1}{7}, \frac{1}{4}, \frac{3}{7}\right)$	$\left(\frac{1}{13}, \frac{1}{3}, \frac{6}{13}\right)$	$\left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}\right)$	$\left(\frac{1}{8}, \frac{9}{32}, \frac{7}{16}\right)$	$\left(\frac{3}{25}, \frac{7}{25}, \frac{11}{25}\right)$
5	$\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{7}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{5}, \frac{4}{15}, \frac{1}{3}\right)$	$\left(\frac{1}{5}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{2}{11}, \frac{3}{11}, \frac{1}{3}\right)$
5	$\left(\frac{1}{9}, \frac{5}{18}, \frac{4}{9}\right)$	$\left(\frac{1}{6}, \frac{5}{18}, \frac{13}{36}\right)$	$\left(\frac{4}{25}, \frac{7}{25}, \frac{9}{25}\right)$	$\left(\frac{3}{29}, \frac{8}{29}, \frac{13}{29}\right)$	
6	$\left(\frac{2}{15}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{5}{32}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{8}, \frac{1}{4}, \frac{7}{16}\right)$	$\left(\frac{1}{6}, \frac{5}{18}, \frac{1}{3}\right)$	$\left(\frac{3}{28}, \frac{1}{4}, \frac{25}{56}\right)$
6	$\left(\frac{3}{16}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{8}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{9}, \frac{1}{4}, \frac{4}{9}\right)$	
7	$\left(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right)$	$\left(\frac{1}{6}, \frac{5}{24}, \frac{19}{48}\right)$	$\left(\frac{4}{25}, \frac{1}{5}, \frac{21}{50}\right)$	$\left(\frac{7}{36}, \frac{2}{9}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{1}{5}, \frac{5}{12}\right)$
7	$\left(\frac{1}{5}, \frac{2}{9}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{1}{5}, \frac{2}{5}\right)$	$\left(\frac{1}{7}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{11}, \frac{1}{4}, \frac{5}{11}\right)$	$\left(\frac{1}{7}, \frac{2}{7}, \frac{5}{14}\right)$
7	$\left(\frac{1}{11}, \frac{3}{11}, \frac{5}{11}\right)$	$\left(\frac{1}{8}, \frac{7}{24}, \frac{17}{48}\right)$	$\left(\frac{1}{6}, \frac{7}{36}, \frac{5}{12}\right)$	$\left(\frac{4}{31}, \frac{9}{31}, \frac{11}{31}\right)$	$\left(\frac{5}{31}, \frac{6}{31}, \frac{13}{31}\right)$

m_0	(r_1, r_2, r_3)				
8	$\left(\frac{2}{15}, \frac{13}{45}, \frac{1}{3}\right)$	$\left(\frac{1}{7}, \frac{1}{5}, \frac{3}{7}\right)$	$\left(\frac{1}{7}, \frac{2}{7}, \frac{1}{3}\right)$	$\left(\frac{1}{13}, \frac{7}{26}, \frac{6}{13}\right)$	$\left(\frac{5}{33}, \frac{7}{33}, \frac{13}{33}\right)$
9	$\left(\frac{1}{9}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{3}{20}, \frac{1}{5}, \frac{2}{5}\right)$	$\left(\frac{1}{7}, \frac{3}{14}, \frac{11}{28}\right)$	$\left(\frac{2}{15}, \frac{1}{5}, \frac{13}{30}\right)$	$\left(\frac{1}{8}, \frac{1}{5}, \frac{7}{16}\right)$
9	$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{3}\right)$	$\left(\frac{3}{20}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{7}, \frac{1}{5}, \frac{2}{5}\right)$	$\left(\frac{1}{13}, \frac{1}{4}, \frac{6}{13}\right)$
9	$\left(\frac{1}{7}, \frac{4}{21}, \frac{3}{7}\right)$	$\left(\frac{2}{17}, \frac{5}{17}, \frac{1}{3}\right)$	$\left(\frac{1}{8}, \frac{7}{24}, \frac{1}{3}\right)$	$\left(\frac{4}{37}, \frac{11}{37}, \frac{13}{37}\right)$	$\left(\frac{5}{37}, \frac{7}{37}, \frac{16}{37}\right)$
10	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{2}{21}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{5}, \frac{11}{45}, \frac{4}{15}\right)$	$\left(\frac{4}{35}, \frac{1}{5}, \frac{31}{70}\right)$	$\left(\frac{1}{9}, \frac{1}{5}, \frac{4}{9}\right)$
10	$\left(\frac{1}{5}, \frac{1}{4}, \frac{4}{15}\right)$	$\left(\frac{1}{7}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{9}, \frac{8}{27}, \frac{1}{3}\right)$		
11	$\left(\frac{1}{9}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{6}, \frac{1}{6}, \frac{5}{12}\right)$	$\left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{6}, \frac{2}{9}, \frac{1}{3}\right)$	$\left(\frac{7}{48}, \frac{1}{6}, \frac{5}{12}\right)$
11	$\left(\frac{1}{8}, \frac{7}{32}, \frac{25}{64}\right)$	$\left(\frac{1}{7}, \frac{6}{35}, \frac{29}{70}\right)$	$\left(\frac{5}{36}, \frac{1}{6}, \frac{31}{72}\right)$	$\left(\frac{3}{16}, \frac{1}{4}, \frac{13}{48}\right)$	$\left(\frac{2}{15}, \frac{1}{4}, \frac{1}{3}\right)$
11	$\left(\frac{1}{5}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{7}, \frac{1}{6}, \frac{3}{7}\right)$	$\left(\frac{1}{7}, \frac{1}{6}, \frac{5}{12}\right)$	$\left(\frac{7}{45}, \frac{2}{9}, \frac{1}{3}\right)$	$\left(\frac{2}{11}, \frac{1}{4}, \frac{3}{11}\right)$
11	$\left(\frac{1}{10}, \frac{3}{10}, \frac{7}{20}\right)$	$\left(\frac{5}{41}, \frac{9}{41}, \frac{16}{41}\right)$	$\left(\frac{6}{41}, \frac{7}{41}, \frac{17}{41}\right)$	$\left(\frac{7}{37}, \frac{9}{37}, \frac{10}{37}\right)$	$\left(\frac{4}{43}, \frac{13}{43}, \frac{15}{43}\right)$
11	$\left(\frac{5}{43}, \frac{8}{43}, \frac{19}{43}\right)$	$\left(\frac{1}{8}, \frac{3}{16}, \frac{7}{16}\right)$			
12	$\left(\frac{1}{6}, \frac{5}{24}, \frac{1}{3}\right)$	$\left(\frac{1}{8}, \frac{1}{5}, \frac{2}{5}\right)$	$\left(\frac{5}{48}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{3}{19}, \frac{4}{19}, \frac{1}{3}\right)$	$\left(\frac{1}{9}, \frac{5}{27}, \frac{4}{9}\right)$
12	$\left(\frac{1}{10}, \frac{3}{10}, \frac{1}{3}\right)$	$\left(\frac{2}{21}, \frac{19}{63}, \frac{1}{3}\right)$			
13	$\left(\frac{1}{10}, \frac{11}{60}, \frac{9}{20}\right)$	$\left(\frac{1}{10}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{5}{49}, \frac{9}{49}, \frac{22}{49}\right)$	$\left(\frac{3}{16}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{6}, \frac{13}{54}, \frac{5}{18}\right)$
13	$\left(\frac{5}{42}, \frac{1}{6}, \frac{37}{84}\right)$	$\left(\frac{1}{10}, \frac{1}{5}, \frac{9}{20}\right)$	$\left(\frac{1}{11}, \frac{1}{5}, \frac{5}{11}\right)$	$\left(\frac{1}{6}, \frac{1}{4}, \frac{5}{18}\right)$	$\left(\frac{4}{25}, \frac{1}{5}, \frac{1}{3}\right)$
13	$\left(\frac{1}{8}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{1}{5}, \frac{1}{3}\right)$	$\left(\frac{1}{8}, \frac{1}{6}, \frac{7}{16}\right)$	$\left(\frac{3}{25}, \frac{1}{5}, \frac{2}{5}\right)$	$\left(\frac{4}{45}, \frac{1}{5}, \frac{41}{90}\right)$
13	$\left(\frac{1}{7}, \frac{2}{9}, \frac{1}{3}\right)$				
14	$\left(\frac{1}{9}, \frac{2}{9}, \frac{7}{18}\right)$	$\left(\frac{1}{10}, \frac{9}{40}, \frac{31}{80}\right)$	$\left(\frac{1}{8}, \frac{7}{40}, \frac{33}{80}\right)$	$\left(\frac{3}{20}, \frac{1}{4}, \frac{17}{60}\right)$	$\left(\frac{1}{12}, \frac{1}{5}, \frac{11}{24}\right)$
14	$\left(\frac{1}{7}, \frac{3}{14}, \frac{1}{3}\right)$	$\left(\frac{1}{9}, \frac{1}{5}, \frac{2}{5}\right)$	$\left(\frac{5}{49}, \frac{11}{49}, \frac{19}{49}\right)$		
15	$\left(\frac{1}{11}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{2}{15}, \frac{13}{60}, \frac{1}{3}\right)$	$\left(\frac{1}{13}, \frac{1}{5}, \frac{6}{13}\right)$	$\left(\frac{1}{7}, \frac{1}{4}, \frac{2}{7}\right)$	$\left(\frac{5}{56}, \frac{1}{4}, \frac{3}{8}\right)$
15	$\left(\frac{3}{23}, \frac{5}{23}, \frac{1}{3}\right)$	$\left(\frac{2}{25}, \frac{1}{5}, \frac{23}{50}\right)$	$\left(\frac{7}{46}, \frac{11}{46}, \frac{13}{46}\right)$	$\left(\frac{7}{54}, \frac{2}{9}, \frac{1}{3}\right)$	
16	$\left(\frac{1}{14}, \frac{1}{5}, \frac{13}{28}\right)$	$\left(\frac{1}{8}, \frac{7}{32}, \frac{1}{3}\right)$	$\left(\frac{3}{28}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{7}, \frac{1}{5}, \frac{1}{3}\right)$	$\left(\frac{1}{9}, \frac{1}{4}, \frac{1}{3}\right)$
16	$\left(\frac{4}{55}, \frac{1}{5}, \frac{51}{110}\right)$	$\left(\frac{1}{8}, \frac{2}{9}, \frac{1}{3}\right)$			
17	$\left(\frac{1}{5}, \frac{1}{5}, \frac{4}{15}\right)$	$\left(\frac{1}{11}, \frac{5}{22}, \frac{17}{44}\right)$	$\left(\frac{1}{7}, \frac{5}{21}, \frac{2}{7}\right)$	$\left(\frac{11}{60}, \frac{1}{5}, \frac{4}{15}\right)$	$\left(\frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$
17	$\left(\frac{2}{11}, \frac{1}{5}, \frac{3}{11}\right)$	$\left(\frac{5}{57}, \frac{13}{57}, \frac{22}{57}\right)$	$\left(\frac{9}{49}, \frac{10}{49}, \frac{13}{49}\right)$		
18	$\left(\frac{1}{10}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{12}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{13}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{5}{64}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{1}{12}, \frac{11}{48}, \frac{37}{96}\right)$
19	$\left(\frac{3}{32}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{3}{20}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{8}, \frac{17}{72}, \frac{7}{24}\right)$	$\left(\frac{3}{16}, \frac{13}{64}, \frac{1}{4}\right)$	$\left(\frac{1}{8}, \frac{1}{4}, \frac{7}{24}\right)$
19	$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{4}\right)$	$\left(\frac{3}{16}, \frac{1}{5}, \frac{1}{4}\right)$	$\left(\frac{1}{6}, \frac{1}{5}, \frac{5}{18}\right)$	$\left(\frac{2}{21}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{2}{17}, \frac{1}{4}, \frac{5}{17}\right)$
19	$\left(\frac{7}{55}, \frac{13}{55}, \frac{16}{55}\right)$				
20	$\left(\frac{1}{11}, \frac{1}{4}, \frac{1}{3}\right)$	$\left(\frac{1}{6}, \frac{5}{24}, \frac{19}{72}\right)$	$\left(\frac{1}{9}, \frac{1}{4}, \frac{8}{27}\right)$	$\left(\frac{1}{7}, \frac{1}{4}, \frac{1}{4}\right)$	
21	$\left(\frac{1}{14}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{5}{72}, \frac{1}{4}, \frac{3}{8}\right)$	$\left(\frac{3}{28}, \frac{1}{4}, \frac{25}{84}\right)$		
22	$\left(\frac{1}{15}, \frac{1}{4}, \frac{3}{8}\right)$				

Table B : The list of 25 weights of quasihomogeneous polynomials of corank=4 with $m_0 = m_a$

The following table contains the list of 25 weights of quasihomogeneous polynomials of corank = 4 with $m_0 = m_a$.

m_0	(r_1, r_2, r_3, r_4)			
5	$\left(\frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{3}{8}\right)$	$\left(\frac{1}{4}, \frac{5}{16}, \frac{1}{3}, \frac{3}{8}\right)$	$\left(\frac{1}{4}, \frac{5}{16}, \frac{11}{32}, \frac{3}{8}\right)$	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
6	$\left(\frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{5}{23}, \frac{7}{23}, \frac{8}{23}, \frac{9}{23}\right)$	$\left(\frac{3}{13}, \frac{4}{13}, \frac{1}{3}, \frac{5}{13}\right)$	$\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right)$
6	$\left(\frac{2}{9}, \frac{11}{36}, \frac{1}{3}, \frac{7}{18}\right)$	$\left(\frac{2}{9}, \frac{1}{3}, \frac{1}{3}, \frac{7}{18}\right)$		
7	$\left(\frac{1}{5}, \frac{3}{10}, \frac{7}{20}, \frac{2}{5}\right)$			
8	$\left(\frac{1}{5}, \frac{1}{4}, \frac{3}{8}, \frac{2}{5}\right)$	$\left(\frac{1}{5}, \frac{3}{10}, \frac{1}{3}, \frac{2}{5}\right)$	$\left(\frac{5}{24}, \frac{1}{4}, \frac{3}{8}, \frac{19}{48}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$
9	$\left(\frac{5}{31}, \frac{9}{31}, \frac{11}{31}, \frac{13}{31}\right)$	$\left(\frac{3}{17}, \frac{5}{17}, \frac{1}{3}, \frac{7}{17}\right)$	$\left(\frac{1}{7}, \frac{1}{3}, \frac{1}{3}, \frac{3}{7}\right)$	$\left(\frac{1}{6}, \frac{7}{24}, \frac{17}{48}, \frac{5}{12}\right)$
9	$\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{5}{12}\right)$	$\left(\frac{2}{9}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$		
10	$\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{8}, \frac{1}{3}, \frac{1}{3}, \frac{7}{16}\right)$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}\right)$	$\left(\frac{2}{15}, \frac{1}{3}, \frac{1}{3}, \frac{13}{30}\right)$

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Appendix A : Example of a program code for corank=3

The following program is written in Mathematica.

```
Clear["Global*"];
exp0 = {{m, 3, 2}};
var0 = Permutations[{x, x, x, y, y, y, z, z, z}, {3}];
data1 = {}; data2 = {}; data3 = {}; amax = {};

For[i = 1, i ≤ Length[exp0], i++,
{m1, m2, m3} = {exp0[[i]][[1]], exp0[[i]][[2]], exp0[[i]][[3]]};

  For[k = 1, k ≤ Length[var0], k++,
    x1 = var0[[k]][[1]]; x2 = var0[[k]][[2]]; x3 = var0[[k]][[3]];
    sol = Solve[{m1x + x1 == 1, m2y + x2 == 1, m3z + x3 == 1}, {x, y, z}];
    {r1, r2, r3} = Simplify[{x/.sol[[1]], y/.sol[[1]], z/.sol[[1]]}];
    rmax = Simplify[Max[{r1, r2, r3}], m > 3];
    a[m_]:=Factor[r1 + r2 + r3 − 1/2rmax − 1/2];  (*criteriaform0 = ma*)
    mu[m_]:=Factor[(1/r1 − 1)(1/r2 − 1)(1/r3 − 1) ]; (* Milnor # *)
    pmu[m_]:=Numerator[mu[m]]; qmu[m_]:=Denominator[mu[m]];
                                rmu[m_]:=PolynomialRemainder[pmu[m], qmu[m], m];
     If[Exponent[Numerator[a[m]], m] == 1&&Coefficient[Numerator[a[m]], m] < 0,
        data1 = Append[data1, {{x1, x2, x3}, {m1, m2, m3}, {r1, r2, r3}, a[m], mu[m],
     Flatten[Solve[a[m] == 0, m], 1]}];
        asol = Solve[a[m] == 0, m];
        amax = Append[amax, m/.asol[[1]][[1]]];
     ];

        If[Exponent[Numerator[a[m]], m] == 0,
```

```
            If[Reduce[rmu[m] − pmu[m] ≥ 0&&m > 1, m],
            data2 = Append[data2, {{x1, x2, x3}, {m1, m2, m3}, {r1, r2, r3}, a[m], mu[m],
                          Flatten[Reduce[rmu[m] − pmu[m] ≥ 0&&m > 1, m]]}],
            data2 = Append[data2, {{x1, x2, x3}, {m1, m2, m3}, {r1, r2, r3}, a[m], mu[m]}];
            ];
          ];

            If[Exponent[Numerator[a[m]], m] == 1&&Coefficient[Numerator[a[m]], m] > 0,
              If[Reduce[rmu[m] − pmu[m] ≥ 0&&m > 1, m],
                data2 = Append[data2, {{x1, x2, x3}, {m1, m2, m3}, {r1, r2, r3}, a[m], mu[m],
                          Flatten[Reduce[rmu[m] − pmu[m] ≥ 0&&m > 1, m]]}],
                data3 = Append[data3, {{x1, x2, x3}, {m1, m2, m3}, {r1, r2, r3}, a[m], mu[m]}];
              ];
            ];

  ];
];
Print[Max[amax]]; Print["data1 =", data1]; Print["data2 =", data2]; Print["data3 =", data3]
```

Appendix B : Example of a program code for corank=4

The following program is written in Mathematica.

```
Clear["Global*"];
exp0 = {{m, 2, 2, 2}};
var0 = Permutations[{x, x, x, x, y, y, y, y, z, z, z, z, w, w, w, w}, {4}];
data1 = {}; data2 = {}; data3 = {}; amax = {};
For[i = 1, i ≤ Length[exp0], i++,
exp1 = exp0[[i]];
  For[k = 1, k ≤ Length[var0], k++,
    m1 = exp1[[1]]; m2 = exp1[[2]]; m3 = exp1[[3]]; m4 = exp1[[4]];
    x1 = var0[[k]][[1]]; x2 = var0[[k]][[2]]; x3 = var0[[k]][[3]]; x4 = var0[[k]][[4]];
    sol = Solve[{m1x + x1 == 1, m2y + x2 == 1, m3z + x3 == 1, m4w + x4 == 1}, {x, y, z, w}];
    weights = {x/.sol[[1]], y/.sol[[1]], z/.sol[[1]], w/.sol[[1]]};
    r1 = Simplify[weights[[1]]]; r2 = Simplify[weights[[2]]]; r3 = Simplify[weights[[3]]];
                                    r4 = Simplify[weights[[4]]];
    rmax = Simplify[Max[{r1, r2, r3, r4}], m > 3];
    mu = Factor[(1/r1 − 1)(1/r2 − 1)(1/r3 − 1)(1/r4 − 1)]; (* Milnor # *)
    a = Factor[r1 + r2 + r3 + r4 − (1/2)rmax − 1]; (*criteriaform0 = ma*)
      If[Exponent[Numerator[a], m] == 1&&Coefficient[Numerator[a], m] < 0,
        ClearAll[m];
        data1 = Append[data1, {{x1, x2, x3, x4}, {m1, m2, m3, m4},
                          Factor[{r1, r2, r3, r4}], a, mu,
```

```
                  Flatten[Solve[r1 + r2 + r3 + r4 − (1/2)rmax − 1 == 0, m], 1]}];
          amax = Append[amax, m/.Flatten[Solve[r1 + r2 + r3 + r4 − (1/2)rmax − 1 == 0, m]]];
        ];
          If[Exponent[Numerator[a], m] == 0,
            ClearAll[m];
            data2 = Append[data2, {{x1, x2, x3, x4}, {m1, m2, m3, m4},
                                  Factor[{r1, r2, r3, r4}], a, mu,
           Reduce[{PolynomialRemainder[Numerator[mu],
              Denominator[mu], m]>=Denominator[mu]&&m > 1}, m, Integers]}];
          ];
            If[Exponent[Numerator[a], m] == 1&&Coefficient[Numerator[a], m] > 0,
              ClearAll[m];
              data3 = Append[data3, {{x1, x2, x3, x4}, {m1, m2, m3, m4},
                                  Factor[{r1, r2, r3, r4}], a, mu,
             Reduce[{PolynomialRemainder[Numerator[mu], Denominator[mu], m]
                        >=Denominator[mu]&&m > 1}, m, Integers]}];
            ];
      ];
   ];
Print[Max[amax]];
Print["data1 :", data1, Max[amax]]; Print["data2 :", data2]; Print["data3 :", data3]//Timing
```

Appendix C : Example of a program code for corank=3

The following program is written in Mathematica.

```
exp0 = {{2, 4, 13}, {4, 2, 13},
   {3, 3, 9},
   {5, 2, 19},
   {6, 2, 11}, {4, 3, 11},
   {7, 2, 9},
   {5, 3, 7}, {4, 4, 7},
   {3, 4, 23},
   {5, 4, 5}};
exp1 = {{2, 2, 4}};
exp2 = {{2, 3, 14}};
exp3 = {{3, 2, 14}};
var0 = Permutations[{x, x, x, y, y, y, z, z, z}, {3}];
var1 = {{x, x, x}, {y, x, x}, {z, x, x}};
var2 = Complement[var0, {{x, x, z}, {x, y, x}, {x, z, x}, {y, x, z}, {y, y, x}, {y, z, x}, {z, x, z},
   {z, y, x}, {z, z, x}}];
var3 = Complement[var0, {{x, x, y}, {x, x, z}, {x, y, x}, {y, x, y}, {y, x, z}, {y, y, x}, {z, x, y},
   {z, x, z}, {z, y, x}}];
```

```
classify[exp_, var_]:=Module[{i, j, sol = {}, m, m1, m2, m3, wt = {}, r1, r2, r3, d, q1, q2,
        q3, p1, p2, p3, Q, A1, A2, A3, CPN, CPD, CP, a, CF, IM, AIM, EXP, wts = {}},
For[i = 1, i ≤ Length[var], i++,
  {x1, x2, x3} = {var[[i]][[1]], var[[i]][[2]], var[[i]][[3]]};
    For[k = 1, k ≤ Length[exp], k++,
    {m, m2, m3} = {exp[[k]][[3]], exp[[k]][[1]], exp[[k]][[2]]};

      Do[
        sol = Solve[{m1x + x1 == 1, m2y + x2 == 1, m3z + x3 == 1}, {x, y, z}];
        wt = Sort[{x/.sol[[1]], y/.sol[[1]], z/.sol[[1]]}];
        {r1, r2, r3} = Sort[{r1 = wt[[1]], r2 = wt[[2]], r3 = wt[[3]]}];
        d = 3 − 2 ∗ (r1 + r2 + r3);
        {q1, q2, q3} = {Denominator[r1], Denominator[r2], Denominator[r3]};
        {p1, p2, p3} = {Numerator[r1], Numerator[r2], Numerator[r3]};
        Q = LCM[q1, q2, q3];
        {A1, A2, A3} = {p1 ∗ Quotient[Q, q1], p2 ∗ Quotient[Q, q2], p3 ∗ Quotient[Q, q3]};

        CPN = Expand[(z^{Q−A1} − 1)(z^{Q−A2} − 1)(z^{Q−A3} − 1)];
        CPD = Expand[(z^{A1} − 1)(z^{A2} − 1)(z^{A3} − 1)];
        CP = PolynomialQuotientRemainder[CPN[[1]], CPD[[1]], z];
        a = r1 + r2 + (1/2)r3 − 1/2;
        CF = Take[CoefficientList[CP[[1]], z], (d − 1)Q + 1];
        IM = Sum_{i=1}^{(d−1)Q+1} CF[[i]];
        AIM = Length[{ToRules[Reduce[{(2 + x)A1 + (2 + y)A2 + (2 + z)A3 − 2Q ≤ 0,
                    x ≥ 0, y ≥ 0, z ≥ 0}, {x, y, z}, Integers]]}];

          If[(CP[[2]] == 0) &&(a > 0)&&(MemberQ[wts, {r1, r2, r3, IM, AIM}] == False),
            wts = Append[wts, {r1, r2, r3, IM, AIM}];
          ];

      , {m1, 2, m}

      ];
    ];
  ];
wts = Sort[wts, #1[[4]] < #2[[4]]&];
Return[wts];
]
wts0 = classify[exp0, var0];
wts1 = classify[exp1, var1];
wts2 = classify[exp2, var2];
```

```
wts3 = classify[exp3, var3];
all = Sort[DeleteDuplicates[Join[wts0, wts1, wts2, wts3]], #1[[4]] < #2[[4]]&];
Print["count=", Length[all], ",", " Max of Inner modality=", Max[Table[all[[i]][[4]],
                                              {i, 1, Length[all]}]], "\n", all];
```

Appendix D : Example of a program code for corank=4

The following program is written in Mathematica.

```
exp0 = {{2, 2, 2, 8},
  {2, 2, 3, 12}, {2, 3, 2, 12}, {3, 2, 2, 12},
  {2, 3, 3, 3}, {3, 2, 3, 3}, {4, 2, 2, 3},
  {5, 2, 2, 19},
  {6, 2, 2, 11}, {3, 3, 2, 11}, {4, 2, 3, 11},
  {7, 2, 2, 9},
  {4, 3, 2, 5}, {3, 3, 3, 5},
  {5, 2, 3, 7}};
var0 = Permutations[{x, x, x, x, y, y, y, y, z, z, z, z, w, w, w, w}, {4}];
classify[exp_, var_]:=Module[{i, j, sol = {}, m, m1, m2, m3, m4, wt = {}, r1, r2, r3, r4, d,
    q1, q2, q3, q4, p1, p2, p3, p4, Q, A1, A2, A3, A4, CPN, CPD, CP, a, CF, IM, AIM, EXP, wts = {}},
For[i = 1, i ≤ Length[var], i++,
  {x1, x2, x3, x4} = {var[[i]][[1]], var[[i]][[2]], var[[i]][[3]], var[[i]][[4]]};
    For[j = 1, j ≤ Length[exp], j++,
    {m, m2, m3, m4} = {exp[[j]][[4]], exp[[j]][[1]], exp[[j]][[2]], exp[[j]][[3]]};

      Do[
        sol = Solve[{m1x + x1 == 1, m2y + x2 == 1, m3z + x3 == 1, m4w + x4 == 1}, {x, y, z, w}];
        wt = Sort[{x/.sol[[1]], y/.sol[[1]], z/.sol[[1]], w/.sol[[1]]}];
        {r1, r2, r3, r4} = Sort[{wt[[1]], wt[[2]], wt[[3]], wt[[4]]}];
        d = 4 − 2 ∗ (r1 + r2 + r3 + r4);
        {q1, q2, q3, q4} = {Denominator[r1], Denominator[r2], Denominator[r3],
                                        Denominator[r4]};
        {p1, p2, p3, p4} = {Numerator[r1], Numerator[r2], Numerator[r3], Numerator[r4]};
        Q = LCM[q1, q2, q3, q4];
        {A1, A2, A3, A4} = {p1 ∗ Quotient[Q, q1], p2 ∗ Quotient[Q, q2], p3 ∗ Quotient[Q, q3],
                                        p4 ∗ Quotient[Q, q4]};

        CPN = Expand[(z^{Q−A1} − 1)(z^{Q−A2} − 1)(z^{Q−A3} − 1)(z^{Q−A4} − 1)];
        CPD = Expand[(z^{A1} − 1)(z^{A2} − 1)(z^{A3} − 1)(z^{A4} − 1)];
        CP = PolynomialQuotientRemainder[CPN[[1]], CPD[[1]], z];
        a = r1 + r2 + r3 + (1/2)r4 − 1;

        CF = Take[CoefficientList[CP[[1]], z], (d − 1)Q + 1];
        IM = Sum_{i=1}^{(d−1)Q+1} CF[[i]];
```

```
        AIM = Length[{ToRules[Reduce[{(2 + x)A1 + (2 + y)A2 + (2 + z)A3 + (2 + w)A4 − 3Q ≤ 0,
                    x ≥ 0, y ≥ 0, z ≥ 0, w ≥ 0}, {x, y, z, w}, Integers]]}];
          If[(CP[[2]] == 0) &&(a > 0)&&(MemberQ[wts, {r1, r2, r3, r4, IM, AIM}] == False),

            wts = Append[wts, {r1, r2, r3, r4, IM, AIM}];
          ];

      , {m1, 2, m}

      ];
    ];
  ];
wts = Sort[wts, #1[[5]] < #2[[5]]&];
Return[wts];
]
all = classify[exp0, var0];
Print["count=", Length[all], ",", " Max of Inner modality=", Max[Table[all[[i]][[5]],
                                                {i, 1, Length[all]}]], "\n", all];
```

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